Remark on Sheffer Polynomials

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Abstract. This paper deals with some theorems on Sheffer A-type zero polynomial sets.

1. Introduction

A polynomial set $p_n(x)$ is said to be of Sheffer A-type zero if and only if it has a generating function in the form \cite{3, 12, 13} as

$$A(t) \exp (xG(t)) = \sum_{n=0}^{\infty} p_n(x)t^n,$$

where $A(t)$ and $G(t)$ are two formal power series

$$A(t) = \sum_{n=0}^{\infty} a_n t^n, \quad a_0 \neq 0;$$
$$G(t) = \sum_{n=0}^{\infty} g_n t^{n+1}, \quad g_0 \neq 0;$$

and $J(D)p_0(x) = 0$ and $Jp_n(x) = p_{n-1}(x), \quad n \geq 1$; where $J(D)$ is defined as

$$J = J(D) = \sum_{k=0}^{\infty} a_k D^{k+1}, \quad a_0 \neq 0 \quad \text{and} \quad D \equiv \frac{d}{dx}.$$

Al Salam and Verma \cite{1} gave the generalized Sheffer polynomials by considering $\phi_n(x)$ as a Sheffer A-type zero

$$\sum_{i=1}^{r} A_i(t) \exp ((xG(\epsilon_i(t))) = \sum_{n=0}^{\infty} \phi_n(x)t^n,$$

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where
\[ J(D) = \sum_{k=0}^{\infty} c_k D^{k+r}, \quad J(D) \phi_n(x) = \phi_{n-r}(x), \quad (n = r, r + 1, \ldots) \]

Thorne [21] obtained an interesting characterization of Appell polynomials by means of Stieltjes integral. Appell sets [17] are hold following equivalent condition:

(i) \( p'_n(x) = p_{n-1}(x), n = 0, 1, 2, \ldots \)

(ii) There exists a formal power series \( A(t) = \sum_{n=0}^{\infty} a_n t^n \), \( (a_0 \neq 0) \) such that
\[ A(t) \exp(\alpha t) = \sum_{n=0}^{\infty} p_n(x) t^n. \]

Osegove [14] gave the generalization of Appell sets in a different direction. He studied polynomial sets and hold the following property
\[ D^r p_n(x) = p_{n-r}(x), \quad n \geq r, \]
where \( r \) is a (fixed) positive integer.

Huff and Rainville [11] proved the necessary and sufficient condition for polynomial \( p_n(x) \). If polynomial \( p_n(x) \) is generated by \( A(t) \exp(\alpha t) \) then a necessary and sufficient condition for \( p_n(x) \) be a Sheffer A-type \( m, m > 0 \), \( \psi(x) = 0 \), \( \psi(x) = 0F_m[\beta; b_1, b_2, \ldots, b_m; \alpha t] \), where \( \alpha \) is a nonzero constant.

Goldberg [10] generalized the above result and proved, if the polynomial set \( p_n(x) \) is generated by \( A(t) \exp(\alpha t) \) then a necessary and sufficient condition for \( p_n(x) \) to be a Sheffer A-type \( m, m > 0 \), is that there exist a positive number \( r \) which divides \( m \) and numbers \( b_1, b_2, \ldots, b_r \) (none zero nor negative integers) such that \( p_n(x) \) is \( \alpha \)-type zero for \( \alpha = D \prod_{i=1}^{r} (x^{D} + b_i - 1) \), \( D = \frac{d}{dx} \).


Let \( p_n^{(\alpha)}(x) \) be a simple polynomial set and has following generating function \( [6,19] \)
\[ (1-t)^{-\alpha} F(x,t) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x) t^n, \quad (1) \]
where \( F(x,t) \) is independent on parameter \( \alpha \).

If \( F(x,t) = (1-t)^{-1} \exp(\frac{\alpha}{1-t}) \) then this gives the generalized Laguerre polynomials \( p_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) \).[16]

2. Main Results

First we prove the following Lemmas.

