The Non-Equivalence of $\tau$-Ultracompactness and $\tau$-Boundedness

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Abstract. The main result presented here is a solution to the following problem of V. Saks: Does there exist $\mathfrak{M} > \aleph_0$ and a Hausdorff $\mathfrak{M}$-ultracompact space which is not $\mathfrak{M}$-bounded? The main result is given in a stronger form than the problem suggests itself: For each infinite cardinal $\tau$ there is a Hausdorff $\tau$-ultracompact not $\tau$-bounded space of density $\tau$.

In [1] A. Bernstein introduced the following definitions: let $p \in \beta\omega \setminus \omega$ be a free ultrafilter on $\omega$, the (discrete space) of positive integers. Now let $(x_n : n \in \omega)$ (for short $(x_n)$) be a sequence of points in a topological space $X$ and $x \in X$. Then $x$ is a $p$-limit point of $(x_n)$ provided that for each neighborhood $U$ of $x$ the set $\{n \in \omega : x_n \in U\}$ belongs to $p$, in this case we write $x = p - \lim x_n$. If every sequence in $X$ has a $p$-limit point then $X$ is called $p$-compact. Each infinite cardinal is identified with the initial ordinal of the same cardinality.

V. Saks [2] generalizes the notion of a $p$-limit point to transfinite sequences in the following way: let $\tau$ be an infinite cardinal; if $p \in \beta\tau \setminus \tau$ is a free ultrafilter on $\tau$ (with the discrete topology) and $(x_\alpha : \alpha \in \tau)$ (for short $(x_\alpha)$) is a $\tau$-sequence in a space $X$, then $x \in X$ is a $p$-limit point of $(x_\alpha)$, denoted by $x = p - \lim x_\alpha$, if for each neighborhood $U$ of $x$, $\{\alpha : x_\alpha \in U\} \in p$ and we can say, in this case, that $(x_\alpha)$ $p$-converges to $x$. Saks further extends $p$-compactness for any ultrafilter $p \in \beta\tau \setminus \tau$ where a space $X$ is $p$-compact if any $\tau$-sequence in $X$ has a $p$-limit point. He proves there that in the class of regular spaces the notions of $\tau$-boundedness and $\tau$-ultracompactness are equivalent for any infinite cardinal $\tau$, where $\tau$-boundedness means that the closure of any subset of cardinality not exceeding $\tau$ is compact and $\tau$-ultracompactness means that $X$ is $\tau$-compact for any $p \in \beta\tau \setminus \tau$. In case of $\tau = \aleph_0$ we obtain the notions of ultracompactness and $\aleph_0$-boundedness which are not equivalent in the class of Hausdorff spaces as demonstrates an example in [2] but the space in this example is not separable so V. Saks asks there: Does there exist a separable Hausdorff ultracompact space which is not compact? The positive answer to the problem is in [4] and the theorem 3 in the present article covers not only this result but also give a positive answer in a stronger form to another question of V. Saks [2]: Does there exist $\mathfrak{M} > \aleph_0$ and a Hausdorff $\mathfrak{M}$-ultracompact space which is not $\mathfrak{M}$-bounded?

A. P. Kombarov introduced in [3] the notion of a $p$-sequential space for $p \in \beta\omega \setminus \omega$ and this notion was extended for any $p \in \beta\tau \setminus \tau$ by L. Kočinac [5] in the context of chain-net spaces but for our goals we prefer here to use the name which offered A. P. Kombarov: a space $X$ is $p$-sequential if for any nonclosed $A \subset X$ there are some $\tau$-sequence $(x_\alpha) \subset A$ and a point $x \notin A$ such that $x = p - \lim x_\alpha$. In this case we can say that $(x_\alpha)$ $p$-converges to $x$.

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Let \((X, \gamma)\) be a topological space, \(O \subset X\) and \(p \in \beta \tau \setminus \tau\), then \(O\) is said to be \(p\)-sequentially open if \(x = p - \lim x_n\) for some \(x \in O\) and some \(\tau\)-sequence \((x_n)\) imply \(|x : x_n \in O| \in p\).

Let \(\gamma_p\) be the set of all \(p\)-sequentially open sets in \((X, \gamma)\). It is clear that the union of any number of \(p\)-sequentially open sets is again \(p\)-sequentially open and the intersection of a finite number of \(p\)-sequentially open sets is \(p\)-sequentially open. Obviously, each open set is \(p\)-sequentially open so we get the following statement.

**Proposition 1.** Let \((X, \gamma)\) be a topological space, then the family \(\gamma_p\) forms a topology on \(X\) and \(\gamma \subset \gamma_p\).

