Graphs with Large Geodetic Number

Hossein Abdollahzadeh Ahangar\textsuperscript{a}, Saeed Kosari\textsuperscript{b}, Seyed Mahmoud Sheikholeslami\textsuperscript{b}, Lutz Volkmann\textsuperscript{c}

\textsuperscript{a}Department of Basic Science, Babol University of Technology, Babol, I.R. Iran
\textsuperscript{b}Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran
\textsuperscript{c}Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany

Abstract. For two vertices \( u \) and \( v \) of a graph \( G \), the set \( [u,v] \) consists of all vertices lying on some \( u-v \) geodesic in \( G \). If \( S \) is a set of vertices of \( G \), then \( [S] \) is the union of all sets \( [u,v] \) for \( u,v \in S \). A subset \( S \) of vertices of \( G \) is a geodetic set if \( [S] = V \). The geodetic number \( g(G) \) is the minimum cardinality of a geodetic set. It was shown in [1] that the determination of \( g(G) \) is an NP-hard problem and its decision problem is NP-complete. The geodetic number and its variants have been studied by several authors (see for example [1, 5–7, 9–16, 18]). Clearly, a connected graph \( G \) of order \( n \geq 2 \) has geodetic number \( n \) if and only if \( G = K_n \). It was shown in [3] that a connected graph \( G \) of order \( n \geq 3 \) has geodetic number \( n-1 \) if and only if \( G \) is the join of \( K_1 \) and pairwise disjoint complete graphs \( K_{n_1}, K_{n_2}, \ldots, K_{n_r} \), that is, \( G = (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_r}) + K_1 \), where \( r \geq 2 \) and all \( n_1, n_2, \ldots, n_r \) are positive integers with \( n_1 + n_2 + \ldots + n_r = n-1 \). In this paper we characterize all connected graphs \( G \) of order \( n \geq 3 \) with \( g(G) = n-2 \).

1. Introduction

Throughout this paper, \( G \) is a simple connected graph with vertex set \( V(G) \) and edge set \( E(G) \) (briefly \( V \) and \( E \)). We refer the reader to the book [17] for graph theory notation and terminology not defined here. For every vertex \( v \in V \), the open neighborhood \( N(v) \) is the set \( \{ u \in V \mid uv \in E \} \) and the closed neighborhood of \( v \) is the set \( N[v] = N(v) \cup \{v\} \). The degree of a vertex \( v \in V \) is \( \deg_G(v) = \deg(v) = |N(v)| \). A vertex \( v \) is called a simplicial vertex in a graph \( G \) if the subgraph induced by its neighbors is complete. For vertices \( x \) and \( y \) in a connected graph \( G \), the distance \( d_G(x,y) \) is the length of a shortest \( x-y \) path in \( G \). A vertex \( v \) of \( G \) is called a diameter vertex if \( d_G(v) = \text{diam}(G) \). An \( x-y \) path of length \( d_G(x,y) \) is called an \( x-y \) geodesic. The geodetic interval \( [x,y] \) consists of all vertices lying in some \( x-y \) geodesic of \( G \), and for a nonempty subset \( S \) of \( V(G) \), we define \( [S] = \bigcup_{x \in S} [x,y] \).

A subset \( S \) of vertices of \( G \) is a geodetic set if \( [S] = V \). The geodetic number \( g(G) \) is the minimum cardinality of a geodetic set of \( G \). A \( g(G) \)-set is a geodetic set of \( G \) of size \( g(G) \). The geodetic sets of a connected graph were introduced by Harary, Loukakis and Tsouros [8], as a tool for studying metric properties of connected graphs. It was shown in [1] that the determination of \( g(G) \) is an NP-hard problem and its decision problem is NP-complete. The geodetic number and its variants have been studied by several authors (see for example [1, 5–7, 9–16, 18]). Clearly, a connected graph \( G \) of order \( n \geq 2 \) has geodetic number \( n \) if and only if \( G = K_n \). It was shown in [3] that a connected graph \( G \) of order \( n \geq 3 \) has geodetic number \( n-1 \) if and only if \( G \) is...
the join of $K_1$ and pairwise disjoint complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$, that is, $G = (K_{n_1} \cup K_{n_2} \cup \ldots K_{n_r}) + K_1$, where $r \geq 2$, $n_1, n_2, \ldots, n_r$ are positive integers with $n_1 + n_2 + \ldots + n_r = n - 1$.

The purpose of this paper is to characterize all connected graphs $G$ of order $n \geq 3$ with $g(G) = n - 2$.

We make use of the following results in this paper.

