On the Existence of Isoperimetric Extremals of Rotation and the Fundamental Equations of Rotary Diffeomorphisms

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Abstract. In this paper we study the existence and the uniqueness of isoperimetric extremals of rotation on two-dimensional (pseudo-) Riemannian manifolds and on surfaces on Euclidean space. We find the new form of their equations which is easier than results by S. G. Leiko. He introduced the notion of rotary diffeomorphisms. In this paper we propose a new proof of the fundamental equations of rotary mappings.

1. Introduction

A rotary diffeomorphism of surfaces $S_2$ on a three-dimensional Euclidean space $\mathbb{E}_3$ and also of two-dimensional Riemannian manifolds $V_2$ is studied in papers of S. G. Leiko [10–18]. These results are local and are based on the known fact that a two-dimensional Riemannian manifold $V_2$ is implemented locally as a surface $S_2$ on $\mathbb{E}_3$. Therefore we will deal more with the study of $V_2$, i.e. the inner geometry of $S_2$. For recent studies of the deformation of surfaces from a different point, see [3, 7, 20–23, 29–31].

In [10, 11, 14] the following notion of special mapping is introduced.

Definition 1.1. A diffeomorphism $f: V_2 \rightarrow V_2$ is called rotary if any geodesic $\gamma$ is mapped onto isoperimetric extremal of rotation.

In our paper we have new proof of the fundamental equations of the rotary mappings (Section 5).

The isoperimetric extremal of rotation is a special curve on $V_2$ (resp. $S_2$) which is extremal of a certain variational problem of geodesic curvature (see [10–17] where the existence of these curves was shown for the case $V_2 \in C^3$, resp. on $S_2 \in C^3$).

The above curves have a physical meaning as can be interpreted as trajectories of particles with a spin, see [10, 12].

Our paper is devoted to the proof of the existence of isoperimetric extremal of rotation on $V_2 \in C^3$, resp. on $S_2 \in C^3$. Besides we find the fundamental equations of these curves in a more simple form of ordinary differential equation of Cauchy type. From the above the problem of a rotary diffeomorphism can be solved for the surfaces with the lower smoothness class.

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Remark. A two-dimensional Riemannian manifold \( V_2 \) belongs to the smoothness class \( C^r \) if its metric \( g_{ij} \in C^r \). We suppose that the differentiability class \( r \) is equal to 0, 1, 2, \ldots, \infty, \omega \), where 0, \infty and \( \omega \) denote continuous, infinitely differentiable, and real analytic functions respectively.

Surface \( S_2: \mathbf{p}(x_1, x_2) \in C^{r+1} \) if the vector function \( \mathbf{p}(x_1, x_2) \in C^{r+1} \) and evidently inner two-dimensional Riemannian manifold \( V_2 \) belongs to \( C^r \) with induced metric \( g_{ij}(x) = \mathbf{p}_i \cdot \mathbf{p}_j \in C^r \), where \( \mathbf{p}_i = \partial \mathbf{p}/\partial x^i \). There \( x = (x_1, x_2) \) are local coordinates of \( V_2 \), resp. \( S_2 \).

An immersion \( V_2 \) in Euclidean space is studied in detail, for example, in [24, 27]. In the study of surfaces \( S_2 \) we use the notation that is used in the books [1, 6, 9, 25, 26].

2. Isoperimetric extremals of rotation

Let us consider a two-dimensional Riemannian space \( V_2 \in C^3 \) with a metric tensor \( g \). Let \( g_{ij}(x_1, x_2) \in C^3 \) \((i, j = 1, 2)\) be components of \( g \) in some local map.

