Generalized Hyers-Ulam Stability of General Cubic Functional Equation in Random Normed Spaces

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Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of a general cubic functional equation:

\[ f(x + ky) - k f(x + y) + kf(x - y) - f(x - ky) = 2k(k^2 - 1)f(y) \]

for fixed \( k \in \mathbb{Z}^+ \) with \( k \geq 2 \) in random normed spaces.

1. Introduction

A basic question in the theory of functional equations is as follows:

When is it true that a function that approximately satisfies a functional equation must be close to an exact solution of the equation?

If the problem accepts a solution, we say that the equation is stable. The first stability problem concerning group homomorphisms was introduced by Ulam [20] in 1940. The famous Ulam stability problem was partially solved by Hyers [12] for linear functional equation of Banach spaces. Hyers theorem was generalized by Aoki [3] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Gavruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. Cădariu and Radu [5] applied the fixed point method to investigation of the Jensen functional equation. They could present a short and a simple proof (different from the direct method initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation and for quadratic functional equation. Their methods are a powerful tool for studying the stability of several functional equations.
The theory of random normed spaces (briefly, RN-spaces) is important as a generalization of deterministic result of normed spaces (see [1]) and also in the study of random operator equations. The notion of an RN-space corresponds to the situations when we do not know exactly the norm of the point and we know only probabilities of possible values of this norm. The RN-spaces may provide us the appropriate tools to study the geometry of nuclear physics and have usefully application in quantum particle physics. A number of papers and research monographs have been published on generalizations of the stability of different functional equations in RN-spaces [4, 7, 14–16, 22].

In the sequel, we use the definitions and notations of a random normed space as in [2, 18, 19]. A function \( F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1] \) is called a distribution function if it is nondecreasing and left-continuous, with \( F(0) = 0 \) and \( F(+\infty) = 1 \). The class of all probability distribution functions \( F \) is denoted by \( \Lambda \). \( D^+ \) is a subset of \( \Lambda \) consisting of all functions \( F \in \Lambda \) for which \( F^{-}(x)=\lim_{t\to x^{-}}F(t) \). The space \( \Lambda \) is partially ordered by the usual pointwise ordering of functions, that is, \( F \leq G \) if and only if \( F(t) \leq G(t) \) for all \( t \in \mathbb{R} \). For any \( a \geq 0 \), \( \epsilon_a \) is the element of \( D^+ \), which is defined by

\[
\epsilon_a(t) = \begin{cases} 
0 & \text{if } t \leq a, \\
1 & \text{if } t > a. 
\end{cases}
\]

**Definition 1.1.** ([18]) A function \( T : [0, 1] \times [0, 1] \to [0, 1] \) is a continuous triangular norm (briefly, a continuous t-norm) if \( T \) satisfies the following conditions:

1. \( T \) is commutative and associative;
2. \( T \) is continuous;
3. \( T(a, 1) = a \) for all \( a \in [0, 1] \);
4. \( T(a, b) \leq T(c, d) \) whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0, 1] \).

Three typical examples of continuous t-norms are as follows:

\[
T(a, b) = ab, \quad T(a, b) = \max(a + b - 1, 0), \quad T(a, b) = \min(a, b).
\]

Recall that, if \( T \) is a t-norm and \( \{x_n\} \) is a sequence of numbers in \([0,1]\), then \( T^n_{i=1} x_i \) is defined recurrently by

\[
T^n_{i=1} x_i = x_1, \quad T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n) = T(x_1, \ldots , x_n)
\]

for all \( n \geq 2 \), where \( T^n_{i=1} x_i \) is defined as \( T^n_{i=1} x_{i+1} ([11]) \).

