A Hosszú-Gluskin Algebra and a Central Operation of \((sm, m)\)-Groups

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Abstract. In this paper we prove a generalization of the Hosszú-Gluskin theorem for \((sm, m)\)-groups in terms of \(a \in \mathbb{N}_0\). Let \(a_p\) denote the sequence of elements of a set \(Q\) and \(a_{p} = \cdots = a_{q} = a\) are satisfied, then \(a_{p}\) is denoted by \(\underbrace{a_{p} a_{p} \cdots a_{p}}_{m}\). If we have \(a_{m} \in Q^{m}\), then \(s\) sequences \(a_{m}\) denote a sequence of \(s\) sequences \(a_{m}\).

1. Introduction

Firstly, we explain a notation introduced by Janez Ušan which we use in this paper. Let \(p \in \mathbb{N}\) and let \(q \in \mathbb{N}_0\). Then \(a_{p}^{q}\) denotes \(a_{p}, \ldots, a_{q}\) the sequence of elements of a set \(Q\) if \(p < q\). If \(p = q\), then \(a_{p}^{q}\) denotes the element \(a_{p}\) into the set \(Q\) and if \(p > q\), then \(a_{p}^{q}\) denotes the empty sequence. If \(a_{p}^{q}\) is a sequence over a set \(Q\), \(p \leq q\) and the equalities \(a_{p} = \cdots = a_{q} = a\) are satisfied, then \(a_{p}^{q}\) is denoted by \(\underbrace{a_{p} a_{p} \cdots a_{p}}_{m}\). If we have \(a_{m} \in Q^{m}\), then a \(m\) denotes a sequence of \(m\) sequences \(a_{m}\).

The notion of an \((n, m)\)-group, as a generalization of the notion of an \(n\)-group, that is of a group was introduced by G. Ćupona in 1983. Let \(Q\) be a nonempty set and let \(A\) be a mapping of the set \(Q\) into the set \(Q^{m}\), where \(n \geq m + 1\). Then, we say that \((Q, A)\) is an \((n, m)\)-groupoid. Since every groupoid is a group if and only if it is a semigroup and a quasigroup, similarly an \((n, m)\)-group was defined as an \((n, m)\)-semigroup and an \((n, m)\)-quasigroup.

Definition 1.1. \([1]\) Let \((Q, A)\) be an \((n, m)\)-groupoid, \(n \geq m + 1\). \((Q, A)\) is an \((n, m)\)-group if the following statements hold:

\(\forall i, j \in \{1, \ldots, n - m + 1\}\) and for every sequence \(x_{i}^{2n-m} \in Q\) the following equality holds:

\[
A \left( x_{i}^{2n-m}, A \left( x_{i}^{i+n-1}, x_{i+m}^{2n-m} \right) \right) = A \left( x_{i}^{2n-m}, A \left( x_{i}^{i+n-1}, x_{i+m}^{2n-m} \right) \right),
\]

which is called \((i, j)\)-associative law.
b) \(\forall i \in [1, \ldots, n - m + 1]\) and for every sequence \(a_i^1 \in Q\) there is exactly one sequence \(x_i^m \in Q\) such that the following equality holds:

\[
A\left(a_i^{n-1}, x_i^m, a_{i}^{n-m}\right) = a_{n-m+1}^n.
\]

An \((n,m)\)-groupoid \((Q,A)\), where the statement \(a)\) holds, is called an \((n,m)\)-semigroup, so that the \((n,m)\)-groupoid \((Q,A)\), where the statement \(b)\) holds, is called a weak \((n,m)\)-quasigroup.

**Example 1.2.** Let \((Q,\cdot)\), \(Q = \{1,2,3,4\}\) be the Klein group defined with the following table:

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Let \(\psi\) be the permutation of the set \(Q\) defined in the following way: \(\psi \overset{\text{def}}{=} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}\). Further on, let \(A : Q^6 \to Q^2\) be the mapping defined in the following way: \(A(x_i^j) \overset{\text{def}}{=} (x_1 \cdot \psi(x_3) \cdot x_5 \cdot \psi(x_4) \cdot x_6)\). We prove that \((Q,A)\) is a \((6,2)\)-group. Firstly, we prove that the \((1,2)\)-associative law holds.

\[
A(A(x_i^j, x_j^{j'p})) = A(x_1 \cdot \psi(x_3) \cdot x_5 \cdot \psi(x_4) \cdot x_6, x_7, x_8, x_9, x_{10}) = (x_1 \cdot \psi(x_3) \cdot x_5 \cdot \psi(x_4) \cdot x_6, x_10) \quad A(x_1, A(x_2^j, x_j^{j''})) = A(x_1, x_2 \cdot \psi(x_4) \cdot x_6, x_7, x_8, x_9, x_{10}) = (x_1 \cdot \psi(x_3) \cdot x_5 \cdot \psi(x_7) \cdot x_8, x_2 \cdot \psi(x_4) \cdot x_6, x_8, x_{10})
\]

Similarly, we can prove that the \((i, j)\)-associative law holds for all \(i, j \in \{1,2,3,4,5\}\).

Now, we prove the statement \(b)\) of Definition 1.1 for \(i = 1\) holds.

\[
A(x_i^1, a_i^1) = a_i^2 \Leftrightarrow (x_1 \cdot \psi(a_1) \cdot a_3, x_2 \cdot \psi(a_2) \cdot a_4) = (a_5, a_6) \Leftrightarrow x_1 \cdot \psi(a_1) \cdot a_3 = a_5 \text{ and } x_2 \cdot \psi(a_2) \cdot a_4 = a_6
\]

Because \((Q,\cdot)\) is a group, for every sequence \(a_i^1 \in Q\) there is exactly one \(x_1^1 \in Q\) and exactly one \(x_2^m \in Q\) such that the previous sequence of equivalences holds. Likewise, we can prove that the statement \(b)\) of Definition 1.1 for all \(i \in \{1,2,3,4,5\}\) holds.

An interesting research method of \(n\)-structures and \((n,m)\)-structures was inspired by J. Ušan, who in his manifold papers puts emphasis on a neutral operation. These results are systematized in the monograph [16]: \(n\)-groups in the light of the neutral operations. An \((i, j)\)-neutral operation of an \(n\)-groupoid \((Q,A)\) was defined by Ušan in 1988 [10] and it presents a generalization of a neutral element of a groupoid. Then, in 1989 he defined a \(\{1,n-m+1\}\)-neutral operation of an \((n,m)\)-groupoid \((Q,A)\).

