Initial Coefficient Bounds for a General Class of Bi-Univalent Functions

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Abstract. Recently, Srivastava et al. [22] reviewed the study of coefficient problems for bi-univalent functions. Inspired by the pioneering work of Srivastava et al. [22], there has been triggering interest to study the coefficient problems for the different subclasses of bi-univalent functions (see, for example, [1, 3, 6, 7, 27, 29]). Motivated essentially by the aforementioned works, in this paper we propose to investigate the coefficient estimates for a general class of analytic and bi-univalent functions. Also, we obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new class. Further, we discuss some interesting remarks, corollaries and applications of the results presented here.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{D} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$.

For analytic functions $f$ and $g$ in $\mathcal{A}$, $f$ is said to be subordinate to $g$ if there exists an analytic function $w$ such that (see, for example, [13])

$$w(0) = 0, \quad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \mathbb{D}).$$

This subordination will be denoted here by

$$f \prec g \quad (z \in \mathbb{D})$$

or, conventionally, by

$$f(z) < g(z) \quad (z \in \mathbb{D}).$$

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In particular, when \( g \) is univalent in \( \mathbb{U} \),

\[
f < g \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).
\]

Some of the important and well-investigated subclasses of the univalent function class \( \mathcal{S} \) include (for example) the class \( \mathcal{S}'(\alpha) \) of starlike functions of order \( \alpha \) \((0 \leq \alpha < 1)\) in \( \mathbb{U} \) and the class \( \mathcal{K}(\alpha) \) of convex functions of order \( \alpha \) \((0 \leq \alpha < 1)\) in \( \mathbb{U} \), the class \( \mathcal{S}'_\beta(\alpha) \) of \( \beta \)-spirallike functions of order \( \alpha \) \((0 \leq \alpha < 1; |\beta| < \frac{1}{2})\), the class \( \mathcal{S}(\varphi) \) of Ma-Minda starlike functions and the class \( \mathcal{K}(\varphi) \) of Ma-Minda convex functions \( \varphi \) is an analytic function with positive real part in \( \mathbb{U} \), \( \varphi(0) = 1, \varphi'(0) > 0 \) and \( \varphi \) maps \( \mathbb{U} \) onto a region starlike with respect to 1 and symmetric with respect to the real axis) (see [5, 11, 24]). The above-defined function classes have recently been investigated rather extensively in (for example) [9, 17, 25, 26] and the references therein.

It is well known that every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), defined by

\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{U})
\]

and

\[
f(f^{-1}(w)) = w \quad (|w| < r_0(f); \ r_0(f) \geq \frac{1}{4}),
\]

where

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots.
\]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{U} \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \) given by (1). For a brief history and interesting examples of functions which are in (or which are not in) the class \( \Sigma \), together with various other properties of the bi-univalent function class \( \Sigma \) one can refer the work of Srivastava et al. [22] and references therein. In fact, the study of the coefficient problems involving bi-univalent functions was reviewed recently by Srivastava et al. [22]. Various subclasses of the bi-univalent function class \( \Sigma \) were introduced and non-sharp estimates on the first two coefficients \( |a_2| \) and \( |a_3| \) in the Taylor-Maclaurin series expansion (1) were found in several recent investigations (see, for example, [1–4, 6–8, 12, 14, 16, 19–21, 23, 27, 29]). The aforementioned all these papers on the subject were actually motivated by the pioneering work of Srivastava et al. [22]. However, the problem to find the coefficient bounds on \( |a_n| \) \((n = 3, 4, \ldots)\) for functions \( f \in \Sigma \) is still an open problem.

Motivated by the aforementioned works (especially [22] and [3, 7]), we define the following subclass of the function class \( \Sigma \).