For the curve \( \ell: (t_0, t_1) \to V_2 \) with the parametric equation \( x^b = \lambda^b(t) \), we construct the tangent vector \( \lambda^b = dx^b/dt \) and vectors

\[
\lambda_1^b = \nabla_1 \lambda^b \quad \text{and} \quad \lambda_2^b = \nabla_2 \lambda^b.
\]

Here \( \nabla_1 \) is an operator of covariant differentiation along \( \ell \) with respect to the Levi-Civita connection \( \nabla \) of metric \( g \), i.e.

\[
\lambda_1^b = \nabla_1 \lambda^b = \frac{d\lambda^b}{dt} + \lambda^a \Gamma^b_{ab} (x(t)) \frac{dx^a}{dt},
\]

and

\[
\lambda_2^b = \nabla_2 \lambda^b = \frac{d\lambda^b}{dt} + \lambda^a \Gamma^b_{ab} (x(t)) \frac{dx^a}{dt},
\]

where \( \Gamma^b_{ab} \) are the Christoffel symbols of \( V_2 \), i.e. components of \( V \).

It is known that the scalar product of vectors \( \lambda, \xi \) is defined by \( \langle \lambda, \xi \rangle = g_{ij} \lambda^i \xi^j \). We denote

\[
s[\ell] = \int_{t_0}^{t_1} \sqrt{\langle \lambda, \lambda \rangle} \, dt \quad \text{and} \quad \theta[\ell] = \int_{t_0}^{t_1} k_\gamma(s) \, ds
\]

functionals of length and rotation of the curve \( \ell \); \( k_\gamma \) is the Frenet curvature\(^1\) and \( s \) is the arc length. In the case \( S_2 \subset E_3 \) the geodesic curvature of the curve is \( k_\gamma \).

Using these functionals we introduce the following definition

Definition 2.1 (Leiko [11]). A curve \( \ell \) is called the isoperimetric extremal of rotation if \( \ell \) is extremal of \( \theta[\ell] \) and \( s[\ell] = \text{const} \) with fixed ends.

It was shown in [11] that in a (not plain) space \( V_2 \) a curve is an isoperimetric extremal of rotation only if its Frenet curvature \( k_\gamma \) and Gaussian curvature \( K \) are proportional:

\[
k_\gamma = c \cdot K,
\]

where \( c = \text{const} \).

In [11] it is proved that for a canonical parameter \( t = a \cdot s + b \) \((a, b = \text{const})\) the condition (1) can be written in the following form

\[
\lambda_2 = -\frac{\langle \lambda_1, \lambda_1 \rangle}{\langle \lambda, \lambda \rangle} \cdot \lambda + \nabla_x K \cdot \lambda \cdot \frac{\lambda^a}{K} \cdot \lambda_1,
\]

where \( \langle \lambda, \lambda \rangle = 0 \) and \( \nabla_x K = \partial_t K \) is a gradient vector of the Gaussian curvature \((K \neq 0)\).

Using these equations for the case of \( V_2 \in C^4 \) the uniqueness of the existence of isoperimetric extremals of rotation can be shown for the following initial conditions (see [14]):

\[
\ell(0), \lambda(0), \lambda_1(0) \quad \text{such that} \quad \langle \lambda(0), \lambda(0) \rangle = 1 \quad \text{and} \quad \langle \lambda(0), \lambda_1(0) \rangle = 0.
\]

\(^1\)In the original paper \( k_\gamma \) is denoted as \( k \). This fact can lead to confusion between \( k \) and the main curvature of the curve.
3. On new equations of isoperimetric extremals of rotation

First we recall the basic knowledge of theory of surfaces \( S_2 \) and (pseudo-) Riemannian manifolds, see [1, 5, 6, 9, 20, 25, 26].

For simplicity we will consider a two-dimensional Riemannian manifold \( V_2 \) is a subspace of \( S_2 \subset \mathbb{E}_3 \) which is given by the equation \( \mathbf{p} = \mathbf{p}(x^1, x^2) \). It is known that metric of \( S_2 \) is given by the following functions \( g_{ij}(x) = \mathbf{p}_i \cdot \mathbf{p}_j \in C' \), where \( \mathbf{p}_i = \partial_i \mathbf{p} \).