**Observation 1.1.** ([4]) Every geodetic set of a graph contains its simplicial vertices.

**Observation 1.2.** Every connected graph $G$ of order $n$ different from $K_n$, has a geodetic set $S$ of size $n - 1$ such that the vertex not in $S$, belongs to a geodesic path of length two.

**Proof.** Let $G$ be a connected graph of order $n$ different from $K_n$. Since $G \cong K_n$, $G$ has three vertices $u, v$ and $w$ such that $uv, vw \in E(G)$ and $vw \not\in E(G)$ (see Exercise 1.6.14 in [2]). It follows that $d_G(v, w) = 2$ and $u \in I[v, w]$ and hence $S = V(G) - \{u\}$ is a geodetic set of $G$ with desired property. $\Box$

**Observation 1.3.** Let $G$ be a connected graph of order $n$ with $g(G) \leq n - 2$ and let $u$ be an arbitrary vertex of $G$. Then $G$ has a geodetic set $S$ of size $n - 1$ containing $u$ such that the vertex not in $S$, belongs to a geodesic path of length two.

**Proof.** If there is a vertex $v$ at distance 2 from $u$ and $w \in N(u) \cap N(v)$, then $V(G) - \{w\}$ is a geodetic set of $G$ with desired property. Thus, we assume $u$ is adjacent to all vertices of $G$. Since $g(G) \leq n - 2$, $G - u$ has a component $H$ that is not complete. By Observation 1.2, $H$ has a geodetic set $S$ of size $|V(H)| - 1$ such that the vertex not in $S$, say $x$, belongs to a geodesic path of length two. Then obviously $V(G) - \{x\}$ is a geodetic set of $G$ with desired property. $\Box$

**Proposition 1.4.** Let $G$ be a connected graph and $H$ be a connected induced subgraph of $G$ that is not complete. If

1. $G - V(H)$ has a cycle $(v_1v_2v_3v_4)$ in which $v_1v_3 \not\in E(G)$, or
2. $G - V(H)$ has a path $v_1v_2v_3v_4$ in which $d_G(v_1, v_4) = 3$ and there is no edge between the sets $\{v_1, v_4\}$ and $V(H)$,

then $g(G) \leq n - 3$.

**Proof.** By Observation 1.2, $H$ has a geodetic set $S$ of size $|V(H)| - 1$ such that the vertex not in $S$, say $x$, belongs to a $(y,z)$-geodesic path where $d_H(y,z) = 2$. Clearly $d_G(y,z) = 2$. If (1) holds, then clearly $d_G(v_1, v_3) = 2$ and $[v_2, v_4] \subseteq I[v_1, v_3]$. It follows that $V(G) - \{x, v_2, v_4\}$ is a geodetic set of $G$ that implies $g(G) \leq n - 3$. If (2) holds, then $x \in I[y,z]$ and $v_2, v_3 \in I[v_1, v_4]$ and so $V(G) - \{x, v_2, v_3\}$ is a geodetic set of $G$ implying that $g(G) \leq n - 3$. $\Box$

2. **Graphs with Large Geodetic Number**

Chartrand, Harary and Zhang [4] established the following upper bound on geodetic number of a graph in terms of its order and diameter.

**Theorem A.** If $G$ is a nontrivial connected graph of order $n$ and diameter $d$, then $g(G) \leq n - d + 1$.

**Corollary 2.1.** If $G$ is a connected graph of order $n$ with $g(G) = n - i$, then $\text{diam}(G) \leq i + 1$.

In what follows, we characterize all graphs $G$ of order $n \geq 3$ with $g(G) = n - 2$. By Corollary 2.1 we need to consider connected graphs $G$ for which $\text{diam}(G) = 2$ or 3. First we introduce four families of graphs.

Let $F_1$ be the collection of all graphs obtained from a cycle $C_4 = (v_1v_2v_3v_4)$ and complete graphs $K_{n_1}, \ldots, K_{n_r}$ (possibly no complete graphs) by joining $v_1$ and $v_2$ to all vertices of complete graphs. Clearly $C_4 \in F_1$.

Let $F_2$ be the collection of all graphs obtained from a triangle $K_3 = uvw$ and complete graphs $K_{m_1}, \ldots, K_{m_r}$ (possibly no complete graphs of this kind), $K_{m_1}, K_{m_2}, \ldots, K_{m_r}$, by joining $u$ to all vertices of complete graphs, $v$ to all vertices of $K_{m_1}, \ldots, K_{m_r}$, and $w$ to the vertices of $K_{m_i}$. 

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Suppose $\mathcal{F}_3$ is the collection of all graphs obtained from $K_2 = uv$ and two classes of complete graphs $K_{n_1}, \ldots, K_{n_r}$ (possibly no complete graphs of this kind) and $K_{m_1}, \ldots, K_{m_s}$ (at least two complete graphs of this kind if there is no complete graph of the first kind), by joining $u$ to all vertices of complete graphs and $v$ to all vertices of $K_{m_1}, \ldots, K_{m_s}$.