**Definition 1.1.** Let \( h : \mathbb{U} \to \mathbb{C} \), be a convex univalent function such that

\[
h(0) = 1 \quad \text{and} \quad h(z) = \overline{h(\overline{z})} \quad (z \in \mathbb{U} \text{ and } \Re(h(z)) > 0).
\]

Suppose also that the function \( h(z) \) is given by

\[
h(z) = 1 + \sum_{n=1}^{\infty} B_n z^n \quad (z \in \mathbb{U}).
\]

A function \( f(z) \) given by (1) is said to be in the class \( \mathcal{N}_{\Sigma}^{\mu, \lambda}(\beta, h) \) if the following conditions are satisfied:

\[
f \in \Sigma, \ \ e^{\beta} \left( 1 - \lambda \left( \frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \right) < h(z) \cos \beta + i \sin \beta \quad (z \in \mathbb{U}),
\]

and

\[
e^{\beta} \left( 1 - \lambda \left( \frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} \right) < h(w) \cos \beta + i \sin \beta \quad (w \in \mathbb{U}),
\]
where \( \beta \in (-\pi/2, \pi/2) \), \( \lambda \geq 1 \), \( \mu \geq 0 \) and the function \( g \) is given by

\[
g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots
\]

(4)

the extension of \( f \) to \( \mathbb{U} \).

Remark 1.2. If we set \( h(z) = \frac{1 + \cos \alpha}{1 - \cos \alpha} \), \( -1 \leq B < A \leq 1 \), in the class \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, h) \), we have \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, 1 + \cos \alpha) \) and defined as

\[
f \in \Sigma, \quad e^{\beta}(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \lesssim \frac{1 + Az}{1 + Bz} \cos \beta + i \sin \beta \quad (z \in \mathbb{U})
\]

and

\[
e^{\beta} \left( 1 - \lambda \right) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} \lesssim \frac{1 + Aw}{1 + Bw} \cos \beta + i \sin \beta \quad (w \in \mathbb{U}),
\]

where \( \beta \in (-\pi/2, \pi/2) \), \( \lambda \geq 1 \), \( \mu \geq 0 \) and the function \( g \) is given by (4).

Remark 1.3. Taking \( h(z) = \frac{1 + \cos \alpha}{1 - \cos \alpha} \), \( 0 \leq \alpha < 1 \) in the class \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, h) \), we have \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, \alpha) \) and \( f \in \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, \alpha) \) if the following conditions are satisfied:

\[
f \in \Sigma, \quad \Re \left( e^{\beta}(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \right) > \alpha \cos \beta \quad (z \in \mathbb{U})
\]

and

\[
\Re \left( e^{\beta} \left( 1 - \lambda \right) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} \right) > \alpha \cos \beta \quad (w \in \mathbb{U}),
\]

where \( \beta \in (-\pi/2, \pi/2) \), \( 0 \leq \alpha < 1 \), \( \lambda \geq 1 \), \( \mu \geq 0 \) and the function \( g \) is given by (4). It is interest to note that the class \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(0, \alpha) := \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(0, \alpha) \) the class was introduced and studied by Çaglar et al. [3].

Remark 1.4. Taking \( \lambda = 1 \) and \( h(z) = \frac{1 + \cos \alpha}{1 - \cos \alpha} \), \( 0 \leq \alpha < 1 \) in the class \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, h) \), we have \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, \alpha) \) and \( f \in \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, \alpha) \) if the following conditions are satisfied:

\[
f \in \Sigma, \quad \Re \left( e^{\beta} f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \right) > \alpha \cos \beta \quad (z \in \mathbb{U})
\]

and

\[
\Re \left( e^{\beta} g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} \right) > \alpha \cos \beta \quad (w \in \mathbb{U}),
\]

where \( \beta \in (-\pi/2, \pi/2) \), \( 0 \leq \alpha < 1 \), \( \mu \geq 0 \) and the function \( g \) is given by (4). We notice that the class \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(0, \alpha) := \mathcal{F}_{\Sigma}(\mu, \alpha) \) was introduced by Prema and Keerthi [16].