**Definition 1.3.** [11] Let \(n \geq 2m\) and let \((Q,A)\) be an \((n,m)\)-groupoid. Furthermore, let \(e_L, e_R\) and \(e\) be mappings of the set \(Q^{n-2m}\) into the set \(Q^m\). Then:

a) \(e_L\) is a left \(\{1,n-m+1\}\)-neutral operation of the \((n,m)\)-groupoid \((Q,A)\) iff \(\forall a_i^{n-2m}, x_i^m \in Q\) the following equality holds:

\[
A\left(e_L\left(a_i^{n-2m}\right), a_i^{n-2m}, x_i^m\right) = x_i^m;
\]

b) \(e_R\) is a right \(\{1,n-m+1\}\)-neutral operation of the \((n,m)\)-groupoid \((Q,A)\) iff \(\forall a_i^{n-2m}, x_i^m \in Q\) the following equality holds:

\[
A\left(x_i^m, a_i^{n-2m}, e_R\left(a_i^{n-2m}\right)\right) = x_i^m;
\]

c) \(e\) is a \(\{1,n-m+1\}\)-neutral operation of the \((n,m)\)-groupoid \((Q,A)\) iff it is a left \(\{1,n-m+1\}\)-neutral operation and a right \(\{1,n-m+1\}\)-neutral operation.
Example 1.4. Let \((Q, A)\) be the \((6, 2)\)-group defined in Example 1.2. Now we define a mapping \(e : Q^2 \to Q^2\) such that the following equality holds: \(e(a_1^2)^\text{def} = (\psi(a_1), \psi(a_2))\). We prove that \(e\) is a \([1, 5]\)-neutral operation of the \((6, 2)\)-group \((Q, A)\). For all \(x_1^2, x_2^2 \in Q\) two following sequences of equalities hold:

\[
A(e(a_1^2), a_1^2, x_2^2) = A(\psi(a_1), \psi(a_2), a_1^2, x_2^2) = (\psi(a_1) \cdot \psi(a_1) \cdot x_1, \psi(a_2) \cdot \psi(a_2) \cdot x_2) = (x_1, x_2),
\]

\[
A(x_1^2, a_1^2, e(a_2^2)) = A(x_1^2, a_1^2, \psi(a_1), a_2^2) = (x_1 \cdot \psi(a_1) \cdot \psi(a_1), x_2 \cdot \psi(a_2) \cdot \psi(a_2)) = (x_1, x_2).
\]

If we put \((n, m) = (2, 1)\), the definition of a \([1, n - m + 1]\)-neutral operation of an \((n, m)\)-groupoid is the same as the definition of a neutral element of a groupoid. Furthermore, a \([1, n - m + 1]\)-neutral operation has the same properties as a neutral element in binary structures. Some of them are the following statements:

- there is at most one \([1, n - m + 1]\)-neutral operation of the \((n, m)\)-groupoid \((Q, A)\) \(n \geq 2m\);
- if \(e_L\) is a \([1, n - m + 1]\)-neutral operation of the \((n, m)\)-groupoid \((Q, A)\) and \(e_R\) is a \([1, n - m + 1]\)-neutral operation of the \((n, m)\)-groupoid \((Q, A)\), then they are equal and \(e = e_L = e_R\) is a \([1, n - m + 1]\)-neutral operation of the \((n, m)\)-groupoid \((Q, A)\);
- every \((n, m)\)-group, where \(n \geq 2m\), has exactly one \([1, n - m + 1]\)-neutral operation.

The previous statements were proved in [11].

Moreover, one generalization of an inverse element in binary structures was defined in \(n\)-structures by Janez Ţusan 1994 in [12], in terms of a neutral operation. Similarly, he defined an inverse operation in \((n, m)\)-structures.

Proposition 1.5. [14] Let \((Q, A)\) be an \((n, m)\)-groupoid, \(n \geq 2m\) and let the following statements hold:

(a) in \((Q, A)\) a \([1, n - m + 1]\)-associative law holds;
(b) for every sequence \(a_1^n \in Q\) there is at least one \(x_1^n \in Q^m\) such that the following equality holds:

\[
A\left(a_1^n, x_1^n\right) = a_1^{n-m+1};
\]

(c) for every sequence \(a_1^n \in Q\) there is at least one \(y_1^n \in Q^m\) such that the following equality holds:

\[
A\left(y_1^n, a_1^{-m}\right) = a_1^{n-m+1}.
\]

Then, there are mappings \(^{-1} : Q^{n-m} \to Q^m\) and \(e : Q^{n-2m} \to Q^m\) such that in the algebra \((Q, [A, ^{-1}, e])\) the following equalities hold:

(i) \(A\left(b_1^{-2m}, a_1^{-2m}, x_1^n\right) = x_1^n;\)

(ii) \(A\left(a_1^{-2m}, b_1^{-m}, a_1^{-2m}, x_1^n\right) = x_1^n;\)

(iii) \(A\left(\psi_1^{-2m}, a_1^{-2m}, b_1^{-m}\right) = e_1^{-2m};\)

(iv) \(A\left(\psi_1^{-2m}, b_1^{-m}, a_1^{-2m}\right) = e_1^{-2m}\).

Where holds

\[
\begin{align*}
\left(a_1^{-2m}, b_1^{-m}\right)^{-1} & \overset{\text{def}}{=} E\left(a_1^{-2m}, b_1^{-m}, a_1^{-2m}\right), \forall a_1^{-2m}, b_1^{-m} \in Q, \\
\end{align*}
\]

where \(E\) is a \([1, 2n - 2m + 1]\)-neutral operation of the \((2n - m, m)\)-groupoid \((Q, \overline{A})\) and \(\overline{A}\left(x_1^{2m}\right)^{-1} \overset{\text{def}}{=} A\left(x_1^2, x_2^{2n-m}\right)\).

In two following propositions, the equalities, which hold for a \([1, n - m + 1]\)-neutral operation \((n > 2m)\), are given and they are important for further research of \((n, m)\)-groups.
Proposition 1.6. [5] Let \((Q, A)\) be an \((n, m)\)-group, \(n > 2m\) and let \(e\) be its \([1, n - m + 1]\)-neutral operation. Then \(\forall a^{n-2m}_1 \in Q, \forall x^m_1 \in Q^m\) and \(\forall t \in [1, \ldots, n - 2m + 1]\) the following equalities hold:

\[
A\left(x^m_1, a^{n-2m}_1, e\left(a^{n-2m}_1\right), a^{-1}_1\right) = x^m_1;
\]

\[
A\left(a^{n-2m}_1, e\left(a^{n-2m}_1\right), a^{-1}_1, x^m_1\right) = x^m_1.
\]

Proposition 1.7. [6] Let \((Q, A)\) be an \((n, m)\)-group, \(n \geq 2m\), \(e\) its \([1, n - m + 1]\)-neutral operation and let \(-1 : Q^{n-m} \to Q^m\) be its inverse operation. Then \(\forall a^{n-2m}_1, b^{n-2m}_1, x^m_1, y^m_1 \in Q\) the following equality holds:

\[
A\left(x^m_1, b^{n-2m}_1, y^m_1\right) = A\left(A\left(x^m_1, a^{n-2m}_1, e\left(b^{n-2m}_1\right)\right)^{-1}, a^{n-2m}_1, y^m_1\right).
\]

2. A Generalization of the Hosszú-Gluskin Theorem for Some \((n, m)\)-Groups in Terms of a Neutral Operation

Hosszú-Gluskin theorem is very important for the description and systematization of \(n\)-groups.

Theorem 2.1. [8], [9] For every \(n\)-group \((Q, A)\), \(n \geq 3\), there is an algebra \((Q, \cdot, \varphi, b)\) such that the following statements hold: (1) \((Q, \cdot)\) is a group; (2) \(\varphi \in \text{Aut}(Q, \cdot)\); (3) \(\varphi(b) = b\); (4) for every \(x \in Q\), \(\varphi^{-1}(x) \cdot b = b \cdot x\); (5) for every \(x^m_1 \in Q, Ax^m_1 = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{m-1}(x_n) \cdot b\).

In 1995 Ušan proved the Hosszú-Gluskin theorem by using a neutral operation and defined a Hosszú-Gluskin algebra of order \(n, n \geq 3\). One generalization of the Hosszú-Gluskin theorem for \((sm, m)\)-groups was described by Cupona and others in 1988 [2].

Following Ušan’s approach to the research of \((n, m)\)-structures, a generalization of the Hosszú-Gluskin theorem for \((sm, m)\)-groups, \(s > 2\) in terms of a \([1, n - m + 1]\)-neutral operation was proved in the following two theorems.

