Some Classes of $p$-valent Analytic Functions Associated with Hypergeometric Functions

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Abstract. We define a linear operator on the class $A(p)$ of $p$-valent analytic functions in the open unit disc involving Gauss hypergeometric functions and introduce certain new subclasses of $A(p)$ using this operator. Some inclusion results, a radius problem and several other interesting properties of these classes are studied.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{m=1}^{\infty} a_m z^{m+p}, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic and $p$-valent in the open unit disc $E = \{z : |z| < 1\}$. For

$$f(z) = \sum_{m=0}^{\infty} a_m z^m, \quad g(z) = \sum_{m=0}^{\infty} b_m z^m,$$

the Hadamard product (or convolution) is defined by

$$(f \star g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m$$

For $a \in \mathbb{R}$, $c \in \mathbb{R}\setminus\mathbb{Z}^+$, where $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, define $L_p(a, c) : A(p) \to A(p)$ as

$$L_p(a, c)f(z) = \phi_p(a, c; z) \star f(z), \quad z \in E, f \in A(p),$$

where

$$\phi_p(a, c; z) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} z^{m+p}, \quad z \in E.$$
and \((\lambda)\) denotes the Pochhammer symbol (or the shifted factorial) defined (for \(x, \nu \in \mathbb{C}\) and in terms of the Gamma function) by
\[
(x)_\nu = \frac{\Gamma(x + \nu)}{\Gamma(x)} = \begin{cases} 
1, & (\nu = 0; \ x \in \mathbb{C}\setminus\{0\}), \\
x(x + 1)\cdots(x + n - 1), & (\nu = n \in \mathbb{N}, x \in \mathbb{C}).
\end{cases}
\]
The operator \(L_p(a, c)\) was introduced by Saitoh [18]. This operator is an extension of Carlson-Shaffer operator \(L_1(a, c)\), see [2].

For real or complex numbers \(a, b, c\) other than \(-1, -2, \ldots\) the hypergeometric series is defined by
\[
\binom{z}{2} F_1(a, b, c; z) = 1 + \frac{ab}{c!} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots
\]
We note that series (1.1) converges absolutely for all \(z \in E\) so that it represents an analytic function in \(E\). Also
\[
\phi_p(a, c; z)^p = \binom{z}{2} F_1(1, a; c; z).
\]
We now introduce a function \((\binom{z}{2} F_1(a, b, c; z))^{(-1)}\) given by
\[
(\binom{z}{2} F_1(a, b, c; z))^{(-1)} \ast (\binom{z}{2} F_1(a, b, c; z))^{(-1)} = \frac{z^p}{(1-z)^{1+p}}.
\]
and obtain the following linear operator
\[
I_\lambda(a, b, c)f(z) = (\binom{z}{2} F_1(a, b, c; z))^{(-1)} \ast f(z),
\]
where \(a, b, c\) are real other than \(-1, -2, \ldots, \lambda > -p, \ z \in E\) and \(f \in \mathcal{A}(p)\).

In particular, with \(b = 1, p = 1, \ I_\lambda\) was studied in [3] and for \(a = n + p, b = c, \lambda = 1, p = 1, \) see [15]. With some computation, we note that
\[
I_\lambda(a, b, c)f(z) = z^p + \sum_{m=1}^{\infty} \frac{(\lambda)p_m}{(a)_{m}(b)_{m}} \frac{z^{m+p}}{m!}.
\]
From (1.2), it can easily be verified that
\[
z (I_\lambda(a+1, b, c)f(z))' = aI_\lambda(a, b, c)f(z) - (a - p)I_\lambda(a+1, b, c)f(z) \quad (1.4)
\]
\[
z (I_\lambda(a, b, c)f(z))' = (\lambda + p)I_\lambda(a+1, b, c)f(z) - \lambda I_\lambda(a, b, c)f(z). \quad (1.5)
\]
Let \(P_\lambda(\beta)\) be the class of functions \(p(z)\) analytic in the unit disc \(E\) satisfying the properties \(p(0) = 1\) and, for \(z = re^{\theta}, \ k \geq 2\)
\[
\int_{0}^{2\pi} \left|Re \frac{p(z) - \beta}{(1 - \beta)}\right| d\theta \leq k\pi, \quad (0 \leq \beta < 1).
\]
(1.6)
For \(\beta = 0\), we obtain the class \(P_k\) defined by Pinchuk [16] and for \(k = 2, \beta = 0\), we have the class \(P\) of functions with positive real positive real part greater than \(\beta\).