Remark 1.5. Taking \( \mu + 1 = \lambda = 1 \) and \( h(z) = \frac{1 + \cos \alpha}{1 - \cos \alpha} \), \( 0 \leq \alpha < 1 \) in the class \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, h) \), we have \( \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, \alpha) \) and \( f \in \mathcal{N} \mathcal{P}^{\mu,\lambda}_{\Sigma}(\beta, \alpha) \) if the following conditions are satisfied:

\[
f \in \Sigma, \quad \Re \left( e^{\beta} \frac{z'f(z)}{f(z)} \right) > \alpha \cos \beta \quad (z \in \mathbb{U})
\]
Taking Remark 1.7. was introduced and discussed by Frasin and Aouf [6] and

\[ \beta \]

where \( \beta \) was studied by Li and Wang [10] and considered by others.

Remark 1.6. Taking \( \mu = 1 \) and \( h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \), \( 0 < \alpha < 1 \) in the class \( \mathcal{NP}_\Sigma \), we have \( \mathcal{NP}_\Sigma (\beta, \alpha) \) and \( f \) if \( \mathcal{NP}_\Sigma (\beta, \alpha) \) if the following conditions are satisfied:

\[ f \in \Sigma, \ \Re \left( \epsilon^{\beta} \left( 1 - \lambda \right) \frac{f(z)}{z} + \lambda f'(z) \right) > \alpha \cos \beta \quad (z \in \mathbb{U}) \]

and

\[ \Re \left( \epsilon^{\beta} \left( 1 - \lambda \right) \frac{g(w)}{w} + \lambda g'(w) \right) > \alpha \cos \beta \quad (w \in \mathbb{U}), \]

where \( \beta \in (-\pi/2, \pi/2), 0 < 1 < 1 \) and the function \( g \) is given by (4). Further, the class \( \mathcal{NP}_\Sigma (\beta, \alpha) := \mathcal{B}_\Sigma (\lambda) \) was introduced and discussed by Frasin and Aouf [6]

Remark 1.7. Taking \( \mu = \lambda = 1 \) and \( h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \), \( 0 < \alpha < 1 \) in the class \( \mathcal{NP}_\Sigma (\beta, \alpha) \), we have \( \mathcal{NP}_\Sigma (\beta, \alpha) \) and \( f \) if \( \mathcal{NP}_\Sigma (\beta, \alpha) \) if the following conditions are satisfied:

\[ f \in \Sigma, \ \Re \left( \epsilon^{\beta} f'(z) \right) > \alpha \cos \beta \quad (z \in \mathbb{U}) \]

and

\[ \Re \left( \epsilon^{\beta} g'(w) \right) > \alpha \cos \beta \quad (w \in \mathbb{U}), \]

where \( \beta \in (-\pi/2, \pi/2), 0 < 1 < 1 \) and the function \( g \) is given by (4). Also, the class \( \mathcal{NP}_\Sigma (\beta, \alpha) := \mathcal{H}_\Sigma \) was introduced and studied by Srivastava et al. [22].

In order to derive our main result, we have to recall here the following lemmas.

Lemma 1.8. [15] If \( p \in \mathcal{P} \), then \( |p_i| \leq 2 \) for each \( i \), where \( \mathcal{P} \) is the family of all functions \( p \), analytic in \( \mathbb{U} \), for which

\[ \Re \{ p(z) \} > 0 \quad (z \in \mathbb{U}), \]

where

\[ p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}). \]

Lemma 1.9. [18, 28] Let the function \( \varphi(z) \) given by

\[ \varphi(z) = \sum_{n=1}^{\infty} B_n z^n \quad (z \in \mathbb{U}) \]

be convex in \( \mathbb{U} \). Suppose also that the function \( h(z) \) given by

\[ \psi(z) = \sum_{n=1}^{\infty} \psi_n z^n \quad (z \in \mathbb{U}) \]

is holomorphic in \( \mathbb{U} \). If

\[ \psi(z) < \varphi(z) \quad (z \in \mathbb{U}) \]

then

\[ |\psi_n| \leq |B_1| \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}). \]
The object of the present paper is to introduce a general new subclass $NP_{\Sigma}^{\mu,\lambda}(\beta, h)$ of the function class $\Sigma$ and obtain estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in this new class $NP_{\Sigma}^{\mu,\lambda}(\beta, h)$.