Finally assume that $\mathcal{F}_4$ is the collection of all graphs obtained from $K_2 = xy$ and complete graphs $K_{n_1}, \ldots, K_{n_r}$, $K_{m_1}, \ldots, K_{m_s}$, and $K_{l_1}, \ldots, K_{l_t}$ (may be no complete graph of this kind) by joining $x$ and $y$ to all vertices of $K_{l_1}, \ldots, K_{l_t}$, and joining $x$ to all vertices of $K_{m_1}, \ldots, K_{m_s}$ and $y$ to all vertices of $K_{m_1}, \ldots, K_{m_s}$.
Theorem 2.2. Let \( G \) be a connected graph of order \( n \) with \( \text{diam}(G) = 2 \). Then \( g(G) = n - 2 \) if and only if \( G \cong C_3 \) or \( G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \).

Proof. If \( G = C_4 \) or \( G = C_5 \), then clearly \( g(G) = n - 2 \). If \( G \in \mathcal{F}_3 \) then by Observation 1.1, \( g(G) = n - 2 \) because every vertex of \( V(G) - \{u, v\} \) is a simplicial vertex of \( G \). Now let \( G \in \mathcal{F}_2 \) and \( S \) be a \( g(G) \)-set. By Observation 1.1, \( V(G) - \{v_1, v_2, v_3, v_4\} \subseteq S \). If \( v_3, v_4 \in S \), then \( g(G) = |S| \geq n - 2 \). If \( v_3 \notin S \) (the case \( v_4 \notin S \) is similar), then we must have \( v_2, v_4 \in S \) implying that \( g(G) = |S| \geq n - 2 \). On the other hand, \( V(G) - \{v_1, v_2\} \) is a geodetic set of \( G \) that implies \( g(G) = n - 2 \). Finally let \( G \in \mathcal{F}_2 \) and \( S \) be a \( g(G) \)-set. By Observation 1.1, \( V(G) - \{u, v, w\} \subseteq S \). Since \( w \notin I[w_1, w_2] \) for each \( w_1, w_2 \in V(G) - \{u, v, w\} \), we deduce that \( V(G) - \{u, v, w\} \subseteq S \) and so \( g(G) = |S| \geq n - 2 \). On the other hand, \( V(G) - \{u, v\} \) is a geodetic set of \( G \) that yields \( g(G) = n - 2 \).

Conversely, let \( G \) be a connected graph of order \( n \), \( \text{diam}(G) = 2 \) and \( g(G) = n - 2 \). Suppose that \( S = V(G) - \{x_1, x_2\} \) is a \( g(G) \)-set. Since, \( x_i \notin S \), there exists a \( u_i - v_i \) geodesic path containing \( x_i \) for \( i = 1, 2 \). Further, let among the \( g(G) \)-sets \( S \), the one be selected such that \( |[u_1, v_1] \cap [u_2, v_2]| \) is as large as possible and \( x_1, x_2 \notin E(G) \) if possible. We consider two cases:

Case 1: \( |[u_1, v_1] \cap [u_2, v_2]| = 2 \).

We assume that \( u_1 = u_2 \) and \( v_1 = v_2 \). Then \( [x_1, x_2] \subseteq N(u_1) \cap N(v_1) \). On the other hand, since \( V(G) - \{N(u_1) \cap N(v_1)\} \) is a geodetic set of \( G \) and since \( g(G) = n - 2 \), we deduce that

\[
N(u_1) \cap N(v_1) = [x_1, x_2].
\]

If \( n = 4 \), then \( G = C_4 \) and hence \( G \in \mathcal{F}_1 \). Let \( n \geq 5 \). Consider the following subcases.

Subcase 1.1. \( x_1, x_2 \notin E(G) \).

Then \( d_G(x_1, x_2) = 2 \). An argument similar to that described above, we obtain

\[
N(x_1) \cap N(x_2) = \{u_1, v_1\}.
\]

