On Generalized Quasi-Einstein Manifolds Admitting Certain Vector Fields

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Abstract. The object of the present paper is to study some geometric properties of a generalized quasi-Einstein manifold. The existence of such a manifold have been proved by several non-trivial examples.

1. Introduction

A Riemannian or semi-Riemannian manifold \((M^n, g)\), \(n = \dim M \geq 2\), is said to be an Einstein manifold if the following condition
\[
S = \frac{r}{n}g
\]
holds on \(M\), where \(S\) and \(r\) denote the Ricci tensor and the scalar curvature of \((M^n, g)\) respectively. According to Besse\([3]\), p. 432), (1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor\([3]\,p.432-433). For instance, every Einstein manifold belongs to the class of Riemannian or semi-Riemannian manifolds \((M^n, g)\) realizing the following relation:
\[
S(X, Y) = ag(X, Y) + bA(X)A(Y),
\]
where \(a, b \in \mathbb{R}\) and \(A\) is a non-zero 1-form such that
\[
g(X, U) = A(X),
\]
for all vector fields \(X\). Moreover, different structures on Einstein manifolds have been studied by several authors. In 1993, Tamassy and Binh\[29\] studied weakly symmetric structures on Einstein manifolds. A non-flat semi-Riemannian manifold \((M^n, g)\) \((n > 2)\) is defined to be a quasi-Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the condition (2). It is to be noted that Chaki and Maity\[6\] also introduced the notion of quasi-Einstein manifolds in a different
way. They have taken $a$, $b$ as scalars and the vector field $U$ metrically equivalent to the 1-form $A$ as a unit vector field. Such an n-dimensional manifold is denoted by $(\text{QE})_n$. Quasi-Einstein manifolds have been studied by several authors such as Bejan[2], De and Ghosh[11], De and De[12] and De, Ghosh and Binh[13] and many others.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. Also, quasi-Einstein manifolds can be taken as a model of the perfect fluid spacetime in general relativity[10]. So quasi-Einstein manifolds have some importance in the general theory of relativity.

The class consisting of all Riemannian manifolds whose Ricci tensor $S$ of type $(0,2)$ is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y),$$

where $a$, $b$, $c \in \mathbb{R}$ and $A$, $B$ are two non-zero 1-forms such that

$$g(A, B) = 0, \quad ||A|| = ||B|| = 1.$$

The unit vector fields $U$ and $V$ corresponding to the 1-forms $A$ and $B$ respectively, defined by

$$g(X, U) = A(X), \quad g(X, V) = B(X),$$

for every vector field $X$ are orthogonal, that is, $g(U, V) = 0$. Such a manifold is denoted by $G(\text{QE})_n$. If $c = 0$, then the manifold reduces to a quasi-Einstein manifold[6].

Gray[19] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class $A$ consisting of all Riemannian manifolds whose Ricci tensor $S$ is a Codazzi type tensor, i.e.,


The class $B$ consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e.,

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

A non-flat Riemannian or semi-Riemannian manifold $(M^n, g)$ $(n > 2)$ is called a generalized quasi-Einstein manifold if its Ricci tensor $S$ of type $(0,2)$ is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = \gamma(X)S(Y, Z) + \delta(X)g(Y, Z),$$

where $\gamma$ and $\delta$ are non-zero 1-forms. If $\delta = 0$, then the manifold reduces to a Ricci recurrent manifold[25].

The present paper is organized as follows: After introduction in Section 2, it is shown that if the generators $U$ and $V$ are Killing vector fields, then the generalized quasi-Einstein manifold satisfies cyclic parallel Ricci tensor. Section 3 deals with $G(\text{QE})_n$ satisfying Codazzi type of Ricci tensor. In the next two sections we consider $G(\text{QE})_n$ with generators $U$ and $V$ both as concurrent and recurrent vector fields. Finally, we give some examples of generalized quasi-Einstein manifolds.

2. The generators $U$ and $V$ as Killing vector fields

In this section let us consider the generators $U$ and $V$ of the manifold are Killing vector fields. Then we have

$$(\mathcal{L}_U g)(X, Y) = 0$$

(5)
and
\[(\mathcal{L}_V g)(X, Y) = 0,\] (6)

where \(\mathcal{L}\) denotes the Lie derivative.