From (1.6), we can easily verify that \(p \in P_k(\beta)\) if and only if there exists \(p_1, p_2 \in P(\beta)\) such that
\[
p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad z \in E.
\]
(1.7)
We now define the following,
Let \( f \in A(p), z \in E \). Then \( f \in R_k^{I}(a, b, c, \beta, p) \) if and only if, for \( k \geq 2, \quad 0 \leq \beta < 1, \)

\[
\left\{ \frac{z \left( z^{-1}R_1(a, b, c)f \right)'}{z^{-1}R_1(a, b, c)f} \right\} \in P_k(\beta), \quad z \in E.
\]

In particular, \( R_k^{I}(a, \lambda + 1, a, \beta, 1) = R_k(\beta) \) is the class of functions of bounded radius rotation of order \( \beta \), see [8, 12]. Also \( R_k^{I}(a, \lambda + 1, a, \beta, 1) \equiv S(\beta) \), the class of starlike univalent functions of order \( \beta \).

We can define the class \( V_k^{I}(a, b, c, \beta, p) \) as follows.

**Definition 1.2.** Let \( f \in A(p) \). Then, for \( z \in E, \)

\[
f \in V_k^{I}(a, b, c, \beta, p) \iff \frac{zf'(z)}{p} \in R_k^{I}(a, b, c, \beta, p).
\]

We note that \( V_k^{I}(a, \lambda + 1, a, 0, 1) = V_k \) is the class of functions with bounded boundary rotation, and \( V_k^{I}(a, \lambda + 1, a, \beta, 1) = C(\beta) \), the class of convex univalent functions of order \( \beta \).

**Definition 1.3.** Let \( f \in A(p) \). Then, for \( k \geq 0, \alpha \geq 0, 0 \leq \beta < 1, z \in E, \quad f \in M_k^{I}(a, b, c, \beta, p, \alpha) \) if and only if

\[
\left\{ \left( 1 - \alpha \right) z \left( z^{-1}M_1(a, b, c)f \right)' + \alpha \left( z \left( z^{-1}M_1(a, b, c)f \right)' \right)' \right\} \in P_k(\beta).
\]

We note that, for \( \alpha = 1 \), we have the class \( V_k^{I}(a, b, c, \beta, p) \) and \( \alpha = 0 \) gives us the class \( R_k^{I}(a, b, c, \beta, p) \).

Also \( M_k^{I}(a, \lambda + 1, a, 0, 1) \equiv M(\alpha) \) is the class of \( \alpha \)-starlike univalent functions and \( M_k^{I}(a, \lambda + 1, a, 0, 1, \alpha) \) consists entirely of functions of bounded Mocanu variation, see [7].

In the above definitions, we obtain several known subclasses of analytic and multivalent functions by choosing the suitable values of the parameters \( k, \lambda, a, b, c \) and \( \alpha \). We would like to emphasize that a significant and important meromorphic extension of the linear operator \( I_{k}(a, b, c) \), popularly known as the Liu-Srivastava operator has been introduced and studied in [9]. For related work, see [5,6] for the analogous Dziok-Srivastava operator.

In the recent years, several interesting subclasses of analytic functions have been introduced and investigated, see [1,4,12,13,14,15,21,22].

For the sake of simplicity, we shall write \( I_{k}(a) \) in place of \( I_{k}(a, b, c) \) unless required otherwise.

## 2. Preliminary Results

**Lemma 2.1 (17).** Let \( p(z) \) be an analytic function in \( E \) with \( p(0) = 1 \) and \( \text{Re}(p(z)) > 0, \)

\( z \in E. \) Then, for \( s > 0 \) and \( \nu \neq -1 \) (complex),

\[
\text{Re} \left\{ p(z) + \frac{szp'(z)}{p(z) + \nu} \right\} > 0, \quad \text{for} \quad |z| < r_0,
\]

where \( r_0 \) is given by

\[
r_0 = \frac{|\nu + 1|}{\sqrt{A + (A^2 - |\nu^2 - 1|^2)^2}}, \quad A = 2(s + 1)^2 + |\nu|^2 - 1.
\]