**Lemma 1:** The polynomial set \( p_n^{(\alpha-\beta n)}(x) \) is generated by
\[ \frac{(1+u(t))^\beta}{1+\beta u(t)} F(x,u(t)[1+u(t)]^{\beta-1}) = \sum_{n=0}^{\infty} p_n^{(\alpha-\beta n)}(x) t^n, \quad (2) \]
where \( u(t) \) is the inverse of \( v(t) = t(1 + t)^{\beta - 1} \), that is, \( v(u(t)) = u(v(t)) = t \).

**Proof:** Let

\[
(1 - t)^{-\alpha} F(x, t) = \left\{ \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} t^n \right\} \left\{ \sum_{n=0}^{\infty} p_n(x) t^n \right\}
\]

\[
\sum_{n=0}^{\infty} p_n^{(\alpha)}(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{-\alpha}{n} \right) (-1)^n t^{n+k} p_k(x),
\]

On making the use of \((-\alpha)^n = (-1)^n (\alpha + n - 1)\), for positive integers \( \alpha \) and \( n \).

\[
\sum_{n=0}^{\infty} p_n^{(\alpha)}(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{\alpha + n - 1}{n} \right) p_k(x) t^{n+k},
\]

we get

\[
p_n^{(\alpha)}(x) = \sum_{k=0}^{n} \left( \frac{\alpha + n - k - 1}{n - k} \right) p_k(x),
\]

On setting \( \alpha = \beta n \) in equation (3), yields

\[
\sum_{n=0}^{\infty} p_n^{(\alpha-\beta n)}(x) t^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \left( \frac{\alpha + \beta k - 1 - (\beta - 1)n}{n} \right) t^n \right) p_k(x) t^k.
\]

On making the use of following identity \([15]\)

\[
\sum_{n=0}^{\infty} \left( \frac{\alpha + bn}{n} \right) \left[ \frac{z}{(1 + z)^n} \right] = \frac{(1 + z)^{1+a}}{(1 + b)z},
\]

and afterwards setting \( a = \alpha + \beta k - 1, b = -\beta - 1 \) and \( z = u(t) \), this yields

\[
\sum_{n=0}^{\infty} \left( \frac{\alpha + \beta k - 1 - (\beta - 1)n}{n} \right) t^n = \frac{(1 + u(t))^{1+\beta k}}{1 + \beta u(t)}.
\]

This can be easily written in following form as

\[
\sum_{n=0}^{\infty} p_n^{(\alpha-\beta n)}(x) t^n = \frac{(1 + u(t))^{1+\beta}}{1 + \beta u(t)} \sum_{k=0}^{\infty} p_k(x) \left[ t(1 + u(t))^{\beta k} \right],
\]

\[
\sum_{n=0}^{\infty} p_n^{(\alpha-\beta n)}(x) t^n = \frac{(1 + u(t))^{1+\beta}}{1 + \beta u(t)} \sum_{k=0}^{\infty} p_k(x) \left[ u(t)(1 + u(t))^{2\beta - 1} \right]^k.
\]

Thus

\[
\sum_{n=0}^{\infty} p_n^{(\alpha-\beta n)}(x) t^n = \frac{(1 + u(t))^{1+\beta}}{1 + \beta u(t)} F(x, u(t)(1 + u(t))^{2\beta - 1}).
\]

This leads the proof.

**Lemma 2:** The polynomial set \( p_n^{(\alpha-\gamma n, \beta-\delta n)}(x, y) \) is generated by

\[
\frac{(1 + u(t))^{1+\beta}}{1 + (\gamma + \delta) u(t)} F(x, y, u(t)(1 + u(t))^{2\gamma - 1}) = \sum_{n=0}^{\infty} p_n^{(\alpha-\gamma n, \beta-\delta n)}(x, y) t^n,
\]

where \( u(t) \) is the inverse of \( v(t) = t(1 + t)^{\gamma - 3} \), that is, \( v(u(t)) = u(v(t)) = t \).
Proof: Let

\[(1 - t)^{-\alpha - \beta} F(x, y, t) = \left\{ \sum_{n=0}^{\infty} \left( -\frac{\alpha - \beta}{n} \right) (-1)^n t^n \right\} \left\{ \sum_{n=0}^{\infty} p_n(x, y) t^n \right\} \]

\[= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( -\frac{\alpha - \beta}{n} \right) (-1)^n p_k(x, y) t^{n+k}. \]

Since

\[\left( -\frac{\alpha - \beta}{n} \right) = (-1)^n \left( \frac{\alpha + \beta + n - 1}{n} \right),\]

where \(\alpha, \beta\) and \(n\) are positive integers.