From (5) and (6) it follows that
\[g(\nabla_X U, Y) + g(X, \nabla_Y U) = 0\] (7)

and
\[g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0.\] (8)

Since \(g(\nabla_X U, Y) = (\nabla_X A)(Y)\) and \(g(\nabla_X V, Y) = (\nabla_X B)(Y)\), we obtain from (7) and (8) that
\[(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0\] (9)

and
\[(\nabla_X B)(Y) + (\nabla_Y B)(X) = 0,\] (10)

for all \(X, Y\).

Similarly, we have
\[(\nabla_X A)(Z) + (\nabla_Z A)(X) = 0,\] (11)
\[(\nabla_Z A)(Y) + (\nabla_Y A)(Z) = 0,\] (12)
\[(\nabla_X B)(Z) + (\nabla_Z B)(X) = 0,\] (13)
\[(\nabla_Z B)(Y) + (\nabla_Y B)(Z) = 0,\] (14)

for all \(X, Y, Z\).

Now from (4) we have
\[\nabla_Z S(X, Y) = b[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)]
+ c[(\nabla_Z B)(X)B(Y) + B(X)(\nabla_Z B)(Y)].\] (15)

Using (15) we obtain
\[\nabla_S(X, Y) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = b[(\nabla_X A)(Y)
+ (\nabla_Y A)(X)A(Z) + (\nabla_X A)(Z) + (\nabla_Z A)(X)A(Y)
+ (\nabla_Y A)(Z) + (\nabla_Z A)(Y)A(X)]
+ c[(\nabla_X B)(Y)
+ (\nabla_Y B)(X)B(Z) + (\nabla_X B)(Z) + (\nabla_Z B)(X)B(Y)
+ (\nabla_Y B)(Z) + (\nabla_Z B)(Y)B(X)].\] (16)

By virtue of (9)–(14) we obtain from (16) that
\[\nabla_S(X, Y) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.\]

Thus we can state the following theorem:

**Theorem 2.1.** If the generators of a \(G(QE)_n\) are Killing vector fields, then the manifold satisfies cyclic parallel Ricci tensor.
3. $G(QE)_n$ satisfying Codazzi type of Ricci tensor

A Riemannian or semi-Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if its Ricci tensor satisfies the following condition

$$\nabla_XS(Y, Z) = \nabla_YS(X, Z),$$  \hspace{1cm} (17)

for all $X, Y, Z$.

Using (15) and (17), we obtain

$$b[(\nabla_XA)(Y)A(Z) - (\nabla_YA)(X)A(Z) + A(Y)\nabla_XA)(Z) - A(X)\nabla_YA)(Z)] + c[(\nabla_XB)(Y)B(Z) - (\nabla_YB)(X)B(Z) + B(Y)\nabla_XB)(Z) - B(X)\nabla_YB)(Z)] = 0,$$  \hspace{1cm} (18)

Putting $Z = U$ in (18) and using $(\nabla_XA)(U) = 0$ we get

$$(\nabla_XA)(Y) - (\nabla_YA)(X) = 0, \text{ i.e., } dA(X, Y) = 0.$$  

Similarly, putting $Z = V$ in (18) and using $(\nabla_XB)(V) = 0$ yields $dB(X, Y) = 0$.

Thus we can state the following:

**Theorem 3.1.** If a $G(QE)_n$ satisfies the Codazzi type of Ricci tensor, then the associated 1-forms $A$ and $B$ are closed.

Again putting $X = Z = U$ in (18) we have

$$(\nabla_UA)(Y) = 0,$$  \hspace{1cm} (19)

which means that $g(X, \nabla_UU) = 0$ for all $Y$, that is, $\nabla_UU = 0$.

Similarly, putting $X = Z = V$ in (18) we have

$$(\nabla_VB)(Y) = 0,$$  \hspace{1cm} (20)

which yields $\nabla_VV = 0$. This leads to the following theorem:

**Theorem 3.2.** If a generalized quasi-Einstein manifold satisfies Codazzi type of Ricci tensor, then the integral curves of the vector fields $U$ and $V$ are geodesic.