This result is best possible.
Lemma 2.2. Let $\beta_0 > 0, \beta_0 + \gamma > 0$ and $\alpha_1 \in [\alpha_0, 1),$ where $\alpha_0 = \max \left\{ \frac{\ln(1 - n)}{\beta_0}, \frac{1}{n} \right\}$, $n \in \mathbb{N}$. If
\[
\left\{ p(z) + \frac{nzp'(z)}{\beta_0 p(z) + \gamma} \right\} \in P(\alpha_1),
\]
then
\[
\text{Re}(p(z)) \geq \left[ \left( \frac{\beta_0 + \gamma}{\beta_0} \right) \frac{\beta_0(1 + \alpha_1)}{n} - \frac{\gamma}{\beta_0} \right],
\]
and the bound in (2.2) is sharp, extremal function being
\[
p_n(z) = \frac{1}{\beta_0 g_n(z)} - \frac{\gamma}{\beta_0},
\]
where
\[
g_n(z) = \frac{1}{n} \int_0^1 \left[ 1 - z \right]^{\beta_0(1 + \alpha_1)} t^{n - 1} dt
\]
\[
\quad = \left[ \left( \frac{2}{n} \right)^{\beta_0(1 + \alpha_1)} + 1, \beta_0 + \gamma + n, \frac{z}{1 - z} \right] \left( \frac{1}{\beta_0 + \gamma} \right).
\]
The above Lemma is a slightly modified version of Theorem 3.3e in [11, p113].

3. Main Results

Theorem 3.1. Let $a > 0, \lambda \geq 0$ and $\beta \in [\gamma_0, 1)$ with $\gamma_0 = \max \left\{ \frac{1-a}{2}, 0 \right\}$. Then

(i). $M_{k+1}^\lambda(a, b, c, \beta, p, \alpha) \subset R_{k+1}^\lambda(a, b, c, \beta_1, p)$

(ii). $R_{k+1}^\lambda(a, b, c, \beta_1, p) \subset R_{k+1}^\lambda(a, b, c, \beta_2, p)$

(iii). $R_{k+1}^\lambda(a, b, c, \beta_2, p) \subset R_{k+1}^\lambda(a + 1, b, c, \beta_3, p)$,

where
\[
\beta_1 = \frac{1}{\beta_0 \left( \frac{1}{2} \right) \left( 1 - \beta_1, 1, 1 + \frac{1}{2}, \frac{1}{2} \right)}
\]
\[
\beta_2 = \frac{1 + \lambda}{\beta_0 \left( \frac{1}{2} \right) \left( 1 - \beta_1, 1, 2 + \lambda, \frac{1}{2} \right)}
\]
\[
\beta_3 = \frac{1 + a}{\beta_0 \left( \frac{1}{2} \right) \left( 1 - \beta_2, 1, 2 + a, \frac{1}{2} \right)}
\]
The values $\beta_i, \quad i = 1, 2, 3$ are best possible.

Proof. (i). set
\[
\frac{z \left( z^{1-\gamma_0} I_{k+1}(a) f(z) \right)'}{z^{1-\gamma_0} I_{k+1}(a) f(z)} = H(z) = \left( \frac{k}{4} + \frac{1}{2} \right) H_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) H_2(z).
\]
We note that $H$ is analytic in $E$ with $H(0) = 1$ and $H(z) \neq 0$ for all $z \in E.$
Since $f \in M_{k+1}^\lambda(a, b, c, \beta, p, \alpha)$, we have
\[
\left[ H(z) + \frac{a z H'(z)}{H(z)} \right] \in P_k(\beta), \quad z \in E.
\]
Using (3.4) with convolution techniques, it follows from (3.5) that
\[
\frac{\alpha zH_i'(z)}{h_i(z)} \in P(\beta), \quad z \in E, \quad i = 1, 2.
\]

We now use Lemma 2.2 with \(\gamma = 0, \beta_0 = \frac{1}{a}, a_1 = \beta, n = 1\) to have
\[
\text{Re}H_i(z) \geq \frac{1}{zF_1(\frac{2}{5}(1 - \beta), 1, 1 + \frac{1}{a_1/2})}, \quad i = 1, 2.
\]
and this bound is sharp. Consequently \(H \in P_1(\beta)\) and \(f \in R^{\lambda+1}(a, b, c, \beta_1, p)\). This proves (i).

