Properties For an Integral Operator on the Class of Close-to-Convex Functions

Nicoleta Ularu

"Ioan Slavici" University of Timișoara, Aurel Podeanu Str., No. 144, Timișoara, Romania

Abstract. The purpose of this paper is to prove that the functions generated by the integral operator
\[ I(f, g)(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{g_i(t)} \right)^{\gamma_i} \, dt \]
are in the class of close-to-convex functions, considering the analytical functions \( f \) and \( g \) from the classes of starlike and close-to-starlike functions.

1. Introduction and Definitions

Let \( U = \{ z : |z| < 1 \} \) be the open unit disk. By \( \mathcal{A} \) we denote the class of all analytical functions in the open unit disk \( U \) and by \( S \) the class of univalent functions that contains all functions of the form:
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1) \]
which are analytic in \( U \) and satisfy the condition:
\[ f(0) = f'(0) - 1 = 0. \]

To prove our main results we will recall here some known results about some subclasses of analytical functions. First we will recall the classes of starlike and convex functions of order \( \alpha \) denoted by \( S^*(\alpha) \) and \( K(\alpha) \) and defined by:
\[ S^*(\alpha) = \{ f \in \mathcal{A} : \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, z \in U \} \]
\[ K(\alpha) = \{ f \in \mathcal{A} : \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, z \in U \} \]
for \( 0 \leq \alpha < 1. \)

Alexander studied for the first time the class of starlike functions in [1] and the class of convex functions was introduced in [9], by E. Study.
A function \( f \in \mathcal{A} \) is in the class \( S^*(a, A) \) if it satisfy the condition:

\[
\left| \frac{zf''(z)}{f'(z)} - a \right| < A, \ |a - 1| < A \leq a, z \in \mathcal{U}.
\]

(2)

We have that \( a > \frac{1}{2} \) and \( S^*(a, A) \subset S^*(a - A) \subset S^*(0) \equiv S^* \). This class was introduced in [5] by Jakubowski.

The class \( K(a, A) \) contains all the functions \( f \in \mathcal{A} \) such that:

\[
\left| 1 + \frac{zf''(z)}{f'(z)} - a \right| < A, \ |a - 1| < A \leq a, z \in \mathcal{U}.
\]

(3)

Also for this class, \( a > \frac{1}{2} \) and \( K(a, A) \subset K(a - A) \subset K(0) \equiv K \). The relations (2) and (3) are equivalently with

\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > a - A, \ z \in \mathcal{U}, |a - 1| < A \leq a,
\]

respectively

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > a - A, \ z \in \mathcal{U}, |a - 1| < A \leq a.
\]

The class of close-to-convex functions contains all the functions that satisfy the condition:

\[
\int_{\theta_1}^{\theta_2} \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) d\theta > -\pi
\]

where \( 0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta} \) and \( r < 1 \) and is denoted by \( C_c \).

This class was studied for certain analytic functions by Owa et al. in [6].

A function belongs to \( C_s^* \), i.e. the class of close-to-starlike functions if:

\[
\int_{\theta_1}^{\theta_2} \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) d\theta > -\pi,
\]

where \( 0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta} \) and \( r < 1 \).

Shukla and Kumar introduced in [7] some subclasses of \( C_c \) and \( C_s^* \) and proved some important results for these. The class \( C_c(\beta, \rho) \) of close-to-convex functions of order \( \beta \) and type \( \rho \) contains all the functions that for a function \( g \in S^*(\rho) \) satisfies the inequality:

\[
\left| \arg \left( \frac{zf''(z)}{g(z)} \right) \right| < \frac{\beta\pi}{2}, z \in \mathcal{U}, \beta \in [0, 1].
\]

A function \( f \) is in the class of close-to-starlike functions of order \( \beta \) and type \( \rho \), denoted by \( C_s^*(\beta, \rho) \) if for some function \( g \in S^*(\rho) \) we have the following inequality:

\[
\left| \arg \left( \frac{f(z)}{g(z)} \right) \right| < \frac{\beta\pi}{2}, z \in \mathcal{U},
\]

for \( \beta \in [0, 1] \).