4. The generators $U$ and $V$ as concurrent vector fields

A vector field $\xi$ is said to be concurrent if[26]

$$\nabla_X\xi = \rho X,$$  \hspace{1cm} (21)

where $\rho$ is a non-zero constant. If $\rho = 0$, the vector field reduces to a parallel vector field.

In this section we consider the vector fields $U$ and $V$ corresponding to the associated 1-forms $A$ and $B$ respectively are concurrent. Then

$$(\nabla_XA)(Y) = \alpha g(X, Y)$$  \hspace{1cm} (22)

and

$$(\nabla_XB)(Y) = \beta g(X, Y),$$  \hspace{1cm} (23)

where $\alpha$ and $\beta$ are non-zero constants.

Using (22) and (23) in (15) we get

$$(\nabla_ZS)(X, Y) = b[\alpha g(X, Z)A(Y) + \alpha g(Y, Z)A(X)]$$

$$+ c[\beta g(X, Z)B(Y) + \beta g(Y, Z)B(X)].$$  \hspace{1cm} (24)
Contracting (24) over $X$ and $Y$ we obtain

$$dr(Z) = 2[bA(Z) + cβB(Z)],$$

(25)

where $r$ is the scalar curvature of the manifold.

Again from (4) we have

$$r = an + b + c.$$  

(26)

Since, $a, b, c \in \mathbb{R}$, it follows that $dr(X) = 0$, for all $X$. Thus equation (25) yields

$$baA(Z) + cβB(Z) = 0.$$  

(27)

Since $a$ and $β$ are not zero, using (27) in (4), we finally get

$$S(X, Y) = a\eta(X, Y) + (b + \frac{b^2a^2}{cβ^2})A(X)A(Y).$$

Thus the manifold reduces to a quasi-Einstein manifold. Hence we can state the following theorem:

**Theorem 4.1.** If the associated vector fields of a $G(QE)_n$ are concurrent vector fields, then the manifold reduces to a quasi-Einstein manifold.

5. The generators $U$ and $V$ as recurrent vector fields

A vector field $ξ$ corresponding to the associated 1-form $η$ is said to be recurrent if[26]

$$(∇_X η)(Y) = ψ(X)η(Y),$$

(28)

where $ψ$ is a non-zero 1-form.

In this section we suppose that the generators $U$ and $V$ corresponding to the associated 1-forms $A$ and $B$ are recurrent. Then we have

$$(∇_X A)(Y) = λ(X)A(Y)$$

(29)

and

$$(∇_X B)(Y) = μ(X)B(Y),$$

(30)

where $λ$ and $μ$ are non-zero 1-forms.

Now, using (29) and (30) in (15) we get

$$(∇_Z S)(X, Y) = 2bλ(Z)A(X)A(Y) + 2cμ(Z)B(X)B(Y).$$

(31)

We assume that the 1-forms $λ$ and $μ$ are equal, i.e.,

$$λ(Z) = μ(Z),$$

(32)

for all $Z$. Then we obtain from (31) and (32) that

$$(∇_Z S)(X, Y) = 2λ(Z)[bA(X)A(Y) + cB(X)B(Y)].$$

(33)

Using (4) and (33) we have

$$(∇_Z S)(X, Y) = α_1(Z)S(X, Y) + α_2(Z)η(X, Y),$$

where $α_1(Z) = 2λ(Z)$ and $α_2(Z) = -2aλ(Z)$.

Thus we can state the following:

**Theorem 5.1.** If the generators of a $G(QE)_n$ corresponding to the associated 1-forms are recurrent with the same vector of recurrence, then the manifold is a generalized Ricci recurrent manifold.
6. Examples of $G(QE)_n$

In this section we prove the existence of generalized quasi-Einstein manifolds by constructing some non-trivial concrete examples.