(ii). We now prove (ii). Let \(f \in R^{\lambda+1}(a, b, c, \beta_1, p)\) and set
\[
\frac{z(z^{-1-1}I_{\lambda_i}(a)f(z))}{z^{-1-1}I_{\lambda_i}(a)f(z)} = h(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z),
\]
where \(h(z)\) is analytic in \(E\) with \(h(0) = 1\) and \(h(z) \neq 0\) for all \(z \in E\). From (1.5) and (3.6), we have
\[
\frac{z(z^{-1-1}I_{\lambda_1}(a)f(z))}{z^{-1-1}I_{\lambda_1}(a)f(z)} = \left\{h(z) + \frac{zh'(z)}{h(z) + \lambda}\right\} \in P_1(\beta_1),
\]
where \(\beta_1\) is given by (3.1).

Define
\[
\phi_1(z) = \frac{1}{\lambda + 1 + \frac{1}{1 - z} - \frac{z^\lambda}{\lambda + 1 + \frac{1}{(1 - z)^2}}.}
\]
Then
\[
\left(h(z) + \frac{\phi_1(z)}{z^\lambda}\right) = \frac{zh'(z)}{h(z) + \lambda} = \left(\frac{k}{4} + \frac{1}{2}\right)\left[h_1(z) + \frac{zh'(z)}{h_1(z) + \lambda}\right] - \left(\frac{k}{4} - \frac{1}{2}\right)\left[h_2(z) + \frac{zh'(z)}{h_2(z) + \lambda}\right].
\]
Therefore it follows that
\[
\left\{h_i(z) + \frac{zh'(z)}{h_i(z) + \lambda}\right\} \in P(\beta_1), \quad i = 1, 2, \quad z \in E.
\]

Now using Lemma 2.2. with \(n = 1, \beta_0 = 1, \gamma = \lambda, \) we have, for \(i = 1, 2\) \(h_i \in P(\beta_2)\), where the exact value of \(\beta_2\) is given by (3.2). This proves part (ii) of Theorem 3.1. Te last part (iii) of this inclusion result can easily be proved by using (1.4) and similar technique used above. \(\square\)

Special Cases

(i). With \(a = 1\), \(f \in M_k^{\lambda+1}(a, b, c, \beta, p, 1) \equiv V_k^{\lambda+1}(a, b, c, \beta, p)\) and this implies \(f \in R_k^{\lambda+1}(a, b, c, \beta_1, p)\), where \(\beta_1 \in [0, 1)\), and

\[
\beta_1 = \begin{cases} 
\frac{2^{p-1}}{2 - 2^{p-1}}, & \text{if } \beta \neq \frac{1}{2} \\
\frac{1}{2^{p-1}}, & \text{if } \beta = \frac{1}{2}
\end{cases}
\]  
(3.7)
functions. We shall now consider the converse case of Theorem 3.1. 

Let $f$, with

$$q = \frac{z(1-z^\beta p)}{z^{1-\beta} p}.$$ 

where $q$ is analytic in $E$ with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in E$. With some computation, we obtain from (3.8),

$$\frac{z(1-z^\beta p)}{z^{1-\beta} p} = \left[q(z) + \frac{zq'(z)}{q(z) + \mu}\right], \quad z \in E.$$ 

Using similar techniques as in the proof of previous Theorems, we see that $q \in P_1(\delta)$, where $\delta$ is given by (3.9).

Extremal function to show the sharpness is

$$q_1(z) = \frac{1}{\beta_1(z)} - \mu = \int_0^1 \left[1 + \frac{z}{1-z} \right] t^{\mu} dt$$

$$= \left[\frac{2}{F_1(2-\beta), 1+2+\mu; \frac{z}{z-1}} \right] (1+\mu)^{-1}. \quad (3.11)$$

□

In the following, we discuss the special case of (3.8) by choosing $\mu = 1$. We consider

$$F_1(f)(z) = \frac{p+1}{z} \int_0^z f(t) dt. \quad (3.12)$$

For $p = 1$, this integral was discussed by Libera [10]. We have

**Corollary 3.1.** Let $f \in R_k^1(a,b,c,\beta, p)$, then $F_1$, defined by (3.12), belongs to $R_k^1(a,b,c,\beta_1, p)$ where $\beta_1$ is given by (3.9) with $\mu = 1$.