Let \( w \) be an arbitrary vertex in \( V(G) - \{u_1, v_1, x_1, x_2\} \). Since \( g(G) = n - 2 \), \( w \) must be adjacent to some vertex in \( \{u_1, v_1, x_1, x_2\} \). It may assume \( w \in N(u_1) \setminus \{x_1, x_2\} \). It follows from \( N(u_1) \cap N(v_1) = [x_1, x_2] \) and the fact \( d_G(w, v_1) \leq 2 \) that \( wv_1 \notin E(G) \) and \( v_1 \) and \( w \) have a common neighbor, say \( y \). If \( y \notin [x_1, x_2] \), then \( V(G) - [x_1, x_2, y] \) is a geodetic set of \( G \) which is a contradiction. Therefore \( y \in [x_1, x_2] \) and hence \( w \in N(x_1) \) or \( w \in N(x_2) \). By (2), \( w \) in \( N(x_1) \setminus N(x_2) \) or \( w \in N(x_2) \setminus N(x_1) \). We claim that \( N(u_1) - \{x_1, x_2\} \subseteq N(x_1) \setminus N(x_2) \) or \( N(u_1) - \{x_1, x_2\} \subseteq N(x_2) \setminus N(x_1) \). Suppose \( w_1 \in N(u_1) \cap (N(x_1) \setminus N(x_2)) \) and \( w_2 \in N(u_1) \cap (N(x_2) \setminus N(x_1)) \). If \( w_1w_2 \in E(G) \), then \( x_2 \notin I[w_1, w_2] \) and \( [u_1, w_1] \subseteq I[w_1, w_2] \) that implies \( V(G) - [x_2, w_1, u_1] \) is a geodetic set of \( G \), a contradiction. Let \( w_1w_2 \notin E(G) \). Then \( u_1 \notin I[w_1, w_2] \), \( x_1 \notin I[w_1, v_1] \) and \( x_2 \notin I[w_2, v_1] \) and so \( V(G) - [x_1, x_2, u_1] \) is a geodetic set of \( G \), a contradiction. Assume, without loss of generality, that

\[
N(u_1) \setminus \{x_1, x_2\} \subseteq N(x_1) \setminus N(x_2).
\]
Similarly, we have

\[ N(x_1) \setminus \{u_1, v_1\} \subseteq N(u_1) \setminus N(v_1) \quad (4) \]

and

\[ N(v_1) \setminus \{x_1, x_2\} \subseteq N(x_1) \setminus N(x_2) \text{ or } N(v_1) \setminus \{x_1, x_2\} \subseteq N(x_2) \setminus N(x_1). \quad (5) \]

Next we show that \( \deg(v_1) = 2 \). Suppose \( z \in N(v_1) - \{x_1, x_2\} \). By (5), we deduce that \( z \neq E(G) \) and \( z \neq E(G) \) or \( z \neq E(G) \) and \( z \neq E(G) \). First let \( z \neq E(G) \) and \( z \neq E(G) \) or \( z \neq E(G) \). By (1), \( w \neq \#z \). If \( w \neq E(G) \), then \( V(G) - \{x_1, x_2, w\} \) is a geodetic set of \( G \), and if \( \neq E(G) \), then \( u \neq E(G) \) and \( v \neq E(G) \) and \( v \neq E(G) \). Now let \( z \neq E(G) \) and \( z \neq E(G) \). If \( \neq E(G) \) then we get a contradiction as above. Let \( w \neq E(G) \). Then \( w \) and \( w \) have a common neighbor \( y \) not in \( \{x_1, x_2\} \) and so \( u \in I[w, x_2] \) by (3), \( y \in I[w, z] \) and \( v \in I[x_1, z] \). This implies that \( V(G) - \{u_1, v_1, y\} \) is a geodetic set of \( G \), a contradiction. Thus \( \deg(v_1) = 2 \). By symmetry, we must have \( \deg(x_2) = 2 \). Since \( g(G) = n - 2 \), we deduce from Proposition 1.4 that the components of \( G[V - \{u_1, x_1, x_2\}] \) are complete graphs and hence \( G \in \mathcal{F}_1 \).

**Subcase 1.2.** \( x_1, x_2 \in E(G) \).

Consider the components of \( G - \{x_1, x_2\} \). Since \( g(G) = n - 2 \), we conclude from Proposition 1.4 that the components of \( G - \{x_1, x_2\} \) are complete graphs.

Now let \( H_{u_1} \) and \( H_{v_1} \) be the components of \( G - \{u_1, v_1\} \) containing \( u_1 \) and \( v_1 \). Clearly, \( H_{u_1} \cap H_{v_1} = \emptyset \), otherwise \( u_1 \) and \( v_1 \) must have a common neighbor in \( H_{u_1} \), a contradiction. Thus \( H_{u_1} \) and \( H_{v_1} \) are disjoint. If \( |H_{u_1}| \leq |V(H_{u_1})| - 2 \) (the case \( |H_{v_1}| \leq |V(H_{v_1})| - 2 \) is similar), then by Observation 1.3, we can choose a geodetic set \( S \) of \( H_{v_1} \) containing \( u_1 \) and \( v_1 \) and \( |V(H_{v_1})| - 1 \) such that the vertex not in \( S \), say \( a \), belongs to a \( x_1 \)-\( y \) geodesic path where \( x, y \in S \) and \( d_G(x, y) = 2 \). Then obviously \( V(G) - \{a, x_1, x_2\} \) is a geodetic set of \( G \) of size at most \( n - 3 \) which is a contradiction. Hence \( |H_{u_1}| \geq |V(H_{u_1})| - 1 \) and \( |H_{v_1}| \geq |V(H_{v_1})| - 1 \) implying that \( H_{u_1} \) and \( H_{v_1} \) are complete graphs or join of \( K_1 \) and at least two pairwise disjoint complete graphs where \( u_1 \) and \( v_1 \) are adjacent to all vertices of \( H_{u_1} \) and \( H_{v_1} \), respectively (cf. [3]).