Is very clear that \( C_c(0, \rho) = K(\rho) \) and \( C_s^*(0, \rho) = S^*(\rho) \).

We consider the results proved by Shukla and Kumar in [7] about these two subclasses defined before.
Lemma 1.1. [7] If \( f \in S'(\rho) \), then
\[
\rho(\theta_2 - \theta_1) \leq \int_{\theta_1}^{\theta_2} \Re \frac{z f'(z)}{f(z)} d\theta \leq 2\pi(1 - \rho) + \rho(\theta_2 - \theta_1),
\]
where \( z = r e^{i\theta} \) and \( 0 \leq \theta_1 \leq \theta_2 \leq 2\pi \).

Lemma 1.2. [7] If \( f \in C_c(\beta, \rho) \) then
\[
-\beta \pi + \rho(\theta_2 - \theta_1) \leq \int_{\theta_1}^{\theta_2} \Re \frac{z f'(z)}{f(z)} d\theta \leq \beta \pi + 2\pi(1 - \rho) + \rho(\theta_2 - \theta_1),
\]
where \( z = r e^{i\theta} \) and \( 0 \leq \theta_1 \leq \theta_2 \leq 2\pi \).

For the analytical functions \( f_i, g_i \), and the positive real numbers \( \gamma_i \), for \( i = 1, n \), we consider the integral operator:
\[
I(f, g)(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{g_i(t)} \right)^{\gamma_i} dt, \tag{4}
\]
that was introduced by Ularu and Breaz in [10]. Integral operators make the subject of several articles, the authors studying some properties for them, for example the univalence (see for example [2], [8], [4], [11] and [3]).

2. Main Results

Theorem 2.1. Let the analytical functions \( f_i \) from the class \( S'(\eta_i) \), \( g_i \) from the class \( S'(\delta_i) \), and the positive real numbers \( \gamma_i \), for \( i = 1, n \). If \( \sum_{i=1}^n \gamma_i \leq 1 \), then \( I(f, g) \) is in the class of close-to-convex functions \( C_c \).

Proof. From the definitions of \( I(f, g) \) given in (4) by logarithmic differentiations we obtain that
\[
\frac{z I''(f, g)(z)}{I'(f, g)(z)} = \sum_{i=1}^n \gamma_i \left( \frac{zf_i'(z)}{f_i(z)} - \frac{zg_i'(z)}{g_i(z)} \right),
\]
for \( i = 1, n \) and \( z \in \mathcal{U} \).

Using the definition of close-to-convex functions results:
\[
\int_{\theta_1}^{\theta_2} \Re \left( 1 + \frac{z I''(f, g)(z)}{I'(f, g)(z)} \right) d\theta = \int_{\theta_1}^{\theta_2} \Re \left[ \sum_{i=1}^n \gamma_i \left( \frac{zf_i'(z)}{f_i(z)} - \frac{zg_i'(z)}{g_i(z)} \right) \right] d\theta + 1
\]
\[
= \int_{\theta_1}^{\theta_2} \sum_{i=1}^n \gamma_i \Re \left( \frac{zf_i'(z)}{f_i(z)} \right) d\theta - \int_{\theta_1}^{\theta_2} \sum_{i=1}^n \gamma_i \Re \left( \frac{zg_i'(z)}{g_i(z)} \right) d\theta + \int_{\theta_1}^{\theta_2} d\theta.
\]