It is important to note that \(x = p - \lim x_n\) in \(\gamma\) implies \(x = p - \lim x_n\) in \(\gamma_p\). Really, if we have \(x \neq p - \lim x_n\) in \(\gamma_p\) for some \(x\) and some \(\tau\)-sequence \((x_n)\), then there exists some \(W \in \gamma_p\) such that \(x \in W\) and \(|x : x_n \in W| \notin p\). Obviously that \(x \neq p - \lim x_n\) in \(\gamma\) too, otherwise for the sequentially open set \(W\) we would get that \(|x : x_n \in W| \in p\) which is in contradiction with \(|x : x_n \in W| \notin p\).

**Proposition 2.** Topological space \((X, \gamma_p)\) is \(p\)-sequential.

**Proof.** Let \(A\) be a nonclosed subset in \((X, \gamma_p)\), then \(O = X\setminus A\) is not open in \((X, \gamma_p)\), i.e. \(O\) is not \(p\)-sequentially open in \((X, \gamma_p)\). \(X\) which implies that there are some point \(z \in O\) and some \(\tau\)-sequence \((z_n)\) \(p\)-converging to \(z\) such that \(|x : x_n \in O| \notin p\) which implies that \(|x : x_n \in A| \in p\). We put \(x_1 = z_1\) for \(z_1 \in A\) and \(x_n = y\) for some \(y \in A\) if \(z_n \notin A\). Now it is easy to verify that \(z = p - \lim x_n\) for a \(\tau\)-sequence \((x_n)\) \(\subset A\). So \((X, \gamma_p)\) is a \(p\)-sequential space. \(\Box\)

As usually by symbol \(t(X, \gamma)\) we denote the tightness of a topological space \((X, \gamma)\)

**Proposition 3.** The intersection of any family of topologies each of tightness not greater than \(\tau\) has the tightness not greater than \(\tau\) too.

**Proof.** Let \(\gamma = \cap \{\gamma_\alpha : \alpha < k\}\) where each \(\gamma_\alpha\) is a topology on a set \(X\) such that \(t(X, \gamma_\alpha) \leq \tau\) for any \(\alpha < k\). For each \(A \subset X\) we put \(A_1 = \cup \{[A]_\alpha : \alpha < k\}\). Suppose we have constructed \(A_\alpha\) for any ordinal \(\alpha < \beta\) where \(\beta < \tau^+\). Now we construct \(A_\beta\) and there are two cases:

1. \(\beta = \alpha_0 + 1\) for some \(\alpha_0\) then
   \[A_\beta = (A_\alpha_0)_1\]
2. \(\beta\) is a limit ordinal then \(A_\beta = \cup \{A_\alpha : \alpha < \beta\}\).

Finally we put \(A_{\tau^+} = \cup \{A_\alpha : \alpha < \tau^+\}\)

Using the fact that \(\tau \cdot \tau = \tau\) we can state that \([A]_\tau = A_\tau\), and applying transfinite induction on ordinals \(\alpha < \tau^+\) one can see that for any \(z \in [A]_\gamma\) there is \(B \subset A\) such that \(|B| \leq \tau\) and \(z \in [B]_\gamma\). \(\Box\)

**Proposition 4.** The tightness of a \(p\)-sequential space is not greater than \(\tau\).

**Proof.** For each subset \(A\) of \(X\) let \(A_1 = \{x : x = p - \lim x_n\} \subset A\).

Like in the proof of the previous proposition we put \(A_\beta = (A_\alpha)_1\) for \(\beta = \alpha_0 + 1\) and for a limit ordinal \(\beta\) let \(A_\beta = \cup \{A_\alpha : \alpha < \beta\}\). It is easily seen that \(A_{\tau^+} = (A_1)_1\) and thus \([A] = A_{\tau^+}\) which due to the \(\tau \cdot \tau = \tau\) imply the required result. \(\Box\)

The topology \(\gamma_p\) is called a \(p\)-sequential leader of \(\gamma\). Let \(\gamma_\tau = \cap \{\gamma_\beta : p \in \beta \tau \setminus \tau\}\) i.e. \(\gamma_\tau\) is the intersection of all \(p\)-sequential leaders in \((X, \gamma)\). The following theorem is a corollary of the propositions 3 and 4.