Let \( w \neq u_1 \) be an arbitrary vertex of \( H_{u_1} \). Since \( w \neq E(G) \) and \( \text{diam}(G) = 2 \), we have \( d_G(v_1, w) = 2 \). If \( v_1 \) and \( w \) have a common neighbor \( z \) not in \( \{x_1, x_2\} \), then \( V(G) - \{x_1, x_2, z\} \) is a geodetic set of \( G \) which is a contradiction. Therefore, we may assume that \( w \neq E(G) \). We show that \( H_{u_1} \subset N(x_1) \). Suppose \( H_{u_1} \) has a vertex \( y \) which is not adjacent to \( x_1 \). As above we must have \( y \neq E(G) \). If \( y \neq E(G) \), then \( V(G) - \{u_1, x_1, v_1\} \) is a geodetic set of \( G \) and if \( y \neq E(G) \), then \( V(G) - \{u_1, w, x_2\} \) is a geodetic set of \( G \), a contradiction. Hence \( H_{u_1} \subset N(x_1) \). If \( H_{u_1} \) has a vertex \( z \neq u_1 \) adjacent to \( x_2 \), then as above we have \( H_{u_1} \subset N(x_2) \). Thus either \( H_{u_1} \neq \{u_1\} \subset N(x_1) \) or \( H_{v_1} \subset N(x_1) \cap N(x_2) \). Similarly, \( H_{v_1} \neq \{v_1\} \subset N(x_1) \cap N(x_2) \).

**Claim.** \( G - \{x_1, x_2\} \) has at most one component \( H \) of order at least \( 2 \) with \( z \in V(H) \) such that \( z \in N(x_1) \cap N(x_2) \) and \( V(H) - \{z\} \subseteq N(x_1) - N(x_2) \) or \( V(H) - \{z\} \subseteq N(x_2) - N(x_1) \).

**Proof.** Let \( H_1 \) and \( H_2 \) be the components of \( G - \{x_1, x_2\} \) of order at least \( 2 \) with \( z \in V(H) \) such that \( z \in N(x_1) \cap N(x_2) \) and \( V(H_1) - \{z\} \subseteq N(x_1) - N(x_2) \) or \( V(H_2) - \{z\} \subseteq N(x_2) - N(x_1) \) for \( i = 1, 2 \). Let \( \{z_i, z_j\} \in E(H) \) for \( i = 1, 2 \). If \( V(H) - \{z_i\} \subseteq N(x_1) \) or \( N(x_2) \) for \( i = 1, 2 \), then \( x_1 \in I[z_i', z_j'] \) and \( z_1 \in I[z_i', z_j'] \) and \( z_2 \in I[z_i', z_j'] \) and \( z_2 \in I[z_i', z_j'] \) and hence \( V(G) - \{x_1, z_1\} \) is a geodetic set of \( G \), a contradiction. If \( V(H) - \{z_i\} \subseteq N(x_1) \) and \( V(H_2) - \{z_2\} \subseteq N(x_2) - N(x_1) \), then \( d_G(z_i, z_j) = 3 \) which is a contradiction again. The other cases also lead to a contradiction.