Theorem 2.2. Let \((Q, A)\) be an \((sm, m)\)-group, \(s > 2\), \(e : Q^{(s-2)m} \to Q^m\) its \([1, (s - 1) m + 1]\)-neutral operation and let \(a^{(s-2)m}_1 \in Q\) be an arbitrary sequence. \(\forall x^m_1, y^m_1 \in Q^m\), we define the following operations:

\[
(a) \quad x^m_1 \cdot y^m_1 \overset{\text{def}}{=} A\left(x^m_1, a^{(s-2)m}_1, y^m_1\right),
\]

\[
(b) \quad \varphi\left(x^m_1\right) \overset{\text{def}}{=} A\left(e\left(a^{(s-2)m}_1\right), x^m_1, a^{(s-2)m}_1\right),
\]

\[
(c) \quad e^m_1 \overset{\text{def}}{=} A\left(e\left(a^{(s-2)m}_1\right)\right).
\]

Then, the following statements hold:

(i) \((Q^m, \cdot, \varphi)\) is a group;

(ii) \(\varphi \in \text{Aut}(Q^m, \cdot)\);

(iii) \(\varphi\left(e^m_1\right) = e^m_1\);

(iv) \(\varphi^{-1}(b^m_1) \cdot c^m_1 = c^m_1 \cdot b^m_1\), \(\forall b^m_1 \in Q^m\);

(v) \(A\left(x^m_1\right) = x^m_1 \cdot \varphi\left(x^{2m}_{2m+1}\right) \cdot \varphi^2\left(x^{3m}_{2m+1}\right) \cdots \varphi^{s-1}\left(x^{sm}_{(s-1)m+1}\right) \cdot c^m_1, \forall x^m_1 \in Q\).

Proof. (i) \(\forall x^m_1, y^m_1, z^m_1 \in Q^m\), according to (a) and according to definition of the \((sm, m)\)-group, the following sequence of equalities holds:

\[
x^m_1 \cdot (y^m_1 \cdot z^m_1) = A\left(x^m_1, a^{(s-2)m}_1, A\left(y^m_1, a^{(s-2)m}_1, z^m_1\right)\right) = A\left(A\left(x^m_1, a^{(s-2)m}_1, y^m_1\right), a^{(s-2)m}_1, z^m_1\right) = (x^m_1 \cdot y^m_1) \cdot z^m_1,
\]
which proves that \((Q^n, \cdot)\) is a semigroup.

\(\forall x^n, y^n \in Q^n\) and for an arbitrary sequence \(a^{(n-2)m}_1 \in Q\), since \((Q, A)\) is an \((sm, m)\)-group, there is exactly one \(z^n_1 \in Q^n\) and there is exactly one \(k^n_1 \in Q^n\) such that the following implication holds:

\[
A \begin{pmatrix} x^n_1, a^{(n-2)m}_1, z^n_1 \end{pmatrix} = y^n_1 \iff A \begin{pmatrix} x^n_1 \cdot z^n_1 = y^n_1 \end{pmatrix} \Rightarrow (Q^n, \cdot) \text{ is a quasigroup.}
\]

(ii) By (b) and since \((Q, A)\) is an \((sm, m)\)-quasigroup, we conclude that \(\varphi : Q^n \to Q^n\) is a bijection. We prove that \(\varphi\) is a homomorphism.

\[
\varphi \begin{pmatrix} x^n_1 \cdot y^n_1 \end{pmatrix} \equiv \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, A \begin{pmatrix} x^n_1, a^{(n-2)m}_1, y^n_1 \end{pmatrix}, A \begin{pmatrix} s, e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, x^n_1, a^{(n-2)m}_1, y^n_1, a^{(n-2)m}_1 \end{pmatrix} \end{pmatrix} \equiv A \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, x^n_1, a^{(n-2)m}_1, y^n_1, a^{(n-2)m}_1 \end{pmatrix} \equiv \varphi \begin{pmatrix} x^n_1 \end{pmatrix} \cdot \varphi \begin{pmatrix} y^n_1 \end{pmatrix}.
\]

(iii)

\[
\varphi \begin{pmatrix} c^n_1 \end{pmatrix} \equiv \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, c^n_1, a^{(n-2)m}_1 \end{pmatrix} \equiv A \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, A \begin{pmatrix} \frac{s}{e} \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, a^{(n-2)m}_1 \end{pmatrix} \end{pmatrix} \equiv \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, a^{(n-2)m}_1 \end{pmatrix} \equiv e^n_1.
\]

(iv) \(\forall b^n_1 \in Q^n\) the following sequence of equalities holds:

\[
e^n_1 \cdot b^n_1 \overset{(a), (c)}{=} A \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, a^{(n-2)m}_1, b^n_1 \end{pmatrix} \equiv A \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, a^{(n-2)m}_1, b^n_1 \end{pmatrix} \equiv A \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, a^{(n-2)m}_1, b^n_1 \end{pmatrix} \equiv A \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, a^{(n-2)m}_1, b^n_1 \end{pmatrix} = \ldots = \]

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A \begin{pmatrix} e \begin{pmatrix} a^{(n-2)m}_1 \end{pmatrix}, a^{(n-2)m}_1, b^n_1 \end{pmatrix} = \ldots = \]
(v) \( \forall x_1^{(e)} \in Q \) the following sequence of equalities holds:

\[
A \left( A^{(e-1)m} \right) = A \left( x_1^{(e-1)m}, A \left( a_1^{(e-2)m}, x_1^{(e-2)m}, e \left( a_1^{(e-2)m} \right) \right) \right) = A \left( x_1^{(e-1)m}, A \left( a_1^{(e-2)m}, x_1^{(e-2)m}, e \left( a_1^{(e-2)m} \right) \right) \right) = A \left( x_1^{(e-1)m}, A \left( a_1^{(e-2)m}, x_1^{(e-2)m}, e \left( a_1^{(e-2)m} \right) \right) \right) = A \left( x_1^{(e-1)m}, A \left( a_1^{(e-2)m}, x_1^{(e-2)m}, e \left( a_1^{(e-2)m} \right) \right) \right)
\]

Theorem 2.3. Let \((Q, A)\) be an \((sm, m)\)-group, \(s \geq 3\), \(e : Q^{(e-2)m} \to Q^m\) its \(\{1, (s-1)m+1\}\)-neutral operation. Also, let \((Q^m, \cdot)\) be a group and let for every sequence \(x_1^{(e)} \in Q\) holds:

(a) \( A \left( x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)} \right) = A \left( x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)} \right) = A \left( x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)} \right) = A \left( x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)} \right)
\]

(b) \( A \left( x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)} \right) = A \left( x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)} \right) = A \left( x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)} \right) = A \left( x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)}, \cdot, x_1^{(e)} \right)
\]

Then, there is a sequence \(a_1^{(e-2)m} \in Q\), such that \( \forall x_1^{(e)}, y_1^{(e)} \in Q^m \) the following equalities hold:

(i) \( x_1^{(e)} \cdot y_1^{(e)} = A \left( x_1^{(e)}, \cdot, a_1^{(e-2)m}, y_1^{(e)} \right) \),
(ii) \( \varphi \left( x_1^{(e)} \right) = A \left( e \left( a_1^{(e-2)m} \right), x_1^{(e)}, a_1^{(e-2)m} \right) \),
Hence, there is a sequence holds. Then, by the following sequence of equalities, we conclude that there is a sequence Using the two above equalities and the assumption that holds. Let \( A \) be a \((s - 2)m\)-group. Then, the equality: holds.