**Example 6.1.** Let us consider a semi-Riemannian metric $g$ on $\mathbb{R}^4$ by
\[ ds^2 = g_{ij}dx^i dx^j = x^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2, \]
where $i, j = 1, 2, 3, 4$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensors and the derivatives of the components of curvature tensors are
\[ \Gamma^2_{11} = \Gamma^2_{33} = -\frac{1}{2x^2}, \quad \Gamma^1_{12} = \Gamma^3_{23} = \frac{1}{2x^2}, \]
\[ R_{1221} = R_{2332} = -\frac{1}{2x^2}, \quad R_{1331} = \frac{1}{4x^2}, \quad R_{1232} = 0, \]
and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensor $R_{ij}$ are
\[ R_{11} = R_{33} = -\frac{1}{4(x^2)^2}, \quad R_{22} = -\frac{1}{(x^2)^2}. \]
It can be easily shown that the scalar curvature of the resulting manifold $(\mathbb{R}^4, g)$ is $-\frac{3}{2(x^2)^2} \neq 0$. We shall now show that $(\mathbb{R}^4, g)$ is a generalized quasi-Einstein manifold.

Let us now consider the associated scalars as follows:
\[ a = \frac{1}{(x^2)^3}, \quad b = -\frac{5}{2(x^2)^3}, \quad c = -\frac{2}{(x^2)^3}. \]
Again let us choose the associated 1-forms as follows:
\[ A_i(x) = \begin{cases} \frac{1}{\sqrt{x^2}}, & \text{for } i=1, 3 \\ 0, & \text{otherwise}, \end{cases} \]
\[ B_i(x) = \begin{cases} \sqrt{x^2}, & \text{for } i=2 \\ 0, & \text{otherwise}, \end{cases} \]

at any point $x \in \mathbb{R}^4$. To verify the relation (4), it is sufficient to check the following equations:
\[ R_{11} = ag_{11} + bA_1A_1 + cB_1B_1, \]
\[ R_{22} = ag_{22} + bA_2A_2 + cB_2B_2, \]
\[ R_{33} = ag_{33} + bA_3A_3 + cB_3B_3, \]

since for the other cases (4) holds trivially. By virtue of (35), (36), (37) and (38) we get
\[ \text{R.H.S. of (38)} = ag_{11} + bA_1A_1 + cB_1B_1 = \frac{1}{(x^2)^3}x^2 + \left(-\frac{5}{2(x^2)^3}\right)\frac{1}{2}(x^2) + 0 \]
\[ = -\frac{1}{4(x^2)^2} = R_{11} = \text{L.H.S. of (38)}. \]
By similar argument it can be shown that (39) and (40) are also true. We shall now show that the associated vectors \(A_i\) and \(B_i\) are unit.

Here

\[ g^{ij}A_iA_j = 1, \quad g^{ij}B_iB_j = 1, \quad g^{ij}A_iB_j = 0. \]

Therefore the vectors \(A_i\) and \(B_i\) are unit and also they are orthogonal.

So, \((\mathbb{R}^4, g)\) is a generalized quasi-Einstein manifold.

**Example 6.2.** We consider a Riemannian manifold \((M^4, g)\) endowed with the Riemannian metric \(g\) given by

\[ ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 + (dx^4)^2, \]

where \(i, j = 1, 2, 3, 4\). The only non-vanishing components of Christoffel symbols, the curvature tensor and the Ricci tensor are

\[ \Gamma^1_{22} = -x^1, \quad \Gamma^2_{33} = -\frac{x^2}{(x^1)^2}, \quad \Gamma^1_{12} = \frac{1}{x^1}, \quad \Gamma^3_{23} = \frac{1}{x^2}, \]
\[ R_{1332} = -\frac{x^2}{x^1}, \quad S_{12} = -\frac{1}{x^1 x^2}. \]

It can be easily shown that the scalar curvature of the manifold is zero. We shall now show that \((\mathbb{R}^4, g)\) is a generalized quasi-Einstein manifold.

We take the associated scalars as follows:

\[ a = \frac{1}{x^1(x^2)^2}, \quad b = -\frac{8}{3(x^1)^2x^2}, \quad c = -\frac{2}{3(x^1)^2x^2}. \]

We choose the 1-forms as follows:

\[ A_i(x) = \begin{cases} \frac{1}{\sqrt{x^1}}, & \text{for } i=1 \\ \frac{x^2}{\sqrt{x^1}}, & \text{for } i=2 \\ \frac{x^3}{\sqrt{x^1}}, & \text{for } i=3 \\ 0, & \text{for } i=4 \end{cases} \]

and

\[ B_i(x) = \begin{cases} \frac{1}{\sqrt{x^2}}, & \text{for } i=1 \\ \frac{x^1}{\sqrt{x^2}}, & \text{for } i=2 \\ 0, & \text{otherwise} \end{cases} \]

at any point \(x \in M\). In our \((M^4, g)\), (4) reduces with these associated scalars and 1-forms to the following equation:

\[ S_{12} = ag_{12} + bA_1A_2 + cB_1B_2 \quad (42) \]

It can be easily proved that the equation (42) is true.

