Majorization and Doubly Stochastic Operators

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Abstract. We present a close relationship between row, column and doubly stochastic operators and the majorization relation on a Banach space $\ell^p(I)$, where $I$ is an arbitrary non-empty set and $p \in [1, \infty]$. Using majorization, we point out necessary and sufficient conditions that an operator $D$ is doubly stochastic. Also, we prove that if $P$ and $P^{-1}$ are both doubly stochastic then $P$ is a permutation. In the second part we extend the notion of majorization between doubly stochastic operators on $\ell^p(I)$, $p \in [1, \infty)$, and consider relations between this concept and the majorization on $\ell^p(I)$ mentioned above. Moreover, we give conditions that generalized Kakutani’s conjecture is true.

1. Introduction

Theory of majorization plays important role in mathematics and statistics as well as in physics and economics. It has a lot of applications to various fields in mathematics such as matrix and operator theory [1, 9, 12, 13], frame theory [2], graph theory, inequalities involving convex functions [21], etc. We also refer the reader to the classical book of majorization and applications by Marshall, Olkin and Arnold [15].

Given two vectors $x, y \in \mathbb{R}^n$, we say that $x$ is majorized by $y$, and denote it $x \prec y$, if

$$\sum_{i=1}^{k} x_i^* \leq \sum_{i=1}^{k} y_i^* \quad (k = 1, 2, \ldots, n)$$

and

$$\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} y_i^*$$

where $x_1^* \geq x_2^* \geq \ldots \geq x_n^*$ is the decreasing rearrangement of components of a vector $x$.

There are several equivalents for the notion of majorization in finite dimensions. The next well known theorem, which gives an alternative definition for majorization using doubly stochastic matrices, is proved by Hardy, Littlewood and Polya [11]. Namely, $x < y$ if and only if there is doubly stochastic matrix $A$ such that $x = Ay$. Recall that a square matrix with non-negative real entries is called doubly stochastic, if each of its row sums and each of its column sums are equal 1.

2010 Mathematics Subject Classification. Primary 47B60, 15B51.
Keywords. Majorization, Row, column and doubly stochastic operators, Permutation.
Received: 12 June 2015; Accepted: 01 July 2015
Communicated by Dragan S. Djordjević
Research is supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, grant no. 174007.
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Recently, Bahrami, Bayati and Manjegani in their joint papers [4, 5] used doubly stochastic operators to extend the notion of majorization on Banach spaces $\ell^\infty$ and $\ell^p(l)$, where $l$ is an arbitrary non-empty set, $p \in [1, \infty)$, and they characterized the structure of all bounded linear maps which preserve majorization. They presented some helpful results that we will use in this paper. Some topological properties of this structure can be found in [6]. Also, the convex majorization on $\ell^p(l)$ was introduced in [7]. The other extensions of majorization and applications to diagonals of self-adjoint, positive and compact operators are considered in [3, 14, 16].

In finite dimensions, there is a very useful close relationship between majorization and doubly stochastic matrices.

**Theorem 1.1.** [15, Theorem I.2.A.4] An $n \times n$ matrix $A$ is doubly stochastic if and only if $Ax < x$ for all vectors $x \in \mathbb{R}^n$.

The next theorem is proved by Snijders [20], Berge [8], and Farahat [10] using totally different approaches.

**Theorem 1.2.** Let $P$ be an invertible matrix. If $P$ and $P^{-1}$ are both doubly stochastic then $P$ is a permutation matrix.

In Section 3 we will present generalized versions of Theorem 1.1 and Theorem 1.2. More precisely, let $p \in [1, \infty)$ and $A : \ell^p(l) \rightarrow \ell^p(l)$ be a bounded linear operator. It will be shown that:

- If $A \in DS(\ell^p(l))$ then $Af < f$, $\forall f \in \ell^p(l)$ (Lemma 3.7).
- If $Af < f$, $\forall f \in \ell^p(l)$ then $A \in CS(\ell^p(l))$ (Lemma 3.7).
- $A \in DS(\ell^p(l))$ if and only if $Af < f$, $\forall f \in \ell^p(l)$, and $A^* g < g$, $\forall g \in \ell^p(l)$, where $q$ is the conjugate exponent of $p$ (Theorem 3.8).
- If $P \in DS(\ell^p(l))$ and $P^{-1} \in DS(\ell^p(l))$ then $P$ is permutation (Theorem 3.12).

Sherman [18] introduced a pre-order among $n \times n$ doubly stochastic matrices by defining

$$D_1 < D_2$$

if there is a doubly stochastic matrix $D_3$ such that

$$D_1 = D_3 D_2.$$

It is easy to see that $D_1 < D_2$ implies $D_1 x < D_2 x$, $\forall x \in \mathbb{R}^n$. Kakutani has raised the conjecture that the opposite direction is also true:

**If** $(\forall x \in \mathbb{R}^n)$ $D_1 x < D_2 x$, **then** $D_1 < D_2$.

In [18] it is provided that if doubly stochastic matrix $D_2$ is invertible then Kakutani’s conjecture is true. Horn found counterexample [19] of Kakutani’s conjecture and showed that invertibility of operator $D_2$ cannot be omitted. Using a different technique, Schreiber [17] verified Sherman’s result.