(ii) By (i) the following equality holds:

\[ e_1^m \cdot e_1^m = A\left( e_1^m, a_1^{(s-2)m}, e_1^m \right). \]

By definition of the \(1, (s - 1)m + 1\)-neutral operation of the \((sm, m)\)-group \((Q, A)\), the equality:

\[ e_1^m = A\left( e_1^{(s-2)m}, a_1^{(s-2)m}, e_1^m \right) \]

holds.

Using the two above equalities and the assumption that \((Q, A)\) is an \((sm, m)\)-quasigroup, we conclude that the equality

\[ e\left( a_1^{(s-2)m} \right) = e_1^m \]

holds. Then, by the following sequence of equalities, we conclude that there is a sequence \( a_1^{(s-2)m} \in Q \) such that \( \forall x_1^m \in Q^m \), the equality holds.

\[ A\left( e\left( a_1^{(s-2)m} \right), x_1^m, a_1^{(s-2)m} \right) = A\left( e_1^m, x_1^m, e_1^m \right) \]

\[ = e_1^m \cdot \phi\left( x_1^m \right) \cdot \phi\left( e_1^m \right) \cdot \phi\left( e_1^m \right) \cdot \phi\left( e_1^m \right) \cdot \phi\left( e_1^m \right) \cdot \phi\left( e_1^m \right) \cdot \phi\left( e_1^m \right) = e_1^m. \]

(iii) \( A\left( e\left( a_1^{(s-2)m} \right) \right)^2 = A\left( e_1^m \right)^2 = e_1^m \cdot \phi\left( e_1^m \right)^2 \cdot \phi\left( e_1^m \right)^2 \cdot \phi\left( e_1^m \right)^2 = e_1^m. \]

For the algebra \( \left( Q^m, \cdot, \phi, e_1^m \right) \) which have been described in Theorems 2.2 and 2.3, we say that it is associated to the \((sm, m)\)-group \((Q, A)\).
Example 2.4. Let \((Q, A)\) be a \((6, 2)\)-group defined in Example 1.2 and \(e : Q^2 \to Q^2\) be its neutral operation (see Example 1.4). For an arbitrary sequence \(a_1^m \in Q\) and \(\forall x_1^m, y_1^m \in Q^2\), we define the following operations:

\[ a_1 \ast y_1^m \overset{\text{def}}{=} A\left(x_1^m, a_1^m, y_1^m\right) = (x_1 \cdot \psi(a_1) \cdot y_1, x_2 \cdot \psi(a_2) \cdot y_2); \]

\[ \varphi(x_1^m) = A\left(e(a_1^m), x_1^m, a_1^m\right) = (\psi(a_1) \cdot x_1 \cdot a_1, \psi(a_2) \cdot x_2 \cdot a_2); \]

\[ c_1^m = A\left(e\left(x_1^m\right)\right)^3 = (\psi(a_1), \psi(a_2), \psi(a_1), \psi(a_2), \psi(a_1), \psi(a_2)) = (\psi(a_1) \cdot a_1 \cdot \psi(a_1), \psi(a_2) \cdot a_2 \cdot \psi(a_2)) = (a_1, a_2). \]

Since the afore mentioned operators satisfy the assumptions of Theorem 2.2, the following statements hold: \((Q^2, \ast)\) is a group, \(\varphi \in \text{Aut}(Q^2, \ast)\), \(\varphi(c_1^m) = c_1, \varphi^2(b_1^m) \ast c_1^m = c_1^m \ast b_1^m\), \(\forall b_1^m \in Q^2, A(x_1^m) = x_1^m \ast \varphi(x_1^m) \ast \varphi^2(x_1^m) \ast c_1^m, \forall x_1^m \in Q\).

Therefore, \((Q^2, (\ast, \varphi, c_1^m))\) is the algebra which is associated to the \((6, 2)\)-group \((Q, A)\).

In the following proposition we prove some interesting equalities which relate a \([1, (s - 1) m + 1]\)-neutral operation of an \((sm, m)\)-group \((Q, A)\) and an inverse operation of the binary group \((Q^m, \cdot)\), which also relate an inverse operation of an \((sm, m)\)-group \((Q, A)\) and an inverse operation of the binary group \((Q^m, \cdot)\).

Proposition 2.5. Let \((Q, A)\) be an \((sm, m)\)-group, \(s \geq 3, e : Q^{(e - 2)m} \to Q^m\) its \([1, (s - 1) m + 1]\)-neutral operation and \(-1 : Q^{(e - 1)m} \to Q^m\) its inverse operation. Also, let \((Q^m, [\cdot, \varphi, c_1^m])\) be an algebra associated to the \((sm, m)\)-group \((Q, A)\) and \(f : Q^m \to Q^m\) inverse operation of the group \((Q^m, \cdot)\). Then, \(\forall d_1^{(e-2)m} \in Q\) the following equality holds:

\[ (3) \quad e\left(d_1^{(e-2)m}\right) = f\left(\varphi\left(d_1^m\right) \ast q_2^2\left(d_2^{m+1}\right) \ast \cdots \ast q_{s-2}^2\left(d_{(s-3)m+1}^m\right) \ast c_1^m\right). \]

Proof. Firstly, we will prove that there is a sequence \(a_1^{(e-2)m} \in Q\) such that \(\forall b_1^{(e-2)m} \in Q\) the following equality holds:

\[ (4) \quad f\left(b_1^m\right) = a_1^{(e-2)m} \ast b_1^{(e-2)m}. \]

By Theorem 2.3 there is a sequence \(a_1^{(e-2)m} = \left(c_1^m, f\left(c_1^m\right)\right)\) such that \(e\left(a_1^{(e-2)m}\right) = e_1^m\). Thus,

\[ b_1^{(e-2)m} \ast a_1^{(e-2)m} = c_1^m \ast b_1^m \Rightarrow a_1^{(e-2)m} = f\left(b_1^m\right). \]

From the following sequence of equalities, we can conclude that \(\forall d_1^{(e-2)m} \in Q\) the statement of the proposition holds:

\[ x_1^m \ast f\left(e\left(d_1^{(e-2)m}\right)\right)^{\frac{3}{2}} = A\left(x_1^m, a_1^{(e-2)m}, f\left(e\left(d_1^{(e-2)m}\right)\right)\right) = A\left(x_1^m, a_1^{(e-2)m}, e\left(d_1^{(e-2)m}\right)\right)^{\frac{3}{2}} = A\left(x_1^m, d_1^{(e-2)m}, e\left(d_1^{(e-2)m}\right)\right)^{\frac{2}{2}} = A\left(x_1^m, d_1^{(e-2)m}, e\left(d_1^{(e-2)m}\right)\right) = A\left(x_1^m, d_1^{(e-2)m}, e\left(d_1^{(e-2)m}\right)\right) \Rightarrow \]

\[ x_1^m \ast \varphi\left(d_1^m\right) \ast q_2^2\left(d_2^{m+1}\right) \ast \cdots \ast q_{s-2}^2\left(d_{(s-3)m+1}^m\right) \ast c_1^m \overset{\text{def}}{=} d_1^{(e-2)m} \ast \varphi\left(d_1^m\right) \ast q_2^2\left(d_2^{m+1}\right) \ast \cdots \ast q_{s-2}^2\left(d_{(s-3)m+1}^m\right) \ast c_1^m. \]

3. A Central Operation in Terms of a Hosszú-Glusklin Algebra for an \((sm, m)\) – Group

Further aim of research considering \((n, m)\)-structures was to generalize the notion of a central element in a binary group, i.e. to define mapping whose properties for \((n, m) = (2, 1)\) would correspond to properties of the central element in a binary group. In \(n\)-group this notion was described by Ušan in 2001 [15].
Definition 3.1. [7] Let \((Q, A)\) be an \((n, m)\)-group, \(n \geq 2m\) and let \(\alpha\) be a map of the set \(Q^{n-2m}\) into the set \(Q^m\). We say that \(\alpha\) is a central operation of an \((n, m)\)-group \((Q, A)\) iff \(\forall a_1^{n-2m} \in Q, \forall b_1^{n-2m} \in Q\) and \(\forall x_1^m \in Q^m\) the following equality holds:

\[
A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = A\left(x_1^m, \alpha\left(b_1^{n-2m}\right), b_1^{n-2m}\right).
\]  

(5)

By Proposition 1.6 we see that a \([1, n - m + 1]\)-neutral operation is an example of the central operation of the \((n, m)\)-group.