We shall now show that the associated vectors \(A_i\) and \(B_i\) are unit and also they are orthogonal.

Here,

\[ g^{ij}A_iA_j = 1, \quad g^{ij}B_iB_j = 1, \quad g^{ij}A_iB_j = 0. \]

So, the manifold under consideration is a generalized quasi-Einstein manifold.

**Example 6.3.** [16] A 2- quasi-umbilical hypersurface of a space of constant curvature is a \(G(QE)_n\), which is not a quasi-Einstein manifold.

**Example 6.4.** [16] A quasi-umbilical hypersurface of a Sasakian space form is a \(G(QE)_n\), which is not a quasi-Einstein manifold.
Example 6.5. De and Mallick [16] considered a Riemannian metric $g$ on $R^4$ by

$$ds^2 = g_{ij} dx^i dx^j = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2.$$  \hspace{1cm} (43)

Then they showed that $(M^4, g)$ is a generalized quasi-Einstein manifold, which is not a quasi-Einstein manifold.

Example 6.6. Özgür and Sular [24] assumed an isometrically immersed surface $\bar{M}$ in $E^3$ with non-zero distinct principal curvatures $\lambda$ and $\mu$. Then they considered the hypersurface $M = \bar{M} \times E^{n-2}$ in $E^{n+1}$, $n \geq 4$. The principal curvatures of $M$ are $\lambda, \mu, 0, ..., 0$, where $0$ occurs $(n-2)$-times. Hence the manifold is a 2- quasi umbilical hypersurface and so it is generalized quasi-Einstein.

Example 6.7. Özgür and Sular [24] assumed a sphere $S^2$ in $E^{k+2}$ given by the immersion $f: S^2 \to E^{k+2}$ and $BS^2$ be the bundle of unit normal to $S^2$. The hypersurface $M$ defined by the map $\varphi_t : BS^2 \to E^{k+2}$, $\varphi_t(x, \xi) = F(x, t\xi)$ is called the tube of radius $t$ over $S^2$. It was proved in [5] that if $(\lambda, \lambda)$ are the principal curvature of $S^2$ then the principal curvatures of $M$ are $(\frac{\lambda}{(t^2 + 1)} \frac{1}{(t^2 + 1)}, \frac{1}{(t^2 + 1)}, ..., \frac{1}{(t^2 + 1)})$, where $-\frac{1}{t}$ occurs $(k-1)$-times. So $M$ is 2-quasi umbilical and hence it is generalized quasi-Einstein.

Example 6.8. The study of warped product manifold was initiated by Kručković [20] in 1957. Again in 1969 Bishop and O’Neill [4] also obtained the same notion of the warped product manifolds while they were constructing a large class of manifolds of negative curvature. Warped product are generalizations of the Cartesian product of Riemannian manifolds. Let $(\bar{M}, \bar{g})$ and $(M', \bar{g}')$ be two Riemannian or semi-Riemannian manifolds. Let $\bar{M}$ and $M'$ be covered with coordinate charts $(U, x^1, x^2, ..., x^n)$ and $(V, y^p, y^{p+1}, ..., y^n)$ respectively. Then the warped product $M = \bar{M} \times_f M'$ is the product manifold of dimension $n$ furnished with the metric

$$g = \pi'(\bar{g}) + (f \circ \pi)\sigma'(\bar{g}'),$$  \hspace{1cm} (44)

where $\pi : \bar{M} \to M$ and $\sigma : M \to M'$ are natural projections such that the warped product manifold $\bar{M} \times_f M'$ is covered with the coordinate chart

$$(U \times V, x^1, x^2, ..., x^n, y^{p+1}, y^{p+2}, ..., y^n).$$

Then the local components of the metric $g$ with respect to this coordinate chart are given by

$$g_{ij} = \begin{cases} 
\tilde{g}_{ij} & \text{for } i = a \text{ and } j = b, \\
\frac{\partial g}{\partial x^i} & \text{for } i = a \text{ and } j = b, \\
0 & \text{otherwise},
\end{cases}$$  \hspace{1cm} (45)