**Theorem 1.** The tightness of a topological space \((X, \gamma_\tau)\) does not exceed \(\tau\).

**Theorem 2.** For a topological space \((X, \gamma)\) \(t(X, \gamma) \leq \tau\) iff \(\gamma = \gamma_\tau\).
Proof. We need only to prove the necessity, i.e. that the condition \( t(X, \gamma) \leq \tau \) implies \( \gamma = \gamma_\tau \). It is sufficient to demonstrate that \( \gamma_\tau \subseteq \gamma \). To this end we take any nonopen set in the topology \( \gamma \), say \( M \). Then \( A = X \setminus M \) is a nonclosed set in \( \gamma \) and there are some subset \( B \subseteq A \) with \( |B| \leq \tau \) and some point \( y \in M \) such that \( y \in [B]_\gamma \). Considering \( B \) as a \( \tau \)-sequence \( (x_n) \) one can find some \( q \in \beta \tau \setminus \tau \) such that \( (x_n) \) \( q \)-converges to \( y \) in \( \gamma \). Then \( (x_n) \) \( q \)-converges to \( y \) too. Since \( \gamma_\tau \subseteq \gamma_q \) it follows that \( (x_n) \) \( q \)-converges to \( y \) in \( \gamma_\tau \). Thus \( M \) is not open in \( \gamma_\tau \), implying \( \gamma_\tau \subseteq \gamma \). \( \Box \)

**Theorem 3.** Let \( (X, \gamma) \) be a Hausdorff compact topological space of density \( \tau \) with tightness greater than \( \tau \). Then \( (X, \gamma_\tau) \) is a Hausdorff \( \tau \)-ultracompact not a \( \tau \)-bounded space of density \( \tau \).

**Proof.** Let \( X_0 \) be a dense subset in \( X \) of power \( \tau \). From the proof of the theorem 2 it is clear that two closure operators \([\_\_]\) and \([\_\_]_\gamma \) coincide on subsets of power no more than \( \tau \). So we can see that \((X, \gamma_\tau)\) is a \( \tau \)-ultracompact space and it contains \( X_0 \) as its dense subset. Since \( t(X, \gamma_\tau) \leq \tau \) then the topology \( \gamma_\tau \) is strictly stronger than \( \gamma \) and hence \((X, \gamma_\tau)\) is not a compact space which in its turn implies that it is not \( \tau \)-bounded. Thus \((X, \gamma_\tau)\) is a \( \tau \)-ultracompact not a \( \tau \)-bounded space of density \( \tau \). \( \Box \)

It is known that the Stone-Čech compactification of any discrete space of power \( \tau \geq \aleph_0 \) has a tightness more than \( \tau \) so we get the following result.

**Corollary 1.** For every infinite cardinal \( \tau \) there is a Hausdorff \( \tau \)-ultracompact not a \( \tau \)-bounded space of density \( \tau \).

**Corollary 2.** The notions of \( \tau \)-ultracompactness and \( \tau \)-boundedness are not equivalent in the class of Hausdorff spaces.

**Proposition 5.** The topology \( \gamma_\tau \) is the least one among all topologies of tightness not greater than \( \tau \) and each containing the given topology \( \gamma \).

**Proof.** Let \( \sigma \) be any topology with tightness not greater than \( \tau \) and containing \( \gamma \). Assume that \( A \) is a nonclosed set in \( \sigma \). Then it is nonclosed in \( \gamma \). Fix \( x \in [X] \setminus X \) then there is some \( B \subseteq A \) with \( |B| \leq \tau \) such that \( x \in [B]_\gamma \), and consequently \( x \in [B]_\gamma \). Now we can represent \( B \) as a \( \tau \)-sequence \( q \)-converging in \( \gamma \) to \( x \) for some \( q \in \beta \tau \setminus \tau \) and hence \( q \)-converging to \( x \) in \( \gamma_\tau \). So this \( \tau \)-sequence \( q \)-converges to \( x \) in \( \gamma_\tau \) implying that \( A \) is a nonclosed set in \( \gamma_\tau \) which proves that \( \gamma_\tau \subseteq \sigma \). \( \Box \)

The closure operator in the topological space \( (X, \gamma_\tau) \) can be described more clearly using the following \( \tau \)-closure operator on \( (X, \gamma) \): let \( A \subseteq X \) then we put \([A]_\tau = \{x : \exists B \subseteq A \text{ such that } |B| \leq \tau \text{ and } x \in [B]_\gamma\} \). This operator is well-known and generates some topology, say \( \gamma_{\tau, \tau} \), of tightness not greater than \( \tau \) with \( \gamma_{\tau, \tau} \supsetneq \gamma \) and coinciding with the origin topology \( \gamma \) provided the tightness of the space \((X, \gamma)\) does not exceed \( \tau \).

**Proposition 6.** In any topological space \((X, \gamma)\) the topologies \( \gamma_{\tau} \) and \( \gamma_{\tau, \tau} \), coincide.

**Proof.** From the previous proposition we get that \( \gamma_{\tau} \subseteq \gamma_{\tau, \tau} \), but the converse inclusion can be obtained using the same arguments as in the proof of the proposition 5. \( \Box \)

**References**