We note that $\beta_1(\frac{1}{2}) = 0$ and $\delta(1) = 1$. Also, by choosing $p = 1$ and other parameters appropriately it can easily be seen that Libera integral operator maps starlike functions of order $\left(\frac{1}{2}\right)$ into starlike(univalent) functions. We shall now consider the converse case of Theorem 3.1.
**Theorem 3.3.** Let \( f \in R_k^{a+1}(a,b,c,\beta,p) \). Then, for \( \alpha > 0 \), \( f \in M_k^{a+1}(a,b,c,\beta,p,\alpha) \) for \( |z| < R_{\alpha,\beta} \), where \( R_{\alpha,\beta} \) is given by (3.14) and this value is exact.

**Proof.** Let

\[
\frac{z(z^{1-\tau}I_{k+1}(\alpha)f(z))'}{z^{1-\tau}I_{k+1}(\alpha)f(z)} = (1 - \beta)H(z) + \beta, \quad z \in E, 
\]

where \( H \in P_k \) and

\[
H(z) = \left(\frac{k}{4} + \frac{1}{2}\right)q_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)q_2(z), \quad q_1, q_2 \in P, z \in E. 
\]

(3.13)

Proceeding as in Theorem 3.1, we have

\[
\frac{1}{1 - \beta}[1 - a] \frac{z(z^{1-\tau}I_{k+1}(\alpha)f')}{z^{1-\tau}I_{k+1}(\alpha)f} + a \frac{[z(z^{1-\tau}I_{k+1}(\alpha)f')]'}{[z^{1-\tau}I_{k+1}(\alpha)f']^2} - \beta = H(z) + \frac{a_1zH'(z)}{H(z) + \beta_1}, 
\]

where \( a_1 = \frac{\alpha}{1 - \beta}, \quad \beta_1 = \frac{\beta}{1 - \beta} \).

Now define

\[
\phi_{a_1,\beta_1}(z) = \frac{1}{1 + \beta_1 (1 - z)^{a_1+1}} + \frac{\beta_1}{1 + \beta_1 (1 - z)^{a_1+2}}. 
\]

Then

\[
\left( H \star \frac{\phi_{a_1,\beta_1}(z)}{z^p} \right) = H(z) + \frac{a_1zH'(z)}{H(z) + \beta_1} 
\]

\[
= \left(\frac{k}{4} + \frac{1}{2}\right)\left[q_1(z) + \frac{a_1zq'_1(z)}{q_1(z) + \beta_1}\right] 
\]

\[
- \left(\frac{k}{4} - \frac{1}{2}\right)\left[q_2(z) + \frac{a_1zq'_2(z)}{q_2(z) + \beta_1}\right], \quad q_i \in P, z \in E, \quad i = 1, 2. 
\]

We use Lemma 2.1, with \( \nu = \frac{\beta}{1 - \beta}, s = \frac{\alpha}{1 - \beta} > 0 \) to have

\[
\left( q_i(z) + \frac{a_1zq'_i(z)}{q_i(z) + \beta_1}\right) \in P, \quad i = 1, 2, 
\]

for

\[
|z| < R_{\alpha,\beta} = \frac{|\nu + 1|}{\sqrt{A + (A^2 - |\nu|^2 - 1)^2}}, \quad A = 2(s + 1)^2 + |\nu|^2 - 1 
\]

(3.14)

and this radius is exact. Consequently \( f \in M_k^{a+1}(a,b,c,\beta,p,\alpha) \) for \( |z| < R_{\alpha,\beta} \) and the exact value of \( R_{\alpha,\beta} \) is given by (3.14). \( \Box \)

As a special case, for \( \beta = 0, \alpha = 1, \nu = 0, s = 1 \) and \( A = 7 \), we have

\[
R_{1,0} = \frac{1}{\sqrt{7} + 48} \approx 0.268 \approx 2 - \sqrt{3}. 
\]

**Remark 3.4.** The radii for the converse cases of other parts of Theorem 3.1 and Theorem 3.2 can be obtained by the similar procedure and techniques applied in Theorem 3.3.
Conclusion. In this paper, we have defined a linear operator on the class \( \mathcal{A}(p) \) of \( p \)-valent analytic functions in the open unit disc involving Gauss hypergeometric functions. Using this linear operator, we have introduced and investigated certain new subclasses of \( \mathcal{A}(p) \). Some inclusion results, a radius problem and several other interesting properties of these classes are studied. Results proved in this paper may stimulate further research activities in this dynamic field.

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