First assume that \( G - \{x_1, x_2\} \) has no component \( H \) of order at least \( 2 \) with \( z \in V(H) \) such that \( z \in N(x_1) \cap N(x_2) \) and \( V(H) - \{z\} \subseteq N(x_1) - N(x_2) \) or \( V(H) - \{z\} \subseteq N(x_2) - N(x_1) \). Then \( V(H_1) \subset N(x_1) \cap N(x_2) \) and \( V(H_2) \subset N(x_1) \cap N(x_2) \). It will now be shown that \( H_{u_1} \) and \( H_{v_1} \) are complete graphs. Assume to the contrary that \( H_{u_1} \neq \{u_1\} \). Since \( g(G) \geq n - 1 \), we have \( g(H_{u_1}) = n - 1 \). Then \( H_{u_1} = (K_n \cup K_{n-1} \cup \ldots K_r) \cup K_1 \) where \( r \geq 2 \), \( n_1, n_2, \ldots, n_r \) are positive integers with \( n_1 + n_2 + \ldots + n_r = n - 1 \) and \( u_1 \) is adjacent to all vertices of \( H_{u_1} \). Let \( z_1 \) and \( z_2 \) belong to different components of \( H_{u_1} \). Then \( u_1 \in I[z_1, z_2] \) and \( x_1, x_2 \in I[z_1, v_1] \) that implies \( V(G) - \{u_1, x_1, x_2\} \) is a geodetic set of \( G \), a contradiction. Let \( H \) be a component of \( G - \{x_1, x_2\} \) not containing \( u_1, v_1 \), if any. Since \( d_G(V(H), u_1) \leq 2 \), we must have \( z \in E(G) \).
or \( z \in E(G) \) for each \( z \in V(H) \). Assume, without loss of generality, that \( H \) has a vertex \( x \) that is adjacent to \( x_1 \).

We show that \( V(H) \subseteq N(x_1) \). Assume first \( |V(H)| \geq 3 \). Suppose \( H \) has a vertex \( y \) which is not adjacent to \( x_1 \). Since \( d_c(y, u_1) = 2 \), we must have \( y \not \in E(G) \). If \( H \) has a vertex \( z \neq x \) which is adjacent to \( x_1 \), then \( x_i, z \in I[y, x_1] \subseteq I[y, u_1] \) which leads to a contradiction. This implies that \( V(H) = \{ x \} \subseteq N(x_2) \). If \( x_2 \not \in E(G) \), then \( V(H) = \{ x \} \subseteq I[x, H_2] \) and \( x_1 \in I[x, H_1] \), implying that \( V(G) = \{ x_1 \} \cup (V(H) - \{ x \}) \) is a geodetic set of \( G \) which is a contradiction. Hence \( x \in N(x_1) \cap N(x_2) \) and \( V(H) = \{ x \} \subseteq N(x_2) - N(x_1) \) contradicting the assumption. If \( H \) has a vertex adjacent to \( x_2 \), then as above we have \( V(H) \subseteq N(x_2) \). This implies that \( V(H) \subseteq (N(x_1) - N(x_2)) \) or \( V(H) \subseteq (N(x_1) \cap N(x_2)) \).

Now suppose \( |V(H)| = 2 \) and let \( V(H) = \{ x, y \} \). We have \( xx_1 \in E(G) \). If \( yx_1 \in E(G) \), then by assumption we must have \( V(H) \subseteq (N(x_1) - N(x_2)) \) or \( V(H) \subseteq (N(x_1) \cap N(x_2)) \). Let \( yx_1 \not \in E(G) \). Since \( d_c(v_1, y) = 2 \), we must have \( yx_2 \not \in E(G) \). It follows from assumption that \( xx_2 \not \in E(G) \). Then \( V(G) - \{ x_1, y \} \) is a \( G \)-set which contradicts the choice of \( S \).

If \( H_1 \) and \( H_2 \) are components of \( G - \{ x_1, x_2 \} \) not containing \( u_1, v_1 \), such that \( V(H_1) \subseteq (N(x_1) - N(x_2)) \) and \( V(H_2) \subseteq (N(x_2) - N(x_1)) \), then \( d_c(V(H_1), V(H_2)) = 3 \) which contradicts our assumption. Thus for every component \( K \) of \( G - \{ x_1, x_2 \} \) not containing \( u_1, v_1 \), we have \( V(K) \subseteq (N(x_1) - N(x_2)) \) or \( V(K) \subseteq (N(x_1) \cap N(x_2)) \).

Let \( K_m \), \( m = 1, \ldots, r \) be the components of \( G - \{ x_1, x_2 \} \) not containing \( u_1, v_1 \) such that \( V(K_m) \subseteq (N(x_1) \cap N(x_2)) \) for \( 1 \leq m \leq r \) and let \( K_{m_1}, \ldots, K_{m_s} \) be the components of \( G - \{ x_1, x_2 \} \) not containing \( u_1, v_1 \) such that \( V(K_{m_1}) \subseteq (N(x_1) - N(x_2)) \) for \( 1 \leq j \leq s \). It follows that \( G \in F_5 \).