In Section 4 we will extend the pre-order (1) on doubly stochastic operators on $\ell^p(l)$ and we will provide that Kakutani’s conjecture is true if $D_2^{-1} \in RS(\ell^p(l))$. More precisely, let $p \in [1, \infty)$ and $D_1, D_2 \in DS(\ell^p(l))$. If $D_1 < D_2$ then $D_1 f < D_2 f$, $\forall f \in \ell^p(l)$. Conversely, if $D_2$ is invertible and $D_2^{-1} \in RS(\ell^p(l))$ then

- $(\forall f \in \ell^p(l))$ $D_1 f < D_2 f$ implies $D_1 < D_2$ (Theorem 4.4).
2. Notations and Preliminaries

We will consider real-valued functions \( f : I \rightarrow \mathbb{R} \), where \( I \) is an arbitrary non-empty set. We say that an arbitrary function \( f \) is summable if there exists a real number \( \sigma \) with the following property:

Given \( \epsilon > 0 \), we can find a finite set \( J_0 \subseteq I \) such that

\[
\left| \sigma - \sum_{j \in J} f(j) \right| \leq \epsilon
\]

whenever \( J \) is a finite set and \( J_0 \subseteq J \). Then \( \sigma \) is called the sum of \( f \) and we denote it by \( \sigma = \sum_{i \in I} f(i) \).

We will denote by \( \ell^p(I) \), where \( I \) is non-empty set and \( p \in [1, \infty) \), the Banach space of all functions \( f : I \rightarrow \mathbb{R} \) such that \( \sum_{i \in I} |f(i)|^p < \infty \), equipped with standard p-norm

\[
\|f\|_p := \left( \sum_{i \in I} |f(i)|^p \right)^{\frac{1}{p}} < \infty
\]

As we know, every function \( f \in \ell^p(I), p \in [1, \infty) \), can be represented in the form \( f = \sum_{i \in I} f(i)e_i \), where the function \( e_i : I \rightarrow \mathbb{R} \) is defined by Kronecker delta, i.e., \( e_i(j) = \delta_{ij}, i \in I \). Let \( p, q \in (1, \infty) \). A number \( q \) is conjugate (dual) exponent of \( p \) if \( \frac{1}{p} + \frac{1}{q} = 1 \). Furthermore, the exponents 1 and \( \infty \) are considered to be dual exponents to each other. Let \( p \in [1, \infty) \) and let \( q \) be a dual exponent of \( p \). Then, for every function \( g \in \ell^q(I) \), the rule \( f \mapsto \langle f, g \rangle := \sum_{i \in I} f(i)g(i) \) defines a functional on \( \ell^q(I) \). In this way, the dual Banach space \( \ell^q(I)' \) can be identified with \( \ell^p(I) \) because \( \ell^q(I)' \) is isometrically isomorphic with \( \ell^p(I) \). Also, using the dual pairing \( \langle \cdot, \cdot \rangle : \ell^q(I) \times \ell^p(I) \rightarrow \mathbb{R} \), for functions \( f \in \ell^q(I) \) and \( e_i \), (considering \( e_i \) as an element of the dual space of \( \ell^p(I) \)) we have

\[
f_i = \langle f, e_i \rangle, \quad \forall i \in I.
\]

Hence,

\[
f = \sum_{i \in I} \langle f, e_i \rangle e_i
\]

Let \( A : \ell^q(I) \rightarrow \ell^p(I) \) be a bounded linear operator, where \( p \in [1, \infty) \). The space \( \ell^q(I) \) is an ordered Banach space under the natural partial ordering on the set of real valued functions defined on \( I \), so an operator \( A \) is called positive if \( A g \geq 0 \) for every \( g \geq 0 \).

An operator \( A^* : \ell^q(I) \rightarrow \ell^p(I) \) is the adjoint operator of \( A : \ell^q(I) \rightarrow \ell^p(I), p \in [1, \infty) \), if \( \langle Af, g \rangle = \langle f, A^*g \rangle \), where \( f \in \ell^q(I), g \in \ell^p(I) \) and \( q \) is the conjugate exponent of \( p \).

We recall definitions of row, column, doubly stochastic operators and majorization when \( p \in [1, \infty) \) introduced by Bahrami, Bayati, Manjegani [4].

**Definition 2.1.** [4, Definition 2.1.] Let \( p \in [1, \infty) \) and \( A : \ell^q(I) \rightarrow \ell^p(I) \) be a bounded linear operator. The operator \( A \) is called

- **row stochastic**, if \( A \) is positive and \( \forall i \in I \sum_{j \in I} \langle Ae_j, e_i \rangle = 1 \).
- **column stochastic**, if \( A \) is positive and \( \forall j \in I \sum_{i \in I} \langle Ae_j, e_i \rangle = 1 \).
- **doubly stochastic**, if \( A \) is both row and column stochastic.
- a **permutation**, if there exists a bijection \( \theta : I \rightarrow I \) for which \( Ae_j = e_{\theta(j)} \), for each \( j \in I \).