**Definition 1.2.** ([19]) Let \( X \) be a real linear space, \( \mu \) be a mapping from \( X \) into \( D^+ \) (for any \( x \in X \), \( \mu(x) \) is denoted by \( \mu_x \)) and \( T \) be a continuous t-norm. The triple \((X, \mu, T)\) is called a random normed space (briefly, RN-space) if \( \mu \) satisfies the following conditions:

1. (RN1) \( \mu_x(t) = \epsilon_t(t) \) for all \( t > 0 \) if and only if \( x = 0 \);
2. (RN2) \( \mu_x(t) = \mu_x(\frac{t}{\|x\|}) \) for all \( x \in X, \alpha \neq 0 \) and \( t \geq 0 \);
3. (RN3) \( \mu_x(y + s) \geq T(\mu_x(t), \mu_y(s)) \) for all \( x, y \in X \) and \( t, s \geq 0 \).

Every normed space \((X, \| \cdot \|)\) defines a random normed space \((X, \mu, T_M)\), where

\[
\mu_x(t) = \frac{t}{t + \|x\|}
\]

for all \( x \in X, t > 0 \) and \( T_M \) is the minimum continuous t-norm. This space is called the induced random normed space.
Definition 1.3. Let \((X, \mu, T)\) be an RN-space.

1. A sequence \(\{x_n\}\) in \(X\) is said to be convergent to a point \(x \in X\) if, for all \(t > 0\) and \(\lambda > 0\), there exists a positive integer \(N\) such that
   \[\mu_{x_n-x}(t) > 1 - \lambda\]
   whenever \(n \geq N\). In this case, \(x\) is called the limit of the sequence \(\{x_n\}\), which is denoted by \(\lim_{n \to \infty} \mu_{x_n-x} = 1\).

2. A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence if, for all \(t > 0\) and \(\lambda > 0\), there exists a positive integer \(N\) such that
   \[\mu_{x_n-x_m}(t) > 1 - \lambda\]
   whenever \(n \geq m \geq N\).

3. The RN-space \((X, \mu, T)\) is said to be complete if every Cauchy sequence in \(X\) is convergent to a point in \(X\).

Theorem 1.4. ([18]) If \((X, \mu, T)\) is an RN-space and \(\{x_n\}\) is a sequence of \(X\) such that \(x_n \to x\), then
   \[\lim_{n \to \infty} \mu_{x_n-x} = 1\].

Now, we consider a mapping \(f : X \to Y\) satisfying the following functional equation:
\[f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) = 2k(k^2 - 1)f(y)\]
for fixed \(k \in \mathbb{Z}^+\) with \(k \geq 3\). Then the equation (1) is called the general cubic functional equation since the function \(f(x) = x^3\) is its solution. Every solution of the cubic functional equation is called a cubic mapping.

Note that, if we put \(x = 0\) and \(y = x\) in the equation (1), then \(f(kx) = k^3f(x)\) and \(f(k^n x) = k^{3n}f(x)\) for all \(n \in \mathbb{Z}^+\).

In the case \(k = 3\), Wiwatwanich et al. [21] established the general solution and the general Hyers-Ulam-Rassias stability of cubic functional equation on Banach spaces. The stability of the functional equation (1) in quasi-\(\beta\)-normed spaces and fuzzy normed spaces were investigated by Eskandani et al. [9] and Javdian et al. [13], respectively.

In this paper, using the direct and fixed point methods, we prove the generalized Hyers-Ulam stability problem of the general cubic functional equation (1) in random normed spaces in the sense of Scherstnev under the minimum continuous \(t\)-norm \(T_M\).

Throughout this paper, let \(X\) be a real linear space, \((Z, \mu', T_M)\) be an RN-space and \((Y, \mu, T_M)\) be a complete RN-space. For any mapping \(f : X \to Y\), we define
\[\Delta f(x, y) = f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) - 2k(k^2 - 1)f(y)\]
for all \(x, y \in X\) and \(k \in \mathbb{Z}^+\) with \(k \geq 2\).

2. Random Stability of Functional Equation (1)

In this section, we investigate the generalized Hyers-Ulam stability of the general cubic functional equation \(\Delta f(x, y) = 0\) in random spaces via direct and fixed point methods.