Now let \( G - \{ x_1, x_2 \} \) has exactly one component \( H \) of order at least 2 with \( z \in V(H) \) such that \( z \in N(x_1) \cap N(x_2) \) and \( V(H) \subseteq (N(x_1) - N(x_2)) \) or \( V(H) \subseteq (N(x_1) \cap N(x_2)) \). We may assume, without loss of generality, that \( H = H_m \) and \( V(H_{m_1}) \subseteq N(x_1) - N(x_2) \). An argument similar to that described above shows that for any component \( H = H_{m_1} \) either \( V(H) \subseteq (N(x_1) \cap N(x_2)) \) or \( V(H) \subseteq (N(x_1) \cap N(x_2)) \). Let \( H_{m_1} = K_{m_1}, K_{m_2}, \ldots, K_{m_s} \) be the components of \( G - \{ x_1, x_2 \} \) not containing \( u_1, v_1 \) such that \( V(K_{m_1}) \subseteq (N(x_1) - N(x_2)) \) for \( 1 \leq j \leq r \) and let \( K_{m_1}, \ldots, K_{m_s} \) be the components of \( G - \{ x_1, x_2 \} \) not containing \( u_1, v_1 \) such that \( V(K_{m_1}) \subseteq (N(x_1) \cap N(x_2)) \) for \( 1 \leq j \leq s \). It follows that \( G \in F_2 \).

**Case 2.** ||\( u_1, v_1 \)|| \( \cap ||u_2, v_2|| = 1 \).

Let \( u_1 = u_2 \) and \( v_1 \neq v_2 \).

**Subcase 2.1.** \( x_1, x_2 \not \in E(G) \).

By the choice of \( S \), we have \( v_1, x_1, x_2 \not \in E(G) \). Since \( \text{diam}(G) = 2 \), \( d(v_1, v_2) \leq 2 \). If \( d(v_1, v_2) = 2 \) and \( w \in N(v_1) \cap N(v_2) \), then \( w \not \in \{ x_1, x_2 \} \) and the set \( V(G) - \{ x_1, x_2, w \} \) is a geodetic set of \( G \) which contradicts \( g(G) = n - 2 \). Hence \( v_1, v_2 \not \in E(G) \). Since \( x_1, x_2 \not \in E(G) \), we deduce that the cycle \( (u_1, x_1, v_1, v_2, x_2) \) has no chord. We claim that \( n = 5 \). Suppose \( n \geq 6 \). Since \( G \) is connected, we may choose a vertex \( w \in V(G) - \{ u_1, x_1, v_1, v_2, x_2 \} \) which is adjacent to a vertex in \( \{ u_1, x_1, v_1, v_2, x_1, x_2 \} \). Suppose, without loss of generality, that \( wu_1 \in E(G) \). If \( v_1w \in E(G) \) or \( v_2w \in E(G) \), then \( V(G) - \{ x_1, x_2, w \} \) is a geodetic set of \( G \) which is a contradiction. Therefore \( v_1w \in E(G) \) and \( v_2w \in E(G) \). Since \( d(v_1, w) \leq 2 \), \( w \) and \( v_1 \) have a common neighbor, say \( y \). If \( y \neq x_1 \), then \( V(G) - \{ x_1, x_2, y \} \) is a geodetic set of \( G \) which contradicts \( g(G) = n - 2 \). Hence \( N(w) \cap N(v_1) = \{ x_1 \} \). Similarly, we have \( N(w) \cap N(v_2) = \{ x_2 \} \). Then \( V(G) - \{ v_1, u_1, w \} \) is a geodetic set of \( G \) which contradicts \( g(G) = n - 2 \). Thus \( n = 5 \) and so \( G = C_5 \).

**Subcase 2.2.** \( x_1, x_2 \in E(G) \).

By the choice of \( S \), we must have \( v_1, x_1 \not \in E(G) \) and \( v_2, x_1 \not \in E(G) \). Also, \( v_1 \) and \( v_2 \) are adjacent, for otherwise they have a common neighbor, say \( w \), and \( V(G) - \{ v_1, v_2, w, x_2 \} \) is a geodetic set of \( G \), contradicting the assumption \( g(G) = n - 2 \). Let \( S' = V(G) - \{ v_1, x_2 \} \). Clearly, \( S' \) is a \( G \)-set. Setting \( u_1' = u_2' = x_1 \) and \( v_1' = v_2' = x_2 \), we obtain \( ||u_1', v_1'||, ||u_2', v_2'|| = 2 \) that contradicts the choice of \( S \). Thus this case is impossible.

**Case 3.** \( ||u_1, v_1|| \cap ||u_2, v_2|| = 0 \).

By the choice of \( S \), \( u_1, x_2, v_1, x_2, v_2, x_1 \not \in E(G) \). Since \( \text{diam}(G) = 2 \), \( d(u_1, u_2) \leq 2 \). If \( d(u_1, u_2) = 2 \) and \( w \) is a common neighbor of \( u_1, u_2 \), then \( V(G) - \{ x_1, x_2, w \} \) is a geodetic set of \( G \) which is a contradiction. Hence \( u_1, u_2 \not \in E(G) \). Similarly, we must have \( v_1, v_2 \not \in E(G) \). Then clearly \( V(G) - \{ x_1, u_2, v_2 \} \) is a geodetic set of \( G \), a contradiction. Thus this case is impossible. This completes the proof.
Theorem 2.3. Let G be a connected graph of order n with diam(G) = 3. Then g(G) = n − 2 if and only if G ∈ ℱ₄.