The set of all row stochastic, column stochastic, doubly stochastic operators and permutations on \( \ell^p(I), p \in [1, \infty) \) are denoted, respectively, by \( \text{RS}(\ell^p(I)) \), \( \text{CS}(\ell^p(I)) \), \( \text{DS}(\ell^p(I)) \) and \( \text{P}(\ell^p(I)) \).

**Definition 2.2.** [4, Definition 3.1.] Let \( p \in [1, \infty) \). For two elements \( f, g \in \ell^p(I) \), we say that \( f \) is majorized by \( g \), if there exists a doubly stochastic operator \( D \in \text{DS}(\ell^p(I)) \), such that \( f = Dg \), and denote it by \( f \prec g \).
Also, we will consider the Banach space of all functions \( f : I \to \mathbb{R} \) for which \( \sup_{i \in I} |f(i)| < \infty \), where \( I \) is a non-empty set, equipped with supremum norm
\[
\|f\|_\infty := \sup_{i \in I} |f(i)|
\]
and denote it by \( \ell^\infty(I) \).

Recently, Bahrami, Bayati, Manjegani [5] introduced the notions of majorization and doubly stochastic operators on the Banach space of all bounded real sequences \( \ell^\infty \). This Banach space is a particular case of Banach space \( \ell^\infty(I) \), when \( I \) is a countable set, that is, when \( I = \mathbb{N} \). In the Section 3, we will generalize these notions when \( I \) may be an uncountable set.

**Definition 2.3.** [5, Definition 2.1.] Let \( A : \ell^\infty \to \ell^\infty \) be a linear and bounded operator. The operator \( A \) is called doubly stochastic if there is a doubly stochastic operator \( A_0 \in DS(\ell^1) \) such that \( A = A_0^* \).

**Definition 2.4.** [5, Definition 2.4.] For two elements \( f, g \in \ell^\infty \), we say that \( f \) is majorized by \( g \), if there exists a doubly stochastic operator \( D \in DS(\ell^\infty) \), such that \( f = Dg \), and denote it by \( f \prec g \).

Now, we present useful results that we will use in this paper.

**Lemma 2.5.** [4, Lemma 2.3.] Let \( p \in [1, \infty) \) and \( A : \ell^p(I) \to \ell^p(I) \) be a positive bounded linear operator. Then
\begin{itemize}
  \item \( A \) is row stochastic, if and only if
    \[ \forall f \in \ell^1(I), \quad \sum_{j \in I} (Ae_j, f) = \sum_{i \in I} f(i) \]
  \item \( A \) is column stochastic, if and only if
    \[ \forall f \in \ell^1(I), \quad \sum_{i \in I} (Af, e_i) = \sum_{j \in I} f(j) \]
\end{itemize}

**Theorem 2.6.** [4, Theorem 2.4.] Let \( p \in [1, \infty) \). If \( A \) and \( B \) belong to \( RS(\ell^p(I)) \), then so does \( AB \), i.e. the set \( RS(\ell^p(I)) \) is closed under the composition. The same conclusion holds for sets \( CS(\ell^p(I)) \) and \( DS(\ell^p(I)) \).

**Theorem 2.7.** [4, Theorem 3.5.] For \( f, g \in \ell^p(I), p \in [1, \infty) \) the following conditions are equivalent:
\begin{itemize}
  \item \( f \prec g \) and \( g = f \).
  \item There exists a permutation \( P \in P(\ell^p(I)) \) such that \( f = Pg \).
\end{itemize}

3. Majorization and Doubly Stochastic Operators

We prove the following results.

**Lemma 3.1.** The majorization relation \( \prec \) in Definition 2.2, when \( p \in [1, \infty) \), is reflexive and transitive relation i.e. \( \prec \) is a pre-order. In particular, if we identify all functions which are different up to the permutation then we may consider \( \prec \) as a partial order.

**Proof.** For any \( f \in \ell^p(I), f = I f \) implies \( f \prec f \), because the identity operator \( I \) is doubly stochastic, so \( \prec \) is reflexive.

Transitivity follows from Theorem 2.6. Precisely, if \( f \prec g \) and \( g \prec h \) then there are \( A, B \in DS(\ell^p(I)) \) such that \( f = Ag, g = Bh \) so \( f = ABh \). Since \( DS(\ell^p(I)) \) is closed under the composition hence \( f \prec h \).

If we identify all function which are different up to the permutation, then it follows directly from Theorem 2.7 that \( \prec \) is antisymmetric. \( \square \)
In the next definitions we will introduce row and column stochastic operators, and we also extend the notions of doubly stochastic operators and majorization on $\ell^\infty(I)$.