We use the hypothesis that \( f_i \in S'(\eta_i) \) and \( g_i \in S'(\delta_i) \) and according to Lemma 1.1 it follows that:
\[
\int_{\theta_1}^{\theta_2} \Re \left( 1 + \frac{z I''(f, g)(z)}{I'(f, g)(z)} \right) d\theta \geq \sum_{i=1}^n \gamma_i \eta_i(\theta_2 - \theta_1) - \sum_{i=1}^n \gamma_i \delta_i(\theta_2 - \theta_1) + (\theta_2 - \theta_1)
\]
\[
\geq \left( \sum_{i=1}^n \gamma_i(\eta_i - \delta_i) + 1 \right)(\theta_2 - \theta_1),
\]
where \( \eta_i, \delta_i \) are the radii of univalence for \( f_i \) and \( g_i \), respectively.
for \( z \in \mathcal{U} \) and \( i = \frac{1}{\eta_n} \). Because \( \sum_{i=1}^{n} \gamma_i (\eta_i - \delta_i) + 1 > 0 \), and minimum is for \( \theta_1 = \theta_2 \), results that
\[
\int_{\theta_1}^{\theta_2} \text{Re} \left( 1 + \frac{z''(f, g)(z)}{P(f, g)(z)} \right) \, d\theta > -\pi.
\]
So, from the above inequality we obtain that \( I(f, g) \in C_c \). \( \Box \)

If we consider \( \eta_1 = \eta_2 = \cdots = \eta_n = \eta \) and \( \delta_1 = \delta_2 = \cdots = \delta_n = \delta \) in Theorem 2.1 it follows:

**Corollary 2.2.** Let \( f_i, g_i \in \mathcal{A} \) and the positive real numbers \( \gamma_i \), for \( i = \frac{1}{\eta_n} \). If \( f_i \in S'(\eta), g_i \in S'(\delta) \) and \( \sum_{i=1}^{n} \gamma_i \leq 1 \), then \( I(f, g) \) is in the class of close-to-convex functions \( C_c \).

**Theorem 2.3.** Let the analytical function \( f_i \in C_c, g_i \in S'(\delta_i) \) and \( \gamma_i \) positive real numbers, for \( i = \frac{1}{\eta_n} \). If \( \sum_{i=1}^{n} \gamma_i \leq 1 \), then the functions generated by the operator \( I(f, g) \) are in the class \( C_c \).

**Proof.** The proof follows the same idea as the proof of Theorem 2.1. Results that
\[
\int_{\theta_1}^{\theta_2} \text{Re} \left( 1 + \frac{z''(f, g)(z)}{P(f, g)(z)} \right) \, d\theta = \int_{\theta_1}^{\theta_2} \sum_{i=1}^{n} \text{Re} \left( \frac{z(f_i(z))}{g_i(z)} \right) \, d\theta - \int_{\theta_1}^{\theta_2} \sum_{i=1}^{n} \gamma_i \text{Re} \left( \frac{z(\beta_i(z))}{g_i(z)} \right) \, d\theta,
\]
for all \( z \in \mathcal{U} \) and \( i = \frac{1}{\eta_n} \). We use that \( f_i \in C_c, g_i \in S'(\delta_i) \) and from Lemma 1.1 and Lemma 1.2 it follows that:
\[
\int_{\theta_1}^{\theta_2} \text{Re} \left( 1 + \frac{z''(f, g)(z)}{P(f, g)(z)} \right) \, d\theta \geq -\pi \sum_{i=1}^{n} \gamma_i - \sum_{i=1}^{n} \gamma_i \delta_i (\theta_2 - \theta_1) + (\theta_2 - \theta_1)
\]
\[
\geq -\pi \sum_{i=1}^{n} \gamma_i - (\theta_2 - \theta_1) (\sum_{i=1}^{n} \delta_i \gamma_i + 1),
\]
for all \( z \in \mathcal{U} \) and \( i = \frac{1}{\eta_n} \). Because \( 1 - \sum_{i=1}^{n} \gamma_i \delta_i > 0 \), minimum is for \( \theta_1 = \theta_2 \) we obtain that \( I(f, g) \in C_c \). \( \Box \)

**Theorem 2.4.** Let the analytical functions \( f_i, g_i \), and the positive real numbers \( \gamma_i \), for all \( i = \frac{1}{\eta_n} \). If \( f_i \in C_c(\beta_i, \rho_i), g_i \in C_c(\alpha_i, \eta_i) \) and \( \sum_{i=1}^{n} \gamma_i \beta_i \leq 1, \sum_{i=1}^{n} \gamma_i \alpha_i \leq 1 \), then the integral operator \( I(f, g) \) is in the class \( C_c \).