The existence of the surface \( S_2 \) with metric \( g \) on \( V_2 \) results from the Bonnet Theorem; components \( g_{ij} \) of the first fundamental form belong to the smoothness class \( C^2 \) and components \( b_{ij} \) of the second fundamental form belong to the smoothness class \( C^1 \) both of them satisfy Gauss and Peterson-Codazzi equations.

For the Gaussian curvature \( K \) it holds that

\[
K = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2},
\]

where \( b_{ij} = \partial_i \mathbf{p} \cdot \mathbf{m} \) and \( \mathbf{m} = \frac{\mathbf{p}_1 \times \mathbf{p}_2}{|\mathbf{p}_1 \times \mathbf{p}_2|} \) is a unit normal vector of the surface \( S_2 \). If \( S_2 \in C^3 \) then the curvature \( K \) is differentiable.

Now we recall the geometry of a (pseudo-) Riemannian manifold \( V_2 \) defined by the metric tensor \( g_{ij} \). The Christoffel symbols of the first and the second kind are given by

\[
\Gamma^i_{jk} = \frac{1}{2} (\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}) \quad \text{and} \quad \Gamma^i_{ij} g^{jk},
\]

where \( g^{ij} \) are components of the matrix inverse to \( (g_{ij}) \).

The Riemannian tensors of the first and the second type are given by

\[
R^h_{ijk} = g_{ha} R^a_{ijk} \quad \text{and} \quad R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^a_{ik} \Gamma^h_{aj} - \Gamma^a_{ij} \Gamma^h_{ak}.
\]

Then from Gauss’s Theorema Egregium for surfaces \( S_2 \in C^3 \) it follows that ([5, § 22.2], [9, p. 145]):

\[
K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}.
\]

This formula defines the curvature \( K \) in a (pseudo-) Riemannian manifold \( V_2 \).

Finally, we recall the Gauss equations

\[
\partial_j \mathbf{p} = \Gamma^i_{ij} \cdot \mathbf{p}_k + b_{ij} \cdot \mathbf{m}.
\]

Let a curve \( \ell: \mathbf{p} = \mathbf{p}(s) \) be an isoperimetric extremal of rotation on a surface \( S_2 \) parametrized by arclength \( s \). On the other hand, because \( \ell \subset S_2: \mathbf{p} = \mathbf{p}(x^1, x^2) \) there exist inner equations \( \ell: x^i = x^i(s) \) such that the following is valid

\[
\mathbf{p}(s) = \mathbf{p}(x(s))
\]

for all \( s \in I \), where \( \mathbf{p} \) on the left side is a vector function describing the curve \( \ell \) and \( \mathbf{p} \) on the right side is a vector function describing the surface \( S \). Let us denote \( \mathbf{d} = \partial s \) by a dot. Then \( \mathbf{p}(s) \) is a unit tangent vector of \( \ell \).

We compute the second order derivative for a vector \( \mathbf{p}(s) \):

\[
\mathbf{p}(s) = \mathbf{p}_i(x(s)) \cdot \dot{x}^i(s) \\
\dot{\mathbf{p}}(s) = \partial_i \mathbf{p}(x(s)) \dot{x}^i(s) \cdot \dot{x}^j(s) + \mathbf{p}_k \cdot \ddot{x}^k(s).
\]

Now we apply the Gauss equation (3) and we obtain

\[
\dot{\mathbf{p}}(s) = (\ddot{x}^i(s) + \Gamma^k_{ij} \cdot \dot{x}^i(s) \dot{x}^j(s)) \cdot \mathbf{p}_k + b_{ij} \cdot \mathbf{m}.
\]
It is obvious that vector $\mathbf{p}(s)$ splits into two components: into a normal vector $\mathbf{m}$ and a unit vector $\mathbf{n}$ which is orthogonal to a vector $\mathbf{m}$ and $\mathbf{p}(s)$. This vector is tangent to a surface $S_2$, therefore we can write $\mathbf{n} = n^k \mathbf{p}_k$, where $n^k$ are components of the vector $\mathbf{n}$.