We get

\[\sum_{n=0}^{\infty} p_n^{(\alpha, \beta)}(x, y) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{\alpha + \beta + n - 1}{n} \right) p_k(x, y) t^{n+k},\]

\[\sum_{n=0}^{\infty} p_n^{(\alpha, \beta)}(x, y) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{\alpha + \beta + n - k - 1}{n - k} \right) p_k(x, y) t^{n}.\]

On comparing the coefficient of \(t^n\), gives

\[p_n^{(\alpha, \beta)}(x, y) = \sum_{k=0}^{n} \left( \frac{\alpha + \beta + n - k - 1}{n - k} \right) p_k(x, y).\]

On replacing \(\alpha\) by \(\alpha - \gamma n\) and \(\beta\) by \(\beta - \delta n\), we get

\[\sum_{n=0}^{\infty} p_n^{(\alpha - \gamma n, \beta - \delta n)}(x, y) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{\alpha - \gamma n + \beta - \delta n + n - k - 1}{n - k} \right) p_k(x, y) t^{n}.\]

On further simplification, yields

\[\sum_{n=0}^{\infty} \left( \frac{a + bn}{n} \right) \left[ \frac{z}{(1 + z)^p} \right] = \frac{(1 + z)^{1+a}}{1 + (1 - b)z}.\]

Now, setting \(a = \alpha + \beta + (\gamma + \delta)k - 1, b = -(\gamma + \delta - 1)\) and \(z = u(t)\), this becomes

\[\sum_{n=0}^{\infty} \left( \frac{\alpha + \beta + (\gamma + \delta)k - 1 - (\gamma + \delta - 1)n}{n} \right) [u(t)(1 + u(t))^{\gamma + \delta - 1}] = \frac{(1 + u(t))^{1+\alpha+\beta+(\gamma+\delta)k-1}}{1 + (\gamma + \delta)u(t)}.\]

Or

\[\sum_{n=0}^{\infty} \left( \frac{\alpha + \beta + (\gamma + \delta)k - 1 - (\gamma + \delta - 1)n}{n} \right) t^n = \frac{(1 + u(t))^{1+\alpha+\beta+(\gamma+\delta)k}}{1 + (\gamma + \delta)u(t)} ;\]

this leads to

\[\sum_{n=0}^{\infty} p_n^{(\alpha - \gamma n, \beta - \delta n)}(x, y) t^n = \frac{(1 + u(t))^{1+\beta}}{1 + (\gamma + \delta)u(t)} \sum_{k=0}^{\infty} p_k(x, y) \left[ t(1 + u(t))^{\gamma + \delta} \right]^k.\]

Finally we arrive at conclusion that

\[\sum_{n=0}^{\infty} p_n^{(\alpha - \gamma n, \beta - \delta n)}(x, y) t^n = \frac{(1 + u(t))^{1+\beta}}{1 + (\gamma + \delta)u(t)} F(x, y, u(t)(1 + u(t))^{2(\gamma + \delta) - 1}). \]
This completes the proof.

To prove the theorems, we consider $p_n(x, y)$ is generated by

$$A(t) \phi(xH(t), yG(t)) = \sum_{n=0}^{\infty} p_n(x, y)t^n,$$

where

$$G(t) = \sum_{n=0}^{\infty} g_nt^n, \quad g_0 \neq 0,$$

$$H(t) = \sum_{n=0}^{\infty} h_nt^n, \quad h_0 \neq 0,$$

$$A(t) = \sum_{n=0}^{\infty} a_nt^n, \quad a_0 \neq 0.$$

On taking $F(x, y, t) = A(t) \phi(xH(t), yG(t))$, we get

$$\frac{(1 + u(t))^{a+\beta}}{1 + (\gamma + \delta)u(t)} A(u(t)(1 + u(t))^{2(\gamma+\beta)-1}) \phi \left(xH(u(t)(1 + u(t))^{2(\gamma+\beta)-1}), yG(u(t)(1 + u(t))^{2(\gamma+\beta)-1}) \right)$$

$$= \sum_{n=0}^{\infty} p_n^{(a-\gamma, \beta-bn)}(x, y)t^n.$$

Hence, we can say that if $p_n(x, y)$ is a generalized Appell set then $p_n^{(a-\gamma, \beta-bn)}(x, y)$ is also generalized Appell set.