**Proof.** We follow the same steps as in the proofs of the above theorems, but we use that the functions \( f_i \) are from the class \( C_c(\beta_i, \rho_i) \) and the functions \( g_i \) are from \( C_c(\alpha_i, \eta_i) \). Using these and applying Lemma 1.2 it follows that:
\[
\int_{\theta_1}^{\theta_2} \text{Re} \left( 1 + \frac{z''(f, g)(z)}{P(f, g)(z)} \right) \, d\theta \geq \sum_{i=1}^{n} \gamma_i [(-\beta_i \pi + \rho_i (\theta_2 - \theta_1)) - (-\alpha_i \pi + \eta_i (\theta_2 - \theta_1))] + (\theta_2 - \theta_1)
\]
\[
\geq (\theta_2 - \theta_1) \left[ \sum_{i=1}^{n} \gamma_i (\rho_i - \eta_i) + 1 \right] - \sum_{i=1}^{n} \gamma_i \beta_i \pi + \sum_{i=1}^{n} \gamma_i \alpha_i \pi.
\]
Since $\sum_{i=1}^{n} \gamma_i (\rho_i - \eta_i) + 1 > 0$, minimum is for $\theta_1 = \theta_2$ and results

$$\int_{b_i}^{a_i} \text{Re} \left( 1 + \frac{z^l(f, g)(z)}{P(f, g)(z)} \right) d\theta > -\pi.$$ 

We obtain that $I(f, g) \in C_r$. \hfill $\square$

If we consider $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ in Theorem 2.4 we obtain:

**Corollary 2.5.** Let the analytical functions $f_i, g_i$ and the positive real numbers $\gamma_i$, for all $i = 1, \ldots, n$. If $f_i \in C_r(\beta, \rho_i), g_i \in C_r(\alpha, \eta_i)$ and $\beta \sum_{i=1}^{n} \gamma_i \leq 1$, respectively $\alpha \sum_{i=1}^{n} \gamma_i \leq 1$, then the integral operator $I(f, g)$ is in the class $C_r$.

**Remark 2.6.** If we consider $\beta_i = 0$ and $\alpha_i = 0$, for $i = 1, n$ in Theorem 2.4 we obtain the results from Theorem 2.1.

**Theorem 2.7.** Let $f_i \in S'((\alpha, \beta_i))$, for $|\alpha - 1| < \beta_i \leq \alpha_i$ and $g_i \in S'((\xi, \eta_i))$, for $|\xi - 1| < \eta_i \leq \xi_i$, $\gamma_i > 0$ for all $i = 1, n$ and $z \in \mathcal{U}$. Then the functions generated by the integral operator $I(f, g)$ are in the class $K(a_i, b_i)$, where

$$a_i = 1 + \sum_{i=1}^{n} \gamma_i (\alpha_i - \beta_i), b_i = \sum_{i=1}^{n} \gamma_i (\xi_i - \eta_i)$$

and

$$\sum_{i=1}^{n} \gamma_i (\xi_i - \eta_i - \alpha_i + \beta_i) \leq 1,$$

for all $i = 1, n$ and $z \in \mathcal{U}$.