Therefore from (4) it follows that

$$(\dot{x}^k(s) + \Gamma^k_{ij}(x(s)) \cdot \dot{x}^i(s) \dot{x}^j(s)) \cdot \mathbf{p}_k + b_{ij} \cdot \mathbf{m} = k_g \cdot n^k \cdot \mathbf{p}_k + k_n \cdot \mathbf{m},$$

where $k_n$ is a normal curvature of $S_2$ in the direction of a tangent vector $\lambda = \dot{x}$.

Because vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}$ are linearly independent, the following equation is true

$$\dot{x}^k(s) + \Gamma^k_{ij}(x(s)) \cdot \dot{x}^i(s) \dot{x}^j(s) = k_g \cdot n^k.$$

We can write this equation in the form:

$$\nabla_s \lambda = k_g \cdot n.$$  

(5)

The formula above is an analogue of the Frenet formulas for the flat curves, see [5], [9, § 12], and for the curves with non-isotropic tangent vector $\lambda$ ($|\lambda| \neq 0$) on (pseudo-) Riemannian manifolds $V_2$, see [19, pp. 22–26].

We show efficient construction of a unit vector $n$ which is orthogonal to $\lambda$ using a discriminant tensor $\varepsilon$ and a structure tensor $F$ on $V_2$ defined by relations

$$\varepsilon_{ij} = \sqrt{|g_{11}g_{22} - g_{12}^2|} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad F_{ij}^k = \varepsilon_{ij} \cdot g^{jk}.$$

The tensor $\varepsilon$ is skew-symmetric and covariantly constant and tensor $F$ defined on $V_2$ is a structure, for which it holds

$$F^2 = e \, \text{Id} \quad \text{and} \quad VF = 0,$$

where $e = -1$ for a “properly” Riemannian $V_2$ and $e = +1$ for a pseudo-Riemannian $V_2$.

It can be easily proved that vector $F \lambda$ is also a unit vector orthogonal to a unit vector $\lambda$. Obviously, it holds that $n = \pm F \lambda$. Therefore from (1) and (5) follows the theorem.

**Theorem 3.1.** The equation of isoperimetric extremal of rotation can be written in the form

$$\nabla_s \lambda = c \cdot K \cdot F \lambda,$$  

(6)

where $c = \text{const}$.

**Remark.** Further differentiation of the equation (6) gives the equation (2) by Leiko [10–17]. Note that the equation (6) has more simple form than the equation (2). If $c = 0$ is satisfied then the curve is geodesic.

4. On the uniqueness of the existence of isoperimetric extremals of rotation

Analysis of the equation (6) convinces of the validity of the following theorem which generalizes and refines the results of Leiko [10–17].

**Theorem 4.1.** Let $V_2$ be a (non flat) Riemannian manifold of the smoothness class $C^3$. Then there is precisely one isoperimetric extremal of rotation going through a point $x_0 \in V_2$ in a given non-isotropic direction $\lambda_0 \in TV_2$ and constant $c$.

**Proof.** Let $x_0^k$ be coordinates of a point $x_0$ at $V_2 \in C^3$ and $\lambda_0^k$ ($\neq 0$) be coordinates of a unit tangent vector $\lambda_0$ in a given point $x_0$.

We will find an isoperimetric extremal of rotation $\ell$: $x^k = x^k(s)$, where $s$ is the arc length, on a space $V_2$ such that $x^k(0) = x_0^k$ and $x^k(0) = \lambda_0^k$, i.e. this curve goes through a point $x_0$ in the direction $\lambda_0$. 

Let us write equation (6) as a system of ordinary differential equations:

\[
\begin{align*}
\dot{x}^i(s) &= \lambda^i(s) \\
\dot{\lambda}^i(s) &= -\Gamma^i_{jk}(x(s)) \cdot \lambda^j(s) \cdot \lambda^k(s) + c \cdot K(x(s)) \cdot P^i_j(x(s)) \cdot \lambda^j(s).
\end{align*}
\] (7)

From the theory of differential equations it is known (see [4, 8]) that given the initial conditions \( x^i(0) = x^i_0 \) and \( \lambda^i(0) = \dot{x}^i(0) = \lambda^i_0 \) the system (7) has only one solution when

\[
\Gamma^i_{ij} \in C^1, K \in C^1 \text{ and } P^i_j \in C^1.
\] (8)

These conditions (8) are met on a space \( V_x \in C^3 \) (we consider that \( V_x \) is a metric of some surface \( S_2 \subset \mathbb{R}^3 \) of the smoothness class \( C^3 \)).