Here $a, b, c, ..., \in \{1, 2, ..., p\}$ and $a, \beta, \gamma, ..., \in \{p + 1, p + 2, ..., n\}$ and $i, j, k, ..., \in \{1, 2, ..., n\}$. Here $\bar{M}$ is called the base, $M'$ is called the fiber and $f$ is called warping function of the warped product $M = \bar{M} \times_f M'$. We denote by $\Gamma^i, R_{ijk}$, $\tilde{R}_{ij}$ and $r$ as the components of Levi-Civita connection $\nabla$, the Riemann-Christoffel curvature tensor $K$, Ricci tensor $S$ and the scalar curvature of $(M, \bar{g})$ respectively. Moreover we consider that, when $\Omega$ is a quantity formed with respect to $g$, we denote by $\tilde{\Omega}$ and $\Omega'$, the similar quantities formed with respect to $\bar{g}$ and $g'$ respectively. Then the non-zero local components of Levi-Civita connection $\nabla$ of $(M, \bar{g})$ are given by

$$\Gamma^a_{bc} = \Gamma^a_{bc}, \quad \Gamma^a_{\beta \gamma} = \Gamma^a_{\beta \gamma}, \quad \Gamma^a_{\beta \gamma} = -\frac{1}{2} \tilde{g}^{ab} f_{\beta \gamma}, \quad \Gamma^a_{\alpha \beta} = \frac{1}{2} \frac{\partial f}{\partial x^\tilde{a}},$$  \hspace{1cm} (46)

where $f_{\alpha \beta} = \partial_\alpha f = \frac{\partial f}{\partial x^\alpha}$. The local components $R_{\alpha \beta \gamma \delta} = g_{\alpha \lambda} R^\lambda_{\beta \gamma \delta} = g_{\alpha \lambda} (\partial_\delta \Gamma^\lambda_{\beta \gamma} - \partial_\gamma \Gamma^\lambda_{\beta \delta} + \Gamma^\lambda_{\mu \delta} \Gamma^\mu_{\beta \gamma} - \Gamma^\lambda_{\mu \gamma} \Gamma^\mu_{\beta \delta})$, $\partial_\alpha f = \frac{\partial f}{\partial x^\alpha}$, of the Riemann-Christoffel curvature tensor $R$ of $(M, \bar{g})$ which may not vanish identically are the following:

$$R_{abcd} = R_{abcd}, \quad R_{a \alpha \beta \gamma} = -f T_{a \alpha \beta} g_{\alpha \beta \gamma}, \quad R_{a \alpha \beta \gamma} = f R^*_{a \alpha \beta \gamma} - f^2 G^*_{a \alpha \beta \gamma},$$  \hspace{1cm} (47)
where $G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$ and
\[
T_{ab} = -\frac{1}{2f}(\nabla_a f_b - \frac{1}{2f}f_a f_b), \quad \text{tr}(T) = g^{ab}T_{ab},
\]
\[
P = \frac{1}{4f^2}g^{ab}f_a f_b,
\]
\[
Q = f((n - p - 1)p - \text{tr}(T)).
\]

Again the non-zero local components of the Ricci tensor $R_{jk} = g^{il}R_{ijkl}$ of $(M, g)$ are given by
\[
R_{ab} = \bar{R}_{ab} + (n - p)T_{ab}, \quad R_{a\beta} = R^*_{a\beta} - Qg^*_{a\beta},\tag{48}
\]
The scalar curvature $r$ of $(M, g)$ is given by
\[
r = \bar{r} + \frac{r}{f} - (n - p)(n - p - 1)p - 2\text{tr}(T).\tag{49}
\]

Here we consider warped product manifold $M = I \times f M^*, \dim I = 1, \dim M^* = n - 1 (n \geq 3), f = \exp\{\frac{q}{4}\}$. We take the metric on $I$ as $(dt)^2$ and $M^*$ is a quasi-Einstein manifold.