By means of a series of propositions, in [7] it was proved that a mapping \(\alpha : Q^{n-2m} \rightarrow Q^m\) is a central operation of the \((n, m)\)-group \((Q, A)\) iff the equality

\[
A\left(x_1^m, a_1^{n-2m}, \alpha\left(a_1^{n-2m}\right)\right) = A\left(b_1^{n-2m}, \alpha\left(b_1^{n-2m}\right), x_1^m\right)
\]

holds.

Proposition 3.2. Let \((Q, A)\) be an \((n, m)\)-group, \(n \geq 2m\) and \(\alpha, \beta : Q^{n-2m} \rightarrow Q^m\) its central operations. Then, \(\forall a_1^{n-2m} \in Q\) the following equality holds:

\[
A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, \beta\left(a_1^{n-2m}\right)\right) = A\left(\beta\left(a_1^{n-2m}\right), a_1^{n-2m}, \alpha\left(a_1^{n-2m}\right)\right).
\]

Proof. For the central operation \(\alpha\) of the \((n, m)\)-group \((Q, A)\) the equality

\[
A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = A\left(x_1^m, b_1^{n-2m}, \alpha\left(b_1^{n-2m}\right)\right)
\]

holds \(\forall a_1^{n-2m}, b_1^{n-2m}, x_1^m \in Q\) (see [7]). The proof of the proposition follows directly from the previous equality if we put \(x_1^m\) instead of \(\beta(a_1^{n-2m})\) and \(b_1^{n-2m}\) instead of \(a_1^{n-2m}\). \(\Box\)

Theorem 3.3. Let \((Q, A)\) be an \((n, m)\)-group, \(n \geq 2m\) and \(\alpha\) its central operation. Then, there is a bijection \(\sigma_\alpha : Q^m \rightarrow Q^m\) such that \(\forall x_1^m \in Q^m\) and \(\forall a_1^{n-2m}, b_1^{n-2m} \in Q\) the following equalities hold:

(i) \(A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = \sigma_\alpha\left(x_1^m\right),\)

(ii) \(A\left(x_1^m, \alpha\left(b_1^{n-2m}\right), b_1^{n-2m}\right) = \sigma_\alpha\left(x_1^m\right)\).

Proof. Let \(k_1^{n-2m} \in Q\) be an arbitrary sequence. Then, we define:

\[
A\left(x_1^m, \alpha\left(k_1^{n-2m}\right), k_1^{n-2m}\right) \overset{def}{=} \sigma_\alpha\left(x_1^m\right).
\]

Firstly, we will prove that such a defined mapping \(\sigma_\alpha\) is a bijection. Since \((Q, A)\) is an \((n, m)\)-quasigroup, by Definition 1.1 (b), for \(i = 1\) holds: \(\forall a_i^n \in Q, \exists x_i^m \in Q^m\) such that holds \(A\left(x_i^m, a_i^{n-m}\right) = a_i^{n-m+1}\).

If we replace the sequence \(a_i^n\) in the above statement with \(A\left(\alpha\left(k_1^{n-2m}\right), k_1^{n-2m}, y_1^m\right)\), then the following statement holds: \(\forall y_1^m \in Q, \exists x_1^m \in Q^m\) such that holds:

\[
y_1^m = A\left(x_1^m, \alpha\left(k_1^{n-2m}\right), k_1^{n-2m}\right) = \sigma_\alpha\left(x_1^m\right).
\]

Then, equalities from the theorem hold by Definition 3.1. \(\Box\)

Definition 3.4. Let \((Q, A)\) be an \((n, m)\)-group, \(n \geq 2m\), \(\alpha : Q^{n-2m} \rightarrow Q^m\) its central operation and let \(\sigma_\alpha : Q^m \rightarrow Q^m\) be a bijection. We say that the bijection \(\sigma_\alpha\) is associated to the central operation \(\alpha\) iff \(\forall x_1^m \in Q^m\) and \(\forall a_1^{n-2m} \in Q\) holds:

\[
A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = \sigma_\alpha\left(x_1^m\right).
\]  

(6)
Example 3.5. Let \((Q, \cdot)\), \(Q = \{1, 2, 3, 4\}\) be the Klein group, let \(\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}\) be the permutation of the set \(Q\). In Example 1.2 we proved that \((Q, A)\) is a \((6, 2)\)-group, when \(A : Q^n \to Q^2\) is the mapping defined with \(A(x^n_1) = (x_1 \cdot \psi(x_3) \cdot x_5, x_2 \cdot \psi(x_4) \cdot x_6)\). Let \(\alpha : Q^2 \to Q^2\) be a mapping such that \(\forall x^n_1 \in Q^2\) the following equality holds:

\[
\alpha(x^n_1) \overset{\text{def}}{=} (2 \cdot \psi(x_1), 2 \cdot \psi(x_2)).
\]

We prove that this mapping is a central operation of the \((6, 2)\)-group \((Q, A)\). By Definition 3.1 we need to prove that \(\forall a^n_1 \in Q, \forall b^n_1 \in Q\) and \(\forall x^n_1 \in Q^2\) the equality (5) holds for \(m = 2, n = 6\).

\[
A(x^n_1, a^n_1, \alpha(a^n_1)) = A(b^n_2, \alpha(b^n_2), x^n_1) \Leftrightarrow \quad \Leftrightarrow A(x_1, x_2, a_1, a_2, 2 \cdot \psi(a_1), 2 \cdot \psi(a_2)) = A(b_1, b_2, 2 \cdot \psi(b_1), 2 \cdot \psi(b_2), x_1, x_2) \Leftrightarrow \quad \Leftrightarrow (x_1 \cdot \psi(a_1) \cdot 2 \cdot \psi(a_2), x_2 \cdot \psi(a_2) \cdot 2 \cdot \psi(a_2)) = (b_1 \cdot \psi(b_1) \cdot x_1, b_2 \cdot \psi(b_2) \cdot b_2 \cdot x_2) \Leftrightarrow \quad \Leftrightarrow (2 \cdot x_1, 2 \cdot x_2) = (2 \cdot x_1, 2 \cdot x_2)
\]

With \(\sigma_\alpha(x^n_1) \overset{\text{def}}{=} (2 \cdot x_1, 2 \cdot x_2)\) a bijection \(\sigma_\alpha : Q^2 \to Q^2\) associated to the central operation \(\alpha\) is defined. Let us prove equation (6):

\[
A(\alpha(a^n_1), \alpha(a^n_1), x^n_1) = A(2 \cdot \psi(a_1), 2 \cdot \psi(a_2), a_1, a_2, x_1, x_2) = (2 \cdot \psi(a_1) \cdot \psi(a_1), x_1, 2 \cdot \psi(a_2) \cdot \psi(a_2), x_2) = (2 \cdot x_1, 2 \cdot x_2) = \sigma_\alpha(x^n_1).
\]

Theorem 3.6. Let \((Q, A)\) be an \((n, m)\)-group, \(n \geq 2m\), \(\alpha : Q^{n-2m} \to Q^m\) its central operation and let the bijection \(\sigma_\alpha : Q^m \to Q^m\) be associated to the central operation \(\alpha\). Then, \(\forall x^n_1 \in Q\) the following equalities hold:

(i) \(\sigma_\alpha(A(x^n_1)) = A(\sigma_\alpha(x^n_1), a^n_1, \sigma_\alpha(x^n_1))\),

(ii) \(\sigma_\alpha(A(a^n_1)) = A(a^n_1, a^n_1, A(a^n_1))\),

(iii) \(\sigma_\alpha(A(x^n_1)) = A(x^n_1, a^n_1, \sigma_\alpha(x^n_1))\).