**Definition 3.2.** Let $A : \ell^\infty(I) \to \ell^\infty(I)$ be a bounded linear operator. The operator $A$ is called
- **row stochastic**, if there is a column stochastic operator $A_0 \in CS(\ell^1(I))$ such that $A = A_0^*$.
- **column stochastic**, if there is a row stochastic operator $A_0 \in RS(\ell^1(I))$ such that $A = A_0^*$.
- **doubly stochastic**, if there is a doubly stochastic operator $A_0 \in DS(\ell^1(I))$ such that $A = A_0^*$.

As we know, $A = A_0^*$ if for every $f \in \ell^\infty(I)$ and for every $g \in \ell^1(I)$, $\langle g, Af \rangle = \langle A_0g, f \rangle$, where $\langle \cdot, \cdot \rangle : \ell^1(I) \times \ell^\infty(I) \to \mathbb{R}$ denotes the dual pairing between $\ell^1(I)$ and its dual space $\ell^\infty(I)$.

The set of all row stochastic, column stochastic and doubly stochastic operators on $\ell^\infty(I)$ will denote, respectively, by $RS(\ell^\infty(I))$, $CS(\ell^\infty(I))$ and $DS(\ell^\infty(I))$.

**Theorem 3.3.** Let $p \in [1, \infty]$. Then, $RS(\ell^p(I))$, $CS(\ell^p(I))$ and $DS(\ell^p(I))$ are convex sets.

**Proof.** Let $p \in [1, \infty]$ and $A, C \in RS(\ell^p(I))$. We claim that $B = tA + (1 - t)C \in RS(\ell^p(I))$ for each $t \in [0, 1]$. For $t = 0$ or $t = 1$, $B \in RS(\ell^p(I))$ obviously. Let $t \in (0, 1)$. It is easy to see that $B$ is a bounded linear operator on $\ell^p(I)$.

Firstly, let $p \in [1, \infty)$. Since $A$ and $C$ are positive operators, if $f \in \ell^p(I)$, $f \geq 0$, then $Af \geq 0$, $Cf \geq 0$, so $B$ is also positive. Now, for arbitrary $i \in I$ we have

$$
\sum_{j \in I} \langle Be_j, e_i \rangle = \sum_{j \in I} \langle (tA + (1 - t)C)e_j, e_i \rangle = t \sum_{j \in I} \langle Ae_j, e_i \rangle + (1 - t) \sum_{j \in I} \langle Ce_j, e_i \rangle = 1
$$

since $A$ and $C$ are row stochastic, hence $B$ is row stochastic, so $RS(\ell^p(I))$ is convex set.

We now turn to the case $p = \infty$. As we know, there are $A_0, C_0 \in CS(\ell^1(I))$ such that $A = A_0^*$ and $C = C_0^*$, by Definition 3.2. Combining these facts and the similar argument as in (2), we obtain

$$
\sum_{j \in I} \langle Be_j, e_i \rangle = t \sum_{j \in I} \langle A_0^*e_j, e_i \rangle + (1 - t) \sum_{j \in I} \langle C_0^*e_j, e_i \rangle = t \sum_{j \in I} \langle e_j, A_0e_i \rangle + (1 - t) \sum_{j \in I} \langle e_j, C_0e_i \rangle = 1,
$$

so $RS(\ell^\infty(I))$ is convex set.

In the same way we conclude that if $A, C \in CS(\ell^p(I))$, $p \in [1, \infty)$, then $B \in CS(\ell^p(I))$ so $CS(\ell^p(I))$ is convex set. Clearly, $DS(\ell^p(I))$ is convex set. \(\square\)

**Theorem 3.4.** The set $RS(\ell^\infty(I))$ is closed under the composition, i.e. if $A, B \in RS(\ell^\infty(I))$, then $AB \in RS(\ell^\infty(I))$. The same conclusion holds for sets $CS(\ell^\infty(I))$ and $DS(\ell^\infty(I))$.

**Proof.** Let $A, B \in RS(\ell^\infty(I))$. There exist $A_0, B_0 \in CS(\ell^1(I))$ such that $A = A_0^*$ and $B = B_0^*$ by Definition 3.2. We obtain

$$
\sum_{j \in I} \langle AB e_j, e_i \rangle = \sum_{j \in I} \langle A_0 B_0^* e_j, e_i \rangle = \sum_{j \in I} \langle B_0^* e_j, A_0 e_i \rangle = \sum_{j \in I} \langle e_j, B_0 A_0 e_i \rangle = 1
$$

because $B_0 A_0 \in CS(\ell^1(I))$ by Theorem 2.6. The rest is left to the reader. \(\square\)

**Definition 3.5.** For two elements $f, g \in \ell^\infty(I)$, we say that $f$ is majorized by $g$, if there exists a doubly stochastic operator $D \in DS(\ell^\infty(I))$, such that $f = Dg$, and denote it by $f < g$. 


Lemma 3.6. The majorization relation "≺" introduced in Definition 3.5, is reflexive and transitive relation i.e. "≺" is a pre-order.