Proof. If G ∈ ℱ₄, then clearly g(G) = n − 2, because every vertex of V(G) − {x, y} is a simplicial vertex of G.

Let g(G) = n − 2 and let uwvw be a diametral path in G. Obviously V(G) − {x, y} is a g(G)-set and N(u) ∩ N(y) = {x} and N(v) ∩ N(x) = {y}. We claim that all components of G − {x, y} are complete graphs. It follows from Proposition 1.4 that the components of G − {x, y} not containing u and v are complete graphs. Now let Hₓ and Hᵧ be the component of G − {x, y} containing u and v, respectively.

If g(Hₓ) ≤ |V(Hₓ)| − 2 (the case g(Hᵧ) ≤ |V(Hᵧ)| − 2 is similar), then by Observation 1.3, we can choose a geodetic set S of Hₓ containing u and size |V(Hₓ)| − 1 such that the vertex not in S, say a, belongs to a w₁ − w₂ geodesic path where w₁, w₂ ∈ S and dₒ(w₁, w₂) = 2. Then obviously V(G) − {a, x, y} is a geodetic set of G of size at most n − 3 which is a contradiction. Hence g(Hₓ) ≥ |V(Hₓ)| − 1 and g(Hᵧ) ≥ |V(Hᵧ)| − 1 implying that Hₓ and Hᵧ are complete graphs or join of K₁ and at least two pairwise disjoint complete graphs.

Suppose Hₓ is not a complete graph. Then Hₓ is the join of K₁ and at least two pairwise disjoint complete graphs. Then clearly u is the central vertex of Hₓ, otherwise the central vertex of Hₓ, say w, lies on some u − w₁ geodesic path implying that V(G) − {x, y, w} is a simplicial vertex of G which is a contradiction. Let z₁ and z₂ belong to different components of Hₓ − {u}. Then dₒ(z₁, z₂) = 2 and u ∈ [u, z₁, z₂]. On the other hand, since dₒ(u, v) = 3, z₁, v ∈ E(G). If N(v) ∩ N(z₁) ≠ ∅, then z₁, x, y, z₁ is a geodetic set of G, a contradiction. Hence N(v) ∩ N(z₁) = ∅. It follows that dₒ(z₁, v) = 3. Similarly, dₒ(z₂, v) = 3. If z₁x′y′v is a diametral path in G, then obviously u, z₂ ∉ {x′, y′}. This implies that V(G) − {u, x′, y′} is a geodetic set of G, a contradiction. Thus Hₓ is a complete graph. Now we claim that each vertex of Hₓ is adjacent to x. Assume to the contrary that some vertex of Hₓ, say w, is not adjacent to x. Since dₒ(u, v) = 3, w ∈ E(G). If N(v) ∩ N(w) ≠ ∅, then w, x, y ∈ [u, v] and so V(G) − {x, y, w} is a geodetic set of G, a contradiction. Therefore N[v] ∩ N[w] = ∅ and hence dₒ(w, v) = 3. Let wx′y′v be a diametral path in G. Since wx ∉ E(G), we have x ≠ x′. Then x, y ∈ [u, v] and x′ ∈ [w, v] that yields V(G) − {x, y, x′} is a geodetic set of G, a contradiction. Thus each vertex of Hₓ is adjacent to x. Similarly, Hᵧ is a complete graph and every vertex of Hᵧ is adjacent to y.

Now let H be a component of G − {x, y} different from Hₓ and Hᵧ. Since G is connected, we may assume, without loss of generality, that H has a vertex v₁ which is adjacent to x. If H has a vertex v₂ that is not adjacent to x, then V(G) − {x, y, w₁} is a geodetic set of G, a contradiction. Thus all vertices of H are adjacent to x. Similarly, if a component of G − {x, y} different from Hₓ and Hᵧ has a vertex that is adjacent to y, then all of its vertices must be adjacent to y.

Let K₁, . . . , Kₘ be the components of G − {x, y} whose vertices are adjacent to x, K₁, . . . , Kₘ be the components of G − {x, y} whose vertices are adjacent to y, and K₁, . . . , Kₘ (possibly there is no such component) be the components of G − {x, y} whose vertices are adjacent to x and y. Thus G ∈ ℱ₄ and the proof is complete. □

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References