Proof. (i)

\[
\sigma_\alpha(A(x^n_1)) = A(\alpha(a^n_1), \alpha(a^n_1), A(x^n_1)) = A(A(\alpha(a^n_1), \alpha(a^n_1), A(x^n_1)) = A(\sigma_\alpha(x^n_1), a^n_1, \sigma_\alpha(x^n_1)).
\]

(ii)

\[
\sigma_\alpha(A(a^n_1)) = A(a^n_1, a^n_1, A(a^n_1)) = A(A(a^n_1, a^n_1, A(a^n_1)) = A(\sigma_\alpha(x^n_1), a^n_1, \sigma_\alpha(x^n_1)).
\]

(iii)

\[
\sigma_\alpha(A(x^n_1)) = A(x^n_1, a^n_1, \sigma_\alpha(x^n_1)) = A(x^n_1, \sigma_\alpha(x^n_1), a^n_1, \sigma_\alpha(x^n_1)).
\]

If we have two central operations of an \((n, m)\)-group and if a bijection is associated to each of them as defined in 3.4, then the bijections commute.

Theorem 3.7. Let \((Q, A)\) be an \((n, m)\)-group, \(n \geq 2m\) and \(\alpha, \beta : Q^{n-2m} \to Q^m\) its central operations. Also, let the bijection \(\sigma_\alpha\) be associated to the central operation \(\alpha\) and the bijection \(\sigma_\beta\) be associated to the central operation \(\beta\). Then \(\forall x^n_1 \in Q^m\) the following equality holds:

\[
\sigma_\alpha(\sigma_\beta(x^n_1)) = \sigma_\beta(\sigma_\alpha(x^n_1)).
\]
Proof. \( \sigma_{a}(x_{1}^{m}) = Q \left( a(a_{1}^{m-2m}), a_{1}^{m-2m}, \sigma_{\beta}(x_{1}^{m}) \right) = A \left( a(a_{1}^{m-2m}), a_{1}^{m-2m}, A \beta(a_{1}^{m-2m}, a_{1}^{m-2m}, x_{1}^{m}) \right) = A \left( a(a_{1}^{m-2m}), a_{1}^{m-2m}, \sigma_{\beta}(x_{1}^{m}) \right). \)

Under particular condition, a bijection associated to the central operation \( \alpha \) is an involution which has been proved in the following theorem.

**Theorem 3.8.** Let \((Q, A)\) be an \((n,m)\)-group, \(n \geq 2m, \alpha : Q^{n-2m} \rightarrow Q^{m}\) its central operation, let \(\sigma_{a}\) be the bijection associated to the central operation \(\alpha\) and let \({}^{-1} : Q^{m} \rightarrow Q^{m}\) be an inverse operation of the \((n,m)\)-group \((Q, A)\). If \(\forall a_{1}^{n-2m} \in Q\) the equality

\[
\left( a_{1}^{n-2m}, \alpha(a_{1}^{n-2m}) \right)^{-1} = \alpha(a_{1}^{n-2m})
\]

holds, then \(\forall x_{1}^{m} \in Q^{m}\) the following equality holds:

\[
\sigma_{a}(x_{1}^{m}) = x_{1}^{m}.
\]

Proof. \(\forall a_{1}^{n-2m} \in Q\) the following sequence of equivalences holds:

\[
\begin{aligned}
\sigma_{a}(x_{1}^{m}) &\equiv A \left( a(a_{1}^{m-2m}), a_{1}^{m-2m}, \sigma_{\alpha}(x_{1}^{m}) \right) = A \left( \alpha(a_{1}^{m-2m}), a_{1}^{m-2m}, A \left( \alpha(a_{1}^{m-2m}), a_{1}^{m-2m}, x_{1}^{m} \right) \right) \\
&\equiv A \left( a_{1}^{m-2m}, \alpha(a_{1}^{m-2m}) \right)^{-1}, a_{1}^{m-2m}, A \left( \alpha(a_{1}^{m-2m}), a_{1}^{m-2m}, x_{1}^{m} \right) \equiv x_{1}^{m}.
\end{aligned}
\]

**Theorem 3.9.** Let \((Q, A)\) be an \((sm,m)\)-group, \(s \geq 3\) and \((Q^{m}, \cdot, \varphi, c_{1}^{m})\) an algebra associated to the \((sm,m)\)-group \((Q, A)\). Also, let \(f\) be an inverse operation in a group \((Q^{m}, \cdot)\), \(\alpha : Q^{(s-2)m} \rightarrow Q^{m}\) central operation of the \((sm,m)\)-group \((Q, A)\) and \(\sigma_{a}\) a bijection associated to the central operation \(\alpha\). Then, \(\exists y_{1}^{m} \in Q^{m}\) such that for every sequence \(a_{1}^{(s-2)m} \in Q\) and \(\forall x_{1}^{m} \in Q^{m}\) the following equivalences hold:

\[
\begin{align*}
(i)\quad a_{1}^{(s-2)m} &\equiv y_{1}^{m} \cdot f(\varphi(a_{1}^{m}) \cdot \varphi^{2}(a_{2}^{m}, \ldots, \varphi^{s-2}(a_{(s-2)m}) ) \\
(ii)\quad \sigma_{a}(x_{1}^{m}) &\equiv \left( y_{1}^{m} \cdot c_{1}^{m} \right) \cdot x_{1}^{m} \\
(iii)\quad \varphi(y_{1}^{m}) &\equiv y_{1}^{m} \\
(iv)\quad \varphi^{s-1}(c_{1}^{m}) \cdot x_{1}^{m} &\equiv y_{1}^{m} \cdot c_{1}^{m}.
\end{align*}
\]

Proof. (i) Let us prove that by assumptions of the theorem \(\exists y_{1}^{m} \in Q^{m}\) such that for every sequence \(a_{1}^{(s-2)m} \in Q\) the following equality holds:

\[
\alpha(a_{1}^{(s-2)m}) \cdot \varphi(a_{1}^{m}) \cdot \varphi^{2}(a_{2}^{m}, \ldots, \varphi^{s-2}(a_{(s-3)m+1}) = y_{1}^{m}.
\]

Let \(k_{1}^{(s-2)m} \in Q\) be an arbitrary fixed sequence. Then by the definition of the central operation of an \((sm,m)\)-group, \(\forall k_{1}^{(s-2)m} \in Q\) holds:

\[
A \left( \alpha(a_{1}^{(s-2)m}), k_{1}^{(s-2)m}, x_{1}^{m} \right) = A \left( x_{1}^{m}, \alpha(k_{1}^{(s-2)m}) \right) = A \left( \alpha(k_{1}^{(s-2)m}), k_{1}^{(s-2)m}, x_{1}^{m} \right).
\]