**Proof.** Using that $f_i \in S'((\alpha, \beta_i))$ and $g_i \in S'((\xi, \eta_i))$ results:

$$\text{Re} \left( 1 + \frac{z^l(f, g)(z)}{P(f, g)(z)} \right) = \text{Re} \left( 1 + \sum_{i=1}^{n} \gamma_i \left( \frac{z^l(f_i(z))}{f_i(z)} - \frac{z^l(g_i(z))}{g_i(z)} \right) \right)$$

$$= 1 + \sum_{i=1}^{n} \gamma_i \text{Re} \left( \frac{z^l(f_i(z))}{f_i(z)} \right) - \sum_{i=1}^{n} \gamma_i \text{Re} \left( \frac{z^l(g_i(z))}{g_i(z)} \right)$$

$$> 1 + \sum_{i=1}^{n} \gamma_i (\alpha_i - \beta_i) - \sum_{i=1}^{n} \gamma_i (\xi_i - \eta_i).$$

From the above inequality and the definition of $K(a_i, b_i)$ we obtain that $I(f, g)(z) \in K(a_i, b_i)$, where $a_i$ and $b_i$ are defined as in the theorem hypothesis. \hfill $\square$

For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ and $\xi_1 = \xi_2 = \cdots = \xi_n = \xi$ in Theorem 2.7 we obtain:

**Corollary 2.8.** Let $f_i \in S'((\alpha, \beta_i))$, for $|\alpha - 1| < \beta_i \leq \alpha_i$ and $g_i \in S'((\xi, \eta_i))$, for $|\xi - 1| < \eta_i \leq \xi_i$, $\gamma_i > 0$ for all $i = 1, n$ and $z \in \mathcal{U}$. Then the functions generated by the integral operator $I(f, g)$ are in the class $K(a_i, b_i)$, where

$$a_i = 1 + \sum_{i=1}^{n} \gamma_i (\alpha_i - \beta_i), b_i = \sum_{i=1}^{n} \gamma_i (\xi_i - \eta_i)$$

and

$$\sum_{i=1}^{n} \gamma_i (\xi_i - \eta_i - \alpha_i + \beta_i) \leq 1,$$

for all $i = 1, n$ and $z \in \mathcal{U}$.

If in Theorem 2.7 we consider $\gamma_1 = \gamma_2 = \cdots = \gamma_n = \gamma$, then we obtain

**Corollary 2.9.** Let $f_i \in S'((\alpha, \beta_i))$, for $|\alpha - 1| < \beta_i \leq \alpha_i$ and $g_i \in S'((\xi, \eta_i))$, for $|\xi - 1| < \eta_i \leq \xi_i$, $\gamma > 0$ for all $i = 1, n$ and $z \in \mathcal{U}$. Then the functions generated by the integral operator $I(f, g)$ are in the class $K(a_i, b_i)$, where

$$a_i = 1 + \gamma \sum_{i=1}^{n} (\alpha_i - \beta_i), b_i = \gamma \sum_{i=1}^{n} (\xi_i - \eta_i)$$

and

$$\gamma \sum_{i=1}^{n} (\xi_i - \eta_i - \alpha_i + \beta_i) \leq 1,$$

for all $i = 1, n$ and $z \in \mathcal{U}$.

**Theorem 2.10.** Let $f_i \in S'((\alpha, \beta_i))$ and $g_i \in S'((\beta_i))$, for all $i = 1, n$. Then the integral operator $I(f, g) \in K(a_i, b_i)$, where

$$a_i = 1 + \sum_{i=1}^{n} \gamma_i (\alpha_i - \beta_i), b_i = \sum_{i=1}^{n} \gamma_i \beta_i$$

and

$$\sum_{i=1}^{n} \gamma_i (\beta_i - \alpha_i) \leq 1,$$

for all $i = 1, n$ and $z \in \mathcal{U}$. 
Proof. The proof is similar to Theorem 2.7. □

If we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ and $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$ in Theorem 2.10 results:

Corollary 2.11. Let $f_i \in S'(\alpha)$ and $g_i \in S^*(\beta)$, for all $i = 1, n$. Then the integral operator $I(f, g) \in K(a_i, b_i)$, where $a_i = 1 + \alpha \sum_{i=1}^{n} \gamma_i$, $b_i = \beta \sum_{i=1}^{n} \gamma_i$ and $(\beta - \alpha) \sum_{i=1}^{n} \gamma_i \leq 1$, for all $i = 1, n$ and $z \in U$.

References