Proof. Obviously, reflexivity holds. Transitivity follows from Theorem 3.4. □

We can not consider majorization relation "≺" on $ℓ^p(I)$ as a partial order, because in [5, Example 2.6.] there are specific functions such that $f < g$ and $g < h$ do not follow that $f$ and $g$ are permutations of each other.

Lemma 3.7. Let $p \in [1, \infty)$ and $A : ℓ^p(I) \rightarrow ℓ^p(I)$ be a bounded linear operator. If $A \in DS(ℓ^p(I))$ then $Af < f$, $∀f \in ℓ^p(I)$. Conversely, if $Af < f$, $∀f \in ℓ^p(I)$, then $A \in CS(ℓ^p(I))$.

Proof. Let $A \in DS(ℓ^p(I))$. If we define $D := A$ it is easy to see that $Af = Df$ and $D \in DS(ℓ^p(I))$, therefore $Af < f$, $∀f \in ℓ^p(I)$.

On the other hand, let $Af < f$, $∀f \in ℓ^p(I)$. Firstly, we will show that the operator $A$ is positive. Suppose that there exists $0 ≤ f \in ℓ^p(I)$ such that $Af ≥ 0$ is not true. If $f$ has a representation $f = \sum_{i=1}^{\infty} (f, e_i)e_i$ then $Af = \sum_{i=1}^{\infty} (f, e_i)e_i$ by continuity of $A$. Then, from $(f, e_i) ≥ 0$, $∀i \in I$, there exists $m \in I$, such that $Ae_m ≥ 0$ is not true. However, for $e_m$ exists $D_{e_m} \in DS(ℓ^p(I))$ such that $Ae_m = D_{e_m} ≥ 0$ because $e_m ≥ 0$ and $D$ is positive. This is contradiction, so the operator $A$ is positive.

Again, let $Af < f$, $∀f \in ℓ^p(I)$. Therefore, $Ae_j < e_j$, $∀j \in I$, and there exist $D_j \in DS(ℓ^p(I))$ such that $D_i e_j = A e_j$, $∀j \in I$. Let $j \in I$ is arbitrary chosen. Then, $\sum_{i=1}^{\infty} (A e_j, e_i) = \sum_{i=1}^{\infty} (D_i e_j, e_i) = 1$, because $D_j \in DS(ℓ^p(I)) \subset CS(ℓ^p(I))$, so $A \in CS(ℓ^p(I))$. □

The next theorem gives the necessary and sufficient conditions for a bounded linear operator to be doubly stochastic.

Theorem 3.8. Let $p \in [1, \infty)$ and $A : ℓ^p(I) \rightarrow ℓ^p(I)$ be a bounded linear operator. $A \in DS(ℓ^p(I))$ if and only if $Af < f$, $∀f \in ℓ^p(I)$ and $A^* g < g$, $∀g \in ℓ^q(I)$, where $q$ is the conjugate exponent of $p$.

Proof. Let $A \in DS(ℓ^p(I))$ where $p \in [1, \infty)$ is arbitrary chosen. Obviously, $Af < f$, $∀f \in ℓ^p(I)$, by Lemma 3.7.

If $p \in (1, \infty)$ then $(A^* e_j, e_i) = (A e_j, e_i) ≥ 0$, $∀i, j \in I$, and

$$A^* g = \sum_{i=1}^{\infty} (g, e_i)A^* e_i ≥ 0,$$

where $0 ≤ g \in ℓ^q(I)$. Thus, the operator $A^*$ is positive. Using definitions of adjoint operator and doubly stochastic operators it is easy to see that $A^* \in DS(ℓ^q(I))$. Hence $A^* g < g$, $∀g \in ℓ^q(I)$ by Lemma 3.7. If $p = 1$ and $q = \infty$ then $A^* \in DS(ℓ^\infty(I))$ by Definition 3.2 of doubly stochastic operators on $ℓ^\infty(I)$ and fact that $A \in DS(ℓ^\infty(I))$ by assumption. Now, for $D := A^*$ is $A^* g = D g$, $∀g \in ℓ^q(I)$, so $A^* g < g$, $∀g \in ℓ^q(I)$, by Definition 3.5.

Conversely, let $Af < f$, $∀f \in ℓ^p(I)$, $p \in [1, \infty)$, and $A^* g < g$, $∀g \in ℓ^q(I)$. Again, using Lemma 3.7 is $A \in CS(ℓ^p(I))$. It remains to be shown that $A \in CS(ℓ^p(I))$. If $p \neq 1$ then $q \neq \infty$, hence $A^* \in CS(ℓ^q(I))$, by Lemma 3.7. Moreover, $\sum_{i=1}^{\infty} (A e_j, e_i) = \sum_{i=1}^{\infty} (e_j, A^* e_i) = 1$, $∀i \in I$ so $A \in RS(ℓ^p(I))$. If $p = 1$ and $q = \infty$, then there are $D_i \in DS(ℓ^\infty(I))$ and $B_i \in DS(ℓ^1(I))$, $∀i \in I$, such that $D_i e_j = A^* e_i$ and $D_i = B^*_i$, $∀i \in I$. Therefore,

$$\sum_{i=1}^{\infty} (A e_j, e_i) = \sum_{i=1}^{\infty} (e_j, D_i e_i) = \sum_{i=1}^{\infty} (B_i e_j, e_i) = 1$$