Using the above consideration and (48) we get
\[
R_{tt} = \bar{R}_{tt} + (n - 1)T_{tt},
\]
which implies
\[
R_{tt} = -\frac{(n - 1)}{16}\left[q'' + 4q'''\right],\tag{50}
\]
since $R_{tt}$ of $I$ is zero.

Also
\[
R_{a\beta} = R^*_{a\beta} - Qg^*_{a\beta},\tag{51}
\]
which implies
\[
R_{a\beta} = R^*_{a\beta} - \frac{e^q}{16}\left[(2n - 3)(q')^2 + 4(n - 1)q''\right]g^*_{a\beta},\tag{52}
\]
where ' $'$ and ' $''$' denote the 1st order and 2nd order partial derivatives respectively with respect to $t$.

Since $M^*$ is $(QE)_n$, we obtain
\[
R^*_{a\beta} = \lambda g^*_{a\beta} + \mu A^*_{a}A^*_{\beta},\tag{53}
\]
where $\lambda$ and $\mu$ are certain non-zero scalars and $A^*_a$ is unit covariant vector such that $g^{*\alpha\beta}A^*_\alpha A^*_\beta = 1$ and
\[
A_a(x) = \begin{cases} \bar{A}_a, & \text{for } \alpha = 1 \\ A^*_a, & \text{otherwise.} \end{cases}
\]

Using (52) in (51) we get
\[
R_{a\beta} = \lambda g^*_{a\beta} + \mu A^*_aA^*_\beta - \frac{e^q}{16}\left[(2n - 3)(q')^2 + 4(n - 1)q''\right]g^*_{a\beta},\tag{54}
\]
Again, using (45) and (53) in (54) we can write
\[
R_{a\beta} = -\frac{1}{16}\left[(2n - 3)(q')^2 + 4(n - 1)q''\right]g_{a\beta} + \frac{\lambda}{e^q}g_{a\beta} + \mu A^*_aA^*_\beta.\tag{55}
\]
Now if we choose $g_{\alpha\beta} = \delta_{\alpha\beta} + B_\alpha B_\beta$, where

$$B_\alpha(x) = \begin{cases} B_\alpha & \text{if } \alpha = 1 \\ B_\alpha' & \text{otherwise} \end{cases},$$

(56)

then

$$R_{\alpha\beta} = \frac{1}{16} ((2n - 3)(q')^2 + 4(n - 1)q'')g_{\alpha\beta} + \lambda B_\alpha B_\beta + \mu A_\alpha A_\beta.$$  

(57)

Again from (50) we obtain

$$R_{tt} = \frac{1}{16} [(2n - 3)(q')^2 + 4(n - 1)q'']g_{tt} - \frac{1}{16} [(2n - 3)(q')^2 + 4(n - 1)q'']$$

$$- \frac{(n - 1)}{16} [(q')^2 + 4q''],$$  

(58)

since $\tilde{g}_{tt} = 1$ and $g_{tt} = \tilde{g}_{tt}$ in I.

Thus (58) can be written as

$$R_{tt} = \frac{1}{16} [(2n - 3)(q')^2 + 4(n - 1)q'']g_{tt} - \frac{3n - 4}{16} (q')^2$$

$$+ \frac{2(n - 1)}{4} q''.$$  

(59)

Since $\dim I = 1$, we can take

$$\tilde{A}_t = \tilde{q}'$$

(60)

and

$$\tilde{B}_t = \sqrt{q''},$$

(61)

where $\tilde{q}'$ and $\tilde{q}''$ are functions on M.

Then using (53), (56), (60) and (61), equation (59) can be written as follows:

$$R_{tt} = \frac{1}{16} [(2n - 3)(q')^2 + 4(n - 1)q'']g_{tt} - \frac{3n - 4}{16} A_tA_t$$

$$+ \frac{2(n - 1)}{4} B_t B_t.$$  

(62)

Thus from (57) and (62) we can conclude that $M = I \times f M^*$ is a generalized quasi-Einstein manifold if $M^*$ is a quasi-Einstein manifold.

References