2.1 The direct Method
Theorem 2.1. Let \( \phi : X^2 \to Z \) be an even function such that, for some \( 0 < \alpha < k^3 \),

\[
\mu_{\phi(x,x,y)}'(t) \geq \mu_{\phi(x,y)}'(t)
\]

(2)

and \( \lim_{n \to \infty} \mu_{\phi(x,x,y)}'(k^{3n}t) = 1 \) for all \( x, y \in X \) and \( t > 0 \). If \( f : X \to Y \) is a mapping with \( f(0) = 0 \) such that

\[
\mu_{\Delta f(x,y)}(t) \geq \mu_{\phi(x,y)}'(t)
\]

(3)

for all \( x, y \in X \) and \( t > 0 \), then there exists a unique cubic mapping \( C : X \to Y \) such that

\[
\mu_{f(y)-C(y)}(t) \geq \mu_{\phi(0,y)}'(t) \left( \frac{2k(k^2-1)(k^3-\alpha)t}{k^3+\alpha} \right)
\]

(4)

for all \( y \in X \) and \( t > 0 \).

Proof. Substituting \( x = 0 \) in (3), we have

\[
\mu_{\Delta f(0,y)}(t) \geq \mu_{\phi(0,y)}'(t)
\]

(5)

and replacing \( y = -y \) in (5), we have

\[
\mu_{\Delta f(0,-y)}(t) \geq \mu_{\phi(0,y)}'(t)
\]

(6)

for all \( y \in X \) and \( t > 0 \). It follows from (5) and (6) that

\[
\mu_{f(y)+f(-y)}(t) \geq \mu_{\phi(0,y)}'(k(k^2-1)t).
\]

Since \( \mu_{2f(y)}-2k^2f(y)(t) = \mu_{f(y)+f(-k^2y)+f(y)+\Delta f(0,y)}(t) \), we have

\[
\begin{align*}
\mu_{2f(y)}-2k^2f(y)(t) & \geq T_M \left( \frac{\alpha}{k(k^2-1)} t, \mu_{\phi(0,y)}'(t) \right) \\
& \geq T_M \left( \mu_{\phi(0,y)}'(at), \mu_{\phi(0,y)}'(t) \right) \\
& = \mu_{\phi(0,y)}'(t)
\end{align*}
\]

for all \( y \in X \) and \( t > 0 \). Thus we have

\[
\mu_{\mu_{\phi(0,y)}'(t)}(t) \geq \mu_{\phi(0,y)}'(t) \left( \frac{2k^4(k^2-1)(k^3-\alpha)t^2}{k^3+\alpha} \right)
\]

(7)

for all \( y \in X \) and \( t > 0 \). Replacing \( y \) by \( k^2y \) in (7), we have

\[
\mu_{\mu_{\phi(0,y)}'(t)}(t) \geq \mu_{\phi(0,y)}'(t) \left( \frac{2k^4(k^2-1)(k^3-\alpha)t^2}{k^3+\alpha} \right)
\]

for all \( y \in X \) and \( t > 0 \). Since \( f(y) = \sum_{j=0}^{n-1} \frac{f(k^j y)}{k^j} - \frac{f(k^{n-1} y)}{k^{n-1}} \), we have

\[
\mu_{\mu_{\phi(0,y)}'(t)}(t) \left( \sum_{j=0}^{n-1} \frac{k^3+\alpha}{2k^4(k^2-1)} \right) \geq T_M \left( \mu_{\phi(0,y)}'(t) \right) = \mu_{\phi(0,y)}'(t)
\]

(8)
for all \( y \in X \) and \( t > 0 \). Replacing \( y \) by \( k^n y \) in (8), we obtain

\[
\frac{\mu_{(\alpha^{\delta},\gamma_{m\gamma})} - \mu_{\phi_0(y)}}{\mu_{(\alpha^{\delta},\gamma_{m\gamma})} - \mu_{\phi_0(y)}}(t) \geq \int \phi_{(0,y)} \left( \frac{2k^4(k^2 - 1)t}{(k^3 + \alpha) \sum_{j=0}^{\infty} \left( \frac{\alpha}{t} \right)^j} \right)
\]