2. Coefficient Bounds for the Function Class $NP_{\Sigma}^{\mu,\lambda}(\beta, h)$

In this section we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $NP_{\Sigma}^{\mu,\lambda}(\beta, h)$.

**Theorem 2.1.** Let $f(z)$ given by (1) be in the class $NP_{\Sigma}^{\mu,\lambda}(\beta, h)$, $\lambda \geq 1$ and $\mu \geq 0$, then

$$|a_2| \leq \sqrt{\frac{2|B_1|\cos \beta}{(1 + \mu)(2\lambda + \mu)}}$$

and

$$|a_3| \leq \frac{2|B_1|\cos \beta}{(2\lambda + \mu)(1 + \mu)},$$

where $\beta \in (-\pi/2, \pi/2)$.

**Proof.** It follows from (2) and (3) that there exists $p, q \in P$ such that

$$e^{i\beta} \left(1 - \lambda \left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1}\right) = p(z) \cos \beta + i \sin \beta$$

and

$$e^{i\beta} \left(1 - \lambda \left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w}\right)^{\mu-1}\right) = p(w) \cos \beta + i \sin \beta,$$

where $p(z) < h(z)$ and $q(w) < h(w)$ have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \quad (z \in \mathbb{U})$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \ldots \quad (w \in \mathbb{U}).$$

Equating coefficients in (7) and (8), we get

$$e^{i\beta} (\lambda + \mu)a_2 = p_1 \cos \beta$$

$$e^{i\beta} \left[\frac{a_2^2}{2} (\mu - 1) + a_3\right] (2\lambda + \mu) = p_2 \cos \beta$$

$$e^{i\beta} (\lambda + \mu)a_2 = q_1 \cos \beta$$

$$e^{i\beta} \left[\frac{a_2^2}{2} (\mu + 3) - a_3\right] (2\lambda + \mu) = q_2 \cos \beta.$$
and
\[ 2e^{2\beta}(\lambda + \mu)^2 a_2^2 = (p_1^2 + q_1^2) \cos^2 \beta. \] (16)

Also, from (12) and (14), we obtain
\[ a_2^2 = -ie^{i\beta}(p_2 + q_2) \cos \beta \]
\[ (1 + \mu)(2\lambda + \mu) \] (17)
Since \( p, q \in h(U) \), applying Lemma 1.9, we immediately have
\[ |p_m| = \left| \frac{p(m)(0)}{m!} \right| \leq |B_1| \quad (m \in \mathbb{N}), \] (18)
and
\[ |q_m| = \left| \frac{q(m)(0)}{m!} \right| \leq |B_1| \quad (m \in \mathbb{N}). \] (19)

Applying (18), (19) and Lemma 1.9 for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we readily get
\[ |a_2| \leq \sqrt{\frac{2|B_1| \cos \beta}{(1 + \mu)(2\lambda + \mu)}}. \]
This gives the bound on \( |a_2| \) as asserted in (5).

Next, in order to find the bound on \( |a_3| \), by subtracting (14) from (12), we get
\[ 2(a_3 - a_2^2)(2\lambda + \mu) = e^{-i\beta}(p_2 - q_2) \cos \beta. \] (20)
It follows from (17) and (20) that
\[ a_3 = \frac{e^{-i\beta}(p_2 + q_2)}{(1 + \mu)(2\lambda + \mu)} + \frac{e^{-i\beta}(p_2 - q_2) \cos \beta}{2(2\lambda + \mu)}. \] (21)

Applying (18), (19) and Lemma 1.9 once again for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we readily get
\[ |a_3| \leq \frac{2|B_1| \cos \beta}{(2\lambda + \mu)(1 + \mu)}. \]
This completes the proof of Theorem 2.1.