Moreover, by Theorem 2.2 \(\forall x_{1}^{m} \in Q^{m}, \forall a_{1}^{(s-2)m} \in Q\) the following sequence of equivalences holds:

\[
A \left( a_{1}^{(s-2)m}, x_{1}^{m} \right) = A \left( a_{1}^{(s-2)m}, c_{1}^{m} \right) = A \left( k_{1}^{(s-2)m}, c_{1}^{m} \right) \equiv
\]

\[
A \left( k_{1}^{(s-2)m}, x_{1}^{m} \right) = A \left( k_{1}^{(s-2)m}, c_{1}^{m} \right) \equiv
\]
From the above equalities it follows:

\[ a \left( k^{(e-2)m}_1 \right) \cdot \varphi \left( a^m_1 \right) \cdot \varphi^2 \left( a^{2m}_{m+1} \right) \cdots \varphi^{s-2} \left( k^{(e-2)m}_{(3s-1)m+1} \right) = a \left( k^{(e-2)m}_1 \right) \cdot \varphi \left( k^m_1 \right) \cdot \varphi^2 \left( k^{2m}_{m+1} \right) \cdots \varphi^{s-2} \left( k^{(e-2)m}_{(3s-1)m+1} \right). \]

Because \( k^{(e-2)m}_1 \in Q \) is a fixed sequence, we can denote:

\[ a \left( k^{(e-2)m}_1 \right) \cdot \varphi \left( k^m_1 \right) \cdot \varphi^2 \left( k^{2m}_{m+1} \right) \cdots \varphi^{s-2} \left( k^{(e-2)m}_{(3s-1)m+1} \right) = y^m_1, \]

which, due to the last equality from the above sequence of equalities yields the equality:

\[ a \left( a^{(e-2)m}_1 \right) \cdot \varphi \left( a^m_1 \right) \cdot \varphi^2 \left( a^{2m}_{m+1} \right) \cdots \varphi^{s-2} \left( a^{(e-2)m}_{(3s-1)m+1} \right) = y^m_1. \]

Because \( f \) is an inverse operation in the group \( (Q^m, \cdot) \), the last equality is equivalent to:

\[ a \left( a^{(e-2)m}_1 \right) = y^m_1 \cdot f \left( \varphi \left( a^m_1 \right) \cdot \varphi^2 \left( a^{2m}_{m+1} \right) \cdots \varphi^{s-2} \left( a^{(e-2)m}_{(3s-1)m+1} \right) \right). \]

(ii) \( \exists y^m_1 \in Q^m \) such that \( \forall x^m_1 \in Q^m \) \( \forall a^{(e-2)m}_1 \in Q \) the following sequence of equalities holds:

\[
\begin{align*}
\sigma_{\alpha} \left( x^m_1 \right) \overset{(6)}{=} A \left( a \left( a^{(e-2)m}_1 \right) \cdot a^{(e-2)m}_1 \cdot x^m_1 \right) & \overset{2}{=} a \left( a^{(e-2)m}_1 \right) \cdot \varphi \left( a^{m}_1 \right) \cdot \varphi^2 \left( a^{2m}_{m+1} \right) \cdots \varphi^{s-2} \left( a^{(e-2)m}_{(3s-1)m+1} \right) \cdot \varphi^{s-1} \left( x^m_1 \right), c^m_1 \overset{2}{=} \\
&= a \left( a^{(e-2)m}_1 \right) \cdot \varphi \left( a^{m}_1 \right) \cdot \varphi^2 \left( a^{2m}_{m+1} \right) \cdots \varphi^{s-2} \left( a^{(e-2)m}_{(3s-1)m+1} \right) \cdot c^m_1 \cdot x^m_1 = y^m_1 \cdot c^m_1 \cdot x^m_1.
\end{align*}
\]

(iii) \( \forall a^{(e-2)m}_1, k^{(e-2)m}_1 \in Q \) and \( \forall x^m_1 \in Q^m \), by definition of a central operation and Theorem 2.2, the following sequence of equivalences holds:

\[
A \left( a \left( a^{(e-2)m}_1 \right), a^{(e-2)m}_1, x^m_1 \right) = A \left( x^m_1, a \left( k^{(e-2)m}_1 \right), k^{(e-2)m}_1 \right) \Leftrightarrow \\
a \left( a^{(e-2)m}_1 \right) \cdot \varphi \left( a^m_1 \right) \cdot \varphi^2 \left( a^{2m}_{m+1} \right) \cdots \varphi^{s-2} \left( a^{(e-2)m}_{(3s-1)m+1} \right) \cdot c^m_1 \cdot x^m_1 = x^m_1 \cdot \varphi \left( a \left( k^{(e-2)m}_1 \right) \right) \cdot \varphi^2 \left( k^m_1 \right) \cdots \varphi^{s-1} \left( k^{(e-2)m}_{(3s-1)m+1} \right) \cdot c^m_1 \Leftrightarrow \\
a \left( a^{(e-2)m}_1 \right) \cdot \varphi \left( a^m_1 \right) \cdot \varphi^2 \left( a^{2m}_{m+1} \right) \cdots \varphi^{s-2} \left( a^{(e-2)m}_{(3s-1)m+1} \right) \cdot c^m_1 \cdot x^m_1 = x^m_1 \cdot \varphi \left( a \left( k^{(e-2)m}_1 \right) \right) \cdot \varphi \left( k^m_1 \right) \cdots \varphi^{s-2} \left( k^{(e-2)m}_{(3s-1)m+1} \right) \cdot c^m_1.
\]

By (i), the last equality is equivalent to:

\[ y^m_1 \cdot c^m_1 \cdot x^m_1 = x^m_1 \cdot \varphi \left( y^m_1 \right) \cdot c^m_1. \]

Since this equality holds \( \forall x^m_1 \in Q^m \), thus it holds for \( x^m_1 = e^m_1 \) where \( e^m_1 \) is a neutral element of a group \( (Q^m, \cdot) \). Accordingly, it follows:

\[ y^m_1 \cdot c^m_1 = \varphi \left( y^m_1 \right) \cdot c^m_1, \]

that is

\[ y^m_1 = \varphi \left( y^m_1 \right). \]

(iv) In the proof (iii), we have proved that \( \exists y^m_1 \in Q^m \) such that \( \forall x^m_1 \in Q^m \) the following equalities hold:

\[ y^m_1 \cdot c^m_1 \cdot x^m_1 = x^m_1 \cdot \varphi \left( y^m_1 \right) \cdot c^m_1 \]

and

\[ y^m_1 = \varphi \left( y^m_1 \right). \]

From the above equalities it follows:

\[ \left( y^m_1 \cdot c^m_1 \right) \cdot x^m_1 = x^m_1 \cdot \left( y^m_1 \cdot c^m_1 \right). \]
**Theorem 3.10.** Let \((Q, A)\) be an \((sm, m)\)-group, \(s \geq 3\) and \((Q^m, \cdot, \varphi, e^m)\) an algebra which is associated to the \((sm, m)\)-group \((Q, A)\). Also, let \(f\) be an inverse operation in the group \((Q^m, \cdot)\) and \(e^m \in Q^m\) its neutral element. Let \(\alpha\) be a central operation of the \((sm, m)\)-group \((Q, A)\), \(\sigma\), a bijection associated to the central operation \(\alpha\) and let \(^{-1} : Q^{(s-1)m} \to Q^m\) be an inverse operation of the \((sm, m)\)-group \((Q, A)\). If for every sequence \(a_1^{(s-2)m} \in Q\) the equality
\[
\left( a_1^{(s-2)m}, \alpha \left( a_1^{(s-2)m} \right) \right)^{-1} = \alpha \left( a_1^{(s-2)m} \right)
\]
holds, then \(\forall y^m \in Q^m\) such that the following equality holds:
\[
\left( y_1^m \cdot c_1^m \right) \cdot \left( y_1^m \cdot c_1^m \right) = c_1^m.
\]