Correctness of the solution of (7) lies in the fact that the vector \( \lambda(s) \) is unit for all \( s \). Evidently, \( \langle \lambda, \lambda \rangle \) is constant along \( \ell \), i.e. \( V_x(\lambda, \lambda) = 2 \cdot \langle \lambda, V_x \lambda \rangle = 0 \), and from \( \langle \lambda_0, \lambda_0 \rangle = \pm 1 \) it follows \( \langle \lambda, \lambda \rangle = \pm 1 \). \( \square \)

**Remark.** It is possible to substitute the condition (8) by the Lipschitz’s condition for these functions.

Continuity of these functions is guaranteed by the existence of a solution to (7). This is possible when \( V_x \in C^2 \), resp. \( S_2 \in C^3 \).

5. On the fundamental equations of rotary diffeomorphisms of \( V_x \)

Assume to be given two-dimensional (pseudo-) Riemannian manifolds \( V_x = (M, g) \) and \( \overline{V}_x = (\overline{M}, \overline{g}) \) with metrics \( g \) and \( \overline{g} \), Levi-Civita connections \( \nabla \) and \( \overline{\nabla} \), complex structures \( F \) and \( \overline{F} \), respectively.

Assume a rotary diffeomorphism \( f : \overline{V}_x \to V_x \). Since \( f \) is a diffeomorphism, we can compose local coordinate system on \( M \) and \( \overline{M} \), respectively, such that locally \( f : \overline{V}_x \to V_x \) maps points onto points with the same coordinates \( x \), and \( M = \overline{M} \).

From Definition 1.1 it follows that any geodesic \( \overline{x} \) on \( \overline{V}_x \) is mapped onto an isoperimetric extremal of rotation on \( V_x \).

Let \( \overline{x}^i = x^i(\overline{s}) \) be a geodesic on \( \overline{V}_x \) for which the following equation is valid

\[
\frac{d^2 x^i}{d\overline{s}^2} + \overline{\Gamma}^i_{ij}(x(s)) \frac{dx^i}{d\overline{s}} \frac{dx^j}{d\overline{s}} = 0
\] (9)

and let \( x^i = x^i(s) \) be an isoperimetric extremal of rotation on \( V_x \) for which the following equation is valid

\[
\dot{\lambda}^i_1 \equiv d\lambda^i/ds + \Gamma^i_{ij}(x(s)) \lambda^j \lambda^i = c \cdot K(x(s)) \cdot P^i_j(x(s)) \cdot \lambda^j,
\] (10)

where \( \Gamma^i_{ij} \) and \( \overline{\Gamma}^i_{ij} \) are components of \( \nabla \) and \( \overline{\nabla} \), parameters \( s \) and \( \overline{s} \) are arc lengths on \( \gamma \) and \( \overline{\gamma} \), \( \lambda^i = dx^i(s)/ds \).

Suppose that \( \overline{s} = \overline{s}(s) \). In this case we modify equation (9):

\[
d\lambda^i/ds + \overline{\Gamma}^i_{ij}(x(s)) \lambda^j \lambda^i = \rho(s) \cdot \lambda^i,
\] (11)

where \( \rho(s) \) is a certain function of parameter \( s \), i.e. this equation is the equation of a geodesic with an arbitrary parameter.

We denote \( P^i_{ij}(x) = \Gamma^i_{ij}(x) - \overline{\Gamma}^i_{ij}(x) \) the deformation tensor of connections \( \nabla \) and \( \overline{\nabla} \) defined by the rotary diffeomorphism.