3. Corollaries and Consequences

In view of Remark 1.2, if we set
\[ h(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; \ z \in \mathbb{U}) \]
and
\[ h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1; \ z \in \mathbb{U}), \]
in Theorem 2.1, we can readily deduce Corollaries 3.1 and 3.2, respectively, which we merely state here without proof.
Corollary 3.1. Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, \frac{\lambda + 1}{1 + \mu})$, then

$$|a_2| \leq \frac{\sqrt{2(A - B) \cos \beta}}{(1 + \mu)(2\lambda + \mu)}$$

and

$$|a_3| \leq \frac{2(A - B) \cos \beta}{(2\lambda + \mu)(1 + \mu)}$$

where $\beta \in (-\pi/2, \pi/2)$, $\mu \geq 0$ and $\lambda \geq 1$.

Corollary 3.2. Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, \alpha)$, $0 \leq \alpha < 1$, $\mu \geq 0$ and $\lambda \geq 1$, then

$$|a_2| \leq \frac{\sqrt{4(1 - \alpha) \cos \beta}}{(1 + \mu)(2\lambda + \mu)}$$

and

$$|a_3| \leq \frac{4(1 - \alpha) \cos \beta}{(2\lambda + \mu)(1 + \mu)}$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 3.3. When $\beta = 0$ the estimates of the coefficients $|a_2|$ and $|a_3|$ of the Corollary 3.2 are improvement of the estimates obtained in [3, Theorem 3.1].

Corollary 3.4. Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, \alpha)$, $0 \leq \alpha < 1$ and $\mu \geq 0$, then

$$|a_2| \leq \frac{\sqrt{4(1 - \alpha) \cos \beta}}{(1 + \mu)(2 + \mu)}$$

and

$$|a_3| \leq \frac{4(1 - \alpha) \cos \beta}{(2 + \mu)(1 + \mu)}$$

where $\beta \in (-\pi/2, \pi/2)$.

Corollary 3.5. Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, \alpha)$, $0 \leq \alpha < 1$, then

$$|a_2| \leq \sqrt{2(1 - \alpha) \cos \beta}$$

and

$$|a_3| \leq 2(1 - \alpha) \cos \beta$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 3.6. Taking $\beta = 0$ in Corollary 3.5, the estimate (28) reduces to $|a_2|$ of [10, Corollary 3.3] and (29) is improvement of $|a_3|$ obtained in [10, Corollary 3.3].
Corollary 3.7. Let $f(z)$ given by (1) be in the class $NP_{\mathcal{C}}^{1,1}(\beta, \alpha)$, $0 \leq \alpha < 1$ and $\lambda \geq 1$, then

$$|a_2| \leq \sqrt{\frac{2(1 - \alpha) \cos \beta}{2\lambda + 1}}$$

and

$$|a_3| \leq \frac{2(1 - \alpha) \cos \beta}{2\lambda + 1},$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 3.8. Taking $\beta = 0$ in Corollary 3.7, the inequality (31) improves the estimate of $|a_3|$ in [6, Theorem 3.2].

Corollary 3.9. Let $f(z)$ given by (1) be in the class $NP_{\mathcal{C}}^{1,1}(\beta, \alpha)$, $0 \leq \alpha < 1$, then

$$|a_2| \leq \sqrt{\frac{2(1 - \alpha) \cos \beta}{3}}$$

and

$$|a_3| \leq \frac{2(1 - \alpha) \cos \beta}{3},$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 3.10. For $\beta = 0$ the inequality (33) improves the estimate $|a_3|$ of [22, Theorem 2].

References