**Proof.** \(\forall x^m, a_1^{(s-2)m} \in Q, \forall y^m \in Q^m\) such that the following sequence of equalities holds:
\[
x_1^m = \sigma_\alpha \left( x_1^m \right)^{3} = \sigma_\alpha \left( y_1^m \cdot c_1^m \right) \cdot x_1^m = \left( y_1^m \cdot c_1^m \right) \cdot \left( y_1^m \cdot c_1^m \right) \cdot x_1^m.
\]
Since \((Q^m, \cdot, \varphi)\) is a group, from the above sequence of equalities it follows:
\[
\left( y_1^m \cdot c_1^m \right) \cdot \left( y_1^m \cdot c_1^m \right) = c_1^m. \quad \square
\]

**Theorem 3.11.** Let \((Q, A)\) be an \((sm, m)\)-group, \(s \geq 3\) and \((Q^m, \cdot, \varphi, e^m)\) the algebra associated to the \((sm, m)\)-group \((Q, A)\). Also, let \(f\) be an inverse operation of the group \((Q^m, \cdot)\) and \(e^m \in Q^m\) its neutral element. Let \(y_1^m \in Q^m\) be a fixed sequence such that \(\forall x^m \in Q^m\) the following equalities hold:
\[
\begin{align*}
(a) \quad & \left( y_1^m \cdot e_1^m \right) \cdot x_1^m = x_1^m \cdot \left( y_1^m \cdot e_1^m \right), \\
(b) \quad & \varphi \left( y_1^m \right) = y_1^m, \\
(c) \quad & \left( y_1^m \cdot e_1^m \right) \cdot \left( y_1^m \cdot e_1^m \right) = e_1^m.
\end{align*}
\]

We define the mapping \(\alpha : Q^{(s-2)m} \to Q^m\) such that \(\forall a_1^{(s-2)m} \in Q\) holds:
\[
(d) \quad \alpha \left( a_1^{(s-2)m} \right) \overset{\text{def}}{=} y_1^m \cdot f \left( \varphi \left( a_1^m \right) \cdot \varphi^2 \left( a_{2m+1}^m \right) \cdot \cdots \varphi^{(s-2)} \left( a_{(s-3)m+1}^m \right) \right).
\]

Then, the following statements hold:

(i) \(\alpha\) is a central operation of the \((sm, m)\)-group \((Q, A)\);
(ii) for every sequence \(a_1^{(s-2)m} \in Q\) the equality
\[
\left( a_1^{(s-2)m}, \alpha \left( a_1^{(s-2)m} \right) \right)^{-1} = \alpha \left( a_1^{(s-2)m} \right)
\]
holds, where \(-1 : Q^{(s-1)m} \to Q^m\) is an inverse operation of the \((sm, m)\)-group \((Q, A)\).

**Proof.** \(\forall a_1^{(s-2)m} \in Q \cap \forall x_1^m \in Q^m\) the following sequence of equalities holds:
\[
A \left( a_1^{(s-2)m}, a_1^{(s-2)m}, x_1^m \right) = \alpha \left( a_1^{(s-2)m} \right) \cdot \varphi \left( a_1^m \right) \cdot \varphi^2 \left( a_{2m+1}^m \right) \cdot \cdots \cdot \varphi^{(s-2)} \left( a_{(s-3)m+1}^m \right) \cdot c_1^m \cdot x_1^m =
\]
\[
= \left( y_1^m \cdot e_1^m \right) \cdot x_1^m.
\]
Further on, \(\forall b_1^{(s-2)m} \in Q \cap \forall x_1^m \in Q^m\) the following sequence of equalities holds:
\[
A \left( x_1^m, \alpha \left( b_1^{(s-2)m} \right), x_1^m \right) = x_1^m \cdot \varphi \left( \alpha \left( b_1^{(s-2)m} \right) \right) \cdot \varphi^2 \left( b_1^m \right) \cdot \cdots \cdot \varphi^{(s-2)} \left( b_{(s-3)m+1}^m \right) \cdot c_1^m =
\]
\[
= x_1^m \cdot \varphi \left( b_1^{(s-2)m} \right) \cdot \left( x_1^m \cdot \varphi \left( b_1^{(s-2)m} \right) \right) \cdot \varphi^2 \left( b_{2m+1}^m \right) \cdot \cdots \cdot \varphi^{(s-2)} \left( b_{(s-3)m+1}^m \right) \cdot c_1^m \overset{\text{def}}{=}
\]
\[
= x^n_m \cdot \phi \left( y^n_m \cdot f \left( \phi \left( b^n_m \right) \cdot \phi^2 \left( b^{n+1}_m \right) \cdots \phi^s_{n-2} \left( b^{(s-3)+1}_m \right) \right) \cdot \phi \left( b^n_m \right) \cdot \phi^2 \left( b^{n+1}_m \right) \cdots \phi^s_{n-2} \left( b^{(s-3)+1}_m \right) \right) \cdot c^n_1 = \\
= x^n_m \cdot \phi \left( y^n_m \right) \cdot c^n_1 = x^n_m \cdot \phi \left( y^n_m \right) \cdot c^n_1 = \phi \left( y^n_m \cdot c^n_1 \right) \cdot x^n_1.
\]

From the previous two sequences of equalities, we conclude that the statement (i) of the theorem holds.

(ii) By Proposition 1.5, \( \forall a^{(e-2)m}_1 \in Q \) and \( \forall x^n_m \in Q^m \) the equality

\[
A \left( \left( a^{(e-2)m}_1 \cdot x^n_m, x^n_m \right)^{-1}, a^{(e-2)m}_1 \cdot x^n_m \right) = \phi \left( a^{(e-2)m}_1 \right)
\]

holds, where \( \phi : Q^{(e-2)m} \rightarrow Q^m \) is a \([1,(s-1)m+1]\)-neutral operation of the \((sm,m)\)-group \((Q,A)\).

For \( x^n_m = \alpha \left( a^{(e-2)m}_1 \right) \), by Theorem 2.2 and Proposition 2.5, the above equality is:

\[
\left( \left( a^{(e-2)m}_1 \cdot x^n_m, x^n_m \right)^{-1}, a^{(e-2)m}_1 \cdot x^n_m \right) = \phi \left( x^n_m \right) \cdot \phi^2 \left( x^{2m}_m \right) \cdots \phi^s_{n-2} \left( a^{(e-3)+1}_m \right) \cdot c^n_1 \cdot \alpha \left( a^{(e-2)m}_1 \right) = \\
\phi \left( x^n_m \right) \cdot \phi^2 \left( x^{2m}_m \right) \cdots \phi^s_{n-2} \left( a^{(e-3)+1}_m \right) \cdot c^n_1.
\]

which implies the following sequence of equalities:

\[
\left( \left( a^{(e-2)m}_1 \cdot x^n_m, x^n_m \right)^{-1}, a^{(e-2)m}_1 \cdot x^n_m \right) = \\
\phi \left( x^n_m \right) \cdot \phi^2 \left( x^{2m}_m \right) \cdots \phi^s_{n-2} \left( a^{(e-3)+1}_m \right) \cdot c^n_1.
\]

References