**Theorem 1:** if $p_n(x, y)$ is Sheffer A-type zero polynomials in two variables then $p_n^{(a-\gamma, \beta-bn)}(x, y)$ is also Sheffer A-type zero polynomials in two variables.

**Proof:** Let $p_n(x, y)$ be of Sheffer A-type zero polynomials in two variables and there exists a differential operator $J = J(D) = \sum_{k=0}^{\infty} c_k D^{k+1}$, $c_0 \neq 0$, $D = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, where $c_k$ are constants, such that $Jp_n(x, y) = p_{n-1}(x, y)$ for all $n \geq 1$.

Since $p_n(x, y)$ is of A-type zero if $p_n(x, y)$ have the generating relation [2] as

$$A(t) \exp(xH(t)) \exp(yG(t)) = \sum_{n=0}^{\infty} p_n(x, y)t^n.$$

From lemma 2 and equation (6), we get

$$\frac{(1 + u(t))^{a+\beta}}{1 + (\gamma + \delta)u(t)} A(u(t)) \exp \left(xH(u(t)(1 + u(t))^{2(\gamma+\beta)-1}) \exp(yG(u(t)(1 + u(t))^{2(\gamma+\beta)-1}) \right)$$

$$= \sum_{n=0}^{\infty} p_n^{(a-\gamma, \beta-bn)}(x, y)t^n.$$

**Theorem 2:** If $p_n(x, y)$ is a generalized Sheffer set of A-type zero then $p_n^{(a-\gamma, \beta-bn)}(x, y)$ is also generalized Sheffer set of A-type zero.

**Proof:** Since $p_n(x, y)$ is a generalized Sheffer set of A-type zero and the generating function is given by [18]

$$\sum_{i=1}^{\infty} A_i(t) \exp \left((xH(\epsilon_i(t)) \exp \left((yG(\epsilon_i(t)) \right) = \sum_{n=0}^{\infty} p_n(x, y)t^n,$$
Thus, we can say that
\[ p_n(t) = \sum_{k=0}^{\infty} \exp\left(\frac{1 + u(t)\alpha^1}{1 + (\gamma + \delta)u(t)} \sum_{i=1}^{\ell} A_i(u(t)(1 + u(t))^{2(\gamma + \delta) - 1}) \right) \]

On applying lemma 2, equation (7) takes following form
\[
\sum_{n=0}^{\infty} p_n^{(α−γn,β−δn)}(x, y)t^n = \frac{(1 + u(t))^{α+β}}{1 + (\gamma + \delta)u(t)} \sum_{i=1}^{\ell} A_i(u(t)(1 + u(t))^{2(\gamma + \delta) - 1}) \exp\left(xH(\epsilon; u(t)(1 + u(t))^{2(\gamma + \delta) - 1}) \exp\left(yG(\epsilon; u(t)(1 + u(t))^{2(\gamma + \delta) - 1}) \right) \right]
\]

Thus, we can say that \( p_n^{(α−γn,β−δn)}(x, y) \) is also generalized Sheffer set of A-type zero.

The operator \( J = \sum_{k=0}^{\infty} c_k D^{k+1} \) is associated with \( p_n(x, y) \), where \( D = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \). This is generated by the function \( f(t) = \sum_{k=0}^{\infty} c_k t^{k+1} \) and \( f(t) \) is the inverse of the function \( (H + G)(t) \). The \( p_n^{(α−γn,β−δn)}(x, y) \) corresponds to the operator which is generated by the inverse of function \( (H + G)(u(t)) \).

References