$∀i \in I$, so $A \in RS(ℓ^1(I))$. It follows that $A \in DS(ℓ^p(I))$, $p \in [1, \infty)$. □

Corollary 3.9. Let $p \in [1, \infty)$ and $A : ℓ^p(I) \rightarrow ℓ^p(I)$ be a bounded linear operator. If $A \in DS(ℓ^p(I))$ then $A^* \in DS(ℓ^q(I))$, where $q$ is the conjugate exponent of $p$.

Proof. Follows directly from proof of Theorem 3.8. □
Corollary 3.10. Let \( p \in [1, \infty) \) and \( A : \ell^p(I) \to \ell^p(I) \) be a bounded linear operator. If \( Af < f, \forall f \in \{e_i, i \in I\} \), then \( A \in CS(\ell^p(I)) \). If additionally, \( A^*f < f, \forall f \in \{e_i, i \in I\} \), then \( A \in DS(\ell^p(I)) \).

Proof. Analysing the proofs in above results, Lemma 3.7 and Theorem 3.8, it is easy to check that \( Af < f \) and \( A^*f < f \), where \( f \in \{e_i, i \in I\} \) are sufficient conditions that operator \( A \) is doubly stochastic. \( \Box \)

In finite dimensional case, when \( \text{card}(I) = n \in \mathbb{N} \) and when \( p = 2 \), a doubly stochastic operator \( A \) becomes a doubly stochastic matrix \( A = [a_{ij}] \in M_n \) and \( A^* = A^T \). If \( Ax < x \), for all \( x \in \mathbb{R}^n \), applying the majorization relation to \( e = (1, 1, \ldots, 1) \), we know that \( Ae < e \), so \( Ae = e \) and \( \sum_{i=1}^{n} a_{ij} = 1, \forall i \in \mathbb{N} \). Because of this fact and using Lemma 3.7, \( A \) is a doubly stochastic matrix so \( A^* \) is a doubly stochastic, too, by Corollary 3.9. Now using Lemma 3.7 is \( A^*x < x \), for all \( x \in \mathbb{R}^n \). Thus, the following Corollary 3.11 is a consequence of Theorem 3.8. Notice that Theorem 1.1 is the same as Corollary 3.11.

Corollary 3.11. An \( n \times n \) matrix \( A \) is doubly stochastic if and only if \( Ax < x \) for all vectors \( x \in \mathbb{R}^n \).

Theorem 3.12. Let \( p \in [1, \infty) \) and \( P : \ell^p(I) \to \ell^p(I) \in DS(\ell^p(I)) \). If \( P \) is invertible operator and \( P^{-1} \in DS(\ell^p(I)) \) then \( P \) is permutation.

Proof. Since \( P \in DS(\ell^p(I)) \) hence \( Pf < f \) by Lemma 3.7, for every \( f \in \ell^p(I) \). Similarly, \( P^{-1} \in DS(\ell^p(I)) \) and \( f = P^{-1} Pf \) implies \( f < Pf \) for every \( f \in \ell^p(I) \). It follows by Theorem 2.7 that there exist permutations \( Q_f \in \text{DS}(\ell^p(I)) \) such that \( Q_f Pf = Pf, \forall f \in \ell^p(I) \).

Suppose that there are \( r,s \in I \) such that \( \langle Pe_r, e_r \rangle \in (0,1) \). For \( e_r \), there exists a permutation \( Q_{e_r} \in P(\ell^p(I)) \) for which \( Pe_r = Q_{e_r} e_r = e_r \), for any \( k \in I \).

Now, we obtain \( e_r(r) = \sum_{j} e_r(j) = \sum_{j} e_r(j) e_r(r) = \langle Pe_r, e_r \rangle \in (0,1) \),

which is a contradiction. Thus, \( \langle Pe_j, e_j \rangle \in [0,1], \forall i, j \in I \). Since \( P \in DS(\ell^p(I)) \) we get for every \( i \in I \) that there is only one \( j \in I \) such that \( \langle Pe_j, e_i \rangle = 1 \), and conversely for every \( j \in I \) there is only one \( i \in I \) such that \( \langle Pe_j, e_i \rangle = 1 \). We define the function \( \theta : I \to I \) to be \( \theta(j) = i \), if \( \langle Pe_j, e_i \rangle = 1 \). It is easy to check that \( \theta \) is a bijection and \( Pe_i = e_{\theta(i)} \), hence \( P \) is a permutation. \( \Box \)

4. Generalized Kakutani’s Conjecture

We will introduce a majorization relation between doubly stochastic operators on \( \ell^p(I) \), where \( p \in [1, \infty) \), and thus restate Kakutani’s conjecture for these operators.