As a consequence of (11), we have

\[
\lambda^i_1 = \rho \cdot \lambda^i + P^i_0,
\] (12)

where \( P^i_0 = P^i_{ij} \cdot \lambda^j \lambda^i \), and from (12) it follows \( \rho = -\langle \lambda, P \rangle \).
After differentiating (12) along the curve \( \ell \) and substituting the corresponding values in (10), we obtain
\[
P_i = \langle \lambda, P_i \rangle \cdot \lambda^i = \left( 3 \langle \lambda, P \rangle + \nabla_a K \cdot \lambda^a / K \right) \cdot \left( \langle \lambda, P \rangle \cdot \lambda^i - P^i \right),
\]
(13)
where \( P_i = (V_i P_i + 2 P_{ia} \frac{\partial}{\partial \lambda^a} \frac{\partial}{\partial P_j}) \lambda^i \lambda^j \).

Let us study the formula (13) in isothermal coordinates in the fixed point \( x' \) in which \( g_{11} = g_{22} = 1, \]
\( g_{12} = 0. \) As \( (\lambda^2)^2 = 1 - (\lambda^1)^2 \) the formula (13) is the function of value \( \lambda^1. \) The coefficient of \( (\lambda^1)^6 \) is equal \( A^2 + B^2, \) where \( A = P_{11} - 2P_{12}^2 - 2P_{12}^2 \) and \( B = P_{11}^2 - P_{12}^2 - 2P_{12}^1. \) From this it follows \( A^2 + B^2 = 0, \) and evidently \( A = B = 0, \) and hence we have in this coordinate system
\[
p_{11} - P_{12}^1 - 2P_{12}^2 = 0 \quad \text{and} \quad P_{12}^1 - 2P_{11}^1 - 2P_{12}^1 = 0.
\]

In this coordinate system we denote \( \psi_1 = P_{12}, \psi_2 = P_{12}, \theta^1 = P_{12}^1 \) and \( \theta^2 = P_{11}^2. \) We can rewrite the above formula equivalently to the following tensor equation
\[
P_{ij} = \delta^i_j \psi_j + \delta^i_j \psi_i + \theta^i g_{ij},
\]
(14)
where \( \psi_i \) and \( \theta^i \) are covector and vector fields.

On the other hand, from (14) it follows \( 4\psi_2 = \psi_2 = P_{12}, \psi_1 = P_{12}^1 \) and \( \theta^2 = P_{11}^2. \)

As a consequence of (14), the formula (12) obtains following form
\[
\lambda_i^i = \ddot{\psi} \cdot \lambda^i + \theta^i,
\]
(15)
and from (15) it follows \( \ddot{\psi} = -\langle \lambda, \theta \rangle. \) After differentiating (15) along the curve \( \ell \) and substituting the corresponding values in (10), we obtain
\[
\nabla_a \theta^a \lambda^a - \theta^a \theta^a K_a \lambda^a / K = \lambda^i \left( \nabla_a \theta_a \lambda^a \lambda^i - \theta_a \theta^a \lambda^a \lambda^i - \theta_a \lambda^a K_a \lambda^a / K \right),
\]
and by similar way we obtain the following formulas
\[
\nabla_\lambda \theta_j = \theta_i (\theta_j + K_j K_i) + \nu g_{ij},
\]
(16)
where \( \nu \) is a function on \( V_2. \)

The equations (14) and (16) are necessary and sufficient conditions of rotary diffeomorphism by \( V_2 \) onto \( \bar{V}_2. \) Our proof is straightforward and more comprehensive than the one proposed in [11]. We notice that the above considerations are possible when \( V_2 \in C^3 \) and \( \bar{V}_2 \in C^3. \)

The vector field \( \theta_i \) is torse-forming, see [20, 28, 33]. Under further conditions on differentiability of metrics it has been proved in [11] that \( \theta_i \) is concircular. From this follows that \( V_2 \) is isometric to surfaces of revolution. Concircular vector fields were studied by many authors, such as [2, 20, 32].

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