Definition 4.1. Let \( p \in [1, \infty) \), and let \( DS(\ell^p(I)) \) be the set of all doubly stochastic operators. For two operators \( D_1, D_2 \in DS(\ell^p(I)) \) we say that \( D_1 \) is majorized by \( D_2 \) and denote it by \( D_1 \triangleleft D_2 \), if there exists a doubly stochastic operator \( D_2 \in DS(\ell^p(I)) \) such that \( D_1 = D_2 D_2 \).

Lemma 4.2. The majorization relation “\( \triangleleft \)” introduced in Definition 4.1 is reflexive and transitive relation i.e. “\( \triangleleft \)” is a pre-order. In particular, if we consider only surjective operators and identify all operators which are different up to the permutation then “\( \triangleleft \)” is an antisymmetric relation on this subset of \( DS(\ell^p(I)) \), so we may consider “\( \triangleleft \)” as a partial order.

Proof. Notice that \( D = ID, \forall D \in DS(\ell^p(I)) \), where \( I \) is the identity operator, which is obviously doubly stochastic, hence “\( \triangleleft \)” is reflexive. Let \( D_1, D_2, D_3 \in DS(\ell^p(I)) \). If \( D_1 \triangleleft D_2 \) and \( D_2 \triangleleft D_3 \) then there are \( D_4, D_5 \in DS(\ell^p(I)) \) such that \( D_1 = D_4 D_2 \) and \( D_2 = D_2 D_5 \). Now, \( D_1 = D_4 D_2 D_5 \) so \( D_1 \triangleleft D_3 \) by Theorem 2.6. Similarly, if \( D_1 \triangleleft D_2 \) and \( D_2 \triangleleft D_3 \) then \( D_1 = ABD_1 \) and \( D_2 = BD_2 B \), where \( D_1 = AD_2, D_2 = BD_1 \) and \( A, B \in DS(\ell^p(I)) \). If \( R(D_1) = R(D_2) = \ell^p(I) \) then \( A = B^{-1} \), so \( A, B \in P(\ell^p(I)) \) by Theorem 3.12. \( \Box \)

If \( D_1 \triangleleft D_2 \), then by definition there exists \( D_3 \in DS(\ell^p(I)) \) such that \( D_1 = D_2 D_3 \) i.e. \( D_1 f = D_2 D_3 f, \forall f \in \ell^p(I) \), so \( D_1 f < D_2 f, \forall f \in \ell^p(I) \). However, the opposite direction is more complicated and it does not hold in general for an arbitrary operator \( D_2 \in DS(\ell^p(I)) \).
Conjecture 4.3. Let \( p \in [1, \infty) \) and \( D_1, D_2 \in DS(\ell^p(I)) \). If for every \( f \in \ell^p(I) \), \( D_1 f \prec D_2 f \), then \( D_1 \prec D_2 \).

This conjecture is not true in general, because the invertibility of operator \( D_2 \) is essential, and necessity of this condition in finite dimensional case is presented in [19] by an counterexample. In accordance with that, the operator \( D_2 \) has to be invertible in our generalization, too. If we show that the existence of the doubly stochastic operators \( D_2^1 \in DS(\ell^p(I)) \), \( \forall f \in \ell^p(I) \), such that \( D_1 f = D_2^1 D_2 f \) implies the existence of the operator \( D_3 \in DS(\ell^p(I)) \) for which \( D_1 = D_3 D_2 \) is true, then the work is done. In the next theorem we will prove our conjecture if the operator \( D_2^{-1} \) is row stochastic.

Theorem 4.4. Let \( p \in [1, \infty) \), \( D_1, D_2 \in DS(\ell^p(I)) \), \( D_2 \) be invertible operator and \( D_2^{-1} \in RS(\ell^p(I)) \). If for every \( f \in \ell^p(I) \), \( D_1 f \prec D_2 f \), then \( D_1 \prec D_2 \).

Proof. Assume that \( D_1 f \prec D_2 f \), for every \( f \in \ell^p(I) \). We know that for arbitrary \( g \in R(D_2) \) there exists \( f \in \ell^p(I) \) such that \( g = D_2 f \).

Let the map \( \Psi : R(D_2) \longrightarrow \ell^p(I) \), be defined by

\[
\Psi g = D_1 f, \quad \forall f \in \ell^p(I)
\]

where \( g = D_2 f \). The domain of the map \( \Psi \) is the entire Banach space \( \ell^p(I) \), because the operator \( D_2 \) is invertible. Thus, \( R(D_2) = \ell^p(I) \) and \( \Psi : \ell^p(I) \longrightarrow \ell^p(I) \), \( p \in [1, \infty) \). Choose arbitrary \( f, f_1, f_2 \in \ell^p(I) \) and fix \( g = D_2 f, g_1 = D_2 f_1 \) and \( g_2 = D_2 f_2 \).

Let us first check that the map \( \Psi \) is well-defined. Suppose that \( g_1 = g_2 \). We need to show that \( \Psi g_1 = D_1 f_1 = D_1 f_2 = \Psi g_2 \). From \( D_2 f_1 = D_2 f_2 \), it follows \( D_2 (f_1 - f_2) = 0 \) by linearity of \( D_2 \), and \( D_1 (f_1 - f_2) \prec D_2 (f_1 - f_2) \) by our assumption. Now, \( D_1 (f_1 - f_2) = 0 \) so \( \Psi g_1 = \Psi g_2 \).

Using linearity of \( D_1 \) and \( D_2 \) we obtain

\[
\Psi(\alpha g) = \Psi(D_2(\alpha f)) = D_1(\alpha f) = \alpha \Psi g
\]

where \( \alpha \in \mathbb{R} \).

\[
\Psi(g_1 + g_2) = \Psi(D_2(f_1 + f_2)) = D_1(f_1 + f_2) = D_1 f_1 + D_1 f_2 = \Psi g_1 + \Psi g_2
\]

so we conclude that the map \( \Psi \) is linear.

\[
\|\Psi(g)\| = \|\Psi(D_2(f))\| = \|D_1 f\| \leq \|D_1\| \|f\| = \|D_1\| \|D_2^{-1} g\|
\]

\[
\leq \|D_1\| \|D_2^{-1}\| \|g\|.
\]

It follows that \( \|\Psi(g)\| \leq \|D_1\| \|D_2^{-1}\| \|f\| \) and \( \|\Psi\| \leq \|D_1\| \|D_2^{-1}\| \), so we obtain that the map \( \Psi \) is bounded. Moreover, \( \|\Psi\| \leq \|D_2^{-1}\| \) because the norm of a doubly stochastic operator is at most 1, by [4, Lemma 2.5].

If additionally \( g = D_2 f \geq 0 \), it follows that exists \( D_2^1 \in DS(\ell^p(I)) \) such that \( \Psi(g) = D_1 f = D_2^1 D_2 f = D_2^1 g \geq 0 \), because the operator \( D_2^1 \) is positive, so \( \Psi \) is positive.

Let \( j \in I \) be arbitrary chosen, and \( a_j = D_2^{-1} e_j \in \ell^p(I) \). Then for this \( a_j \) there exists \( D_2^1 a_j \in DS(\ell^p(I)) \) such that \( D_1 a_j = D_2^1 a_j D_2 a_j, \) i.e. \( D_1 D_2^{-1} e_j = D_2^1 e_j \). Now,

\[
\sum_{j \in I} (\Psi e_j, e_i) = \sum_{j \in I} (D_1 D_2^{-1} e_j, e_i) = \sum_{j \in I} (D_2^1 e_j, e_i) = 1
\]

since \( D_2^1 \in DS(\ell^p(I)) \), hence \( \Psi \in CS(\ell^p(I)) \). The operator \( D_2^1 \) is doubly stochastic by Corollary 3.9, and \( \sum_{k \in I} D_2^1 e_k(k) = \sum_{j \in I} (D_2^1 e_j) = \sum_{j \in I} (e_i, D_2^1 e_j) = 1, \) so \( D_2^1 e_i \in \ell^1(I), \forall i \in I \).

\[
\sum_{j \in I} (\Psi e_j, e_i) = \sum_{j \in I} (D_1 D_2^{-1} e_j, e_i) = \sum_{j \in I} (D_2^1 e_j, D_1^* e_i) = (D_2^* e_i, e_k) = (e_i, D_1 e_k) = 1, \forall i \in I,
\]
where
\[ \sum_{j \in I} \langle D_2^{-1} e_j, D_1^* e_i \rangle = \sum_{k \in I} D_1^* e_i(k) \]  
(3)

by Lemma 2.5, because $D_1^* e_i \in \ell^1(I)$ and $D_2^{-1}$ is a row stochastic operator. Finally, $\Psi \in DS(\ell^p(I))$, thus $D_3 := \Psi$ is a desired operator for which $D_1 = D_3D_2$. It follows that $D_1 \prec D_2$. \[ \square \]

Suppose that $D_2^{-1} \not\in RS(\ell^p(I))$. In this case, there exists $g \in \ell^1(I)$ such that $\sum_{j \in I} \langle D_2^{-1} e_j, g \rangle \neq \sum_{k \in I} g(k)$ by Lemma 2.5. If there is a doubly stochastic operator $D_1$ such that $D_1^* e_i = g$ (or $(D_2^{-1})^* g = e_i$ if $D_1$ is invertible) for any $i \in I$, then $\sum_{j \in I} \langle \Psi^* e_j, e_i \rangle \neq 1$ by (3). Therefore, under the last assumption, the condition $D_2^{-1} \in RS(\ell^p(I))$ cannot be omitted in the last theorem. In accordance with the above consideration, we pose the following question:

**Question:** Is it true that for any given $D_1, D_2 \in DS(\ell^p(I))$ where $D_2$ is invertible, the majorization $D_1 f < D_2 f$ for all $f \in \ell^p(I)$, implies $D_1 \prec D_2$?

References