(9)

for all \( y \in X \) and \( m, n \in \mathbb{Z}^+ \) with \( n > m \). Since \( \alpha < k^3 \), the sequence \( \{ \frac{\mu_{(\alpha^{\delta},\gamma_{m\gamma})}}{\mu_{\phi_0(y)}} \} \) is a Cauchy sequence in a complete RN-space \((X, \mu, T_M)\) and so it converges to some point \( C(y) \in Y \). Fix \( y \in X \) and put \( m = 0 \) in (9). Then we obtain

\[
\frac{\mu_{(\alpha^{\delta},\gamma_{m\gamma})} - \mu_{\phi_0(y)}}{\mu_{(\alpha^{\delta},\gamma_{m\gamma})} - \mu_{\phi_0(y)}}(t) \geq \int \phi_{(0,y)} \left( \frac{2k^4(k^2 - 1)t}{(k^3 + \alpha) \sum_{j=0}^{n-1} \left( \frac{\alpha}{t} \right)^j} \right)
\]

and so, for any \( \delta > 0 \), it follows that

\[
\mu_{C(y)-f(y)}(\delta + t) \geq T_M \left( \mu_{C(y)-f(y)}(\delta), \mu_{\phi_0(y)}(t) \right)
\]

(10)

for all \( y \in X \) and \( t > 0 \). Taking the limit as \( n \to \infty \) in (10), we obtain

\[
\mu_{C(y)-f(y)}(\delta + t) \geq \mu_{\phi_0(y)}(t) \left( \frac{2k(k^2 - 1)(k^3 - \alpha)t}{k^3 + \alpha} \right)
\]

(11)

Since \( \delta \) is arbitrary, by taking \( \delta \to 0 \) in (11), we have

\[
\mu_{C(y)-f(y)}(t) \geq \mu_{\phi_0(y)}(t) \left( \frac{2k(k^2 - 1)(k^3 - \alpha)t}{k^3 + \alpha} \right)
\]

(12)

for all \( y \in X \) and \( t > 0 \). Therefore, we conclude that the condition (4) holds. Replacing \( x \) and \( y \) by \( k^n x \) and \( k^n y \) in (3), respectively, we have

\[
\mu_{\phi_0(x,y)} \rightarrow_n \mu_{\phi_0(x,y)}(k^n t)
\]

for all \( x, y \in X \) and \( t > 0 \). Since \( \lim_{n\to\infty} \mu'_{\phi_0(x,y)}(k^n t) = 1 \), it follows that \( C \) satisfies the equation (1), which implies that \( C \) is a cubic mapping.

To prove the uniqueness of the cubic mapping \( C \), let us assume that there exists another mapping \( D : X \to Y \) which satisfies (4). Fix \( y \in X \). Then \( C(k^n y) = k^n C(y) \) and \( D(k^n y) = k^n D(y) \) for all \( n \in \mathbb{Z}^+ \). It follows from (2.3) that

\[
\mu_{C(y)-D(y)}(t) = \mu_{\phi_0(x,y)}(t)
\]

\[
\geq T_M \left( \mu_{\phi_0(x,y)}(t), \mu_{\phi_0(x,y)}(t) \right)
\]

(13)

Since \( \lim_{n\to\infty} \frac{2k^2(k^2 - 1)(k^3 - \alpha)t^2}{(k^3 + \alpha)k^n} = \infty \), we have \( \mu_{C(y)-D(y)}(t) = 1 \) for all \( t > 0 \). Thus the cubic mapping \( C \) is unique. This completes the proof. □
Theorem 2.2. Let $\phi : X^2 \to Z$ be an even function such that, for some $0 < k^3 < \alpha$,
\begin{equation}
\mu'_{\phi(t)}(t) \geq \mu'_{\phi(0)}(at)
\end{equation}
and $\lim_{n \to \infty} \mu'_{\phi(0)}(t) = 1$ for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (3), then there exists a unique cubic mapping $C : X \to Y$ such that
\begin{equation}
\mu_{f(y) - C(y)}(t) \geq \mu'_{\phi(0)}(y) \left( \frac{2\alpha k^2 - 1)(\alpha - k^3)t}{k^3 + \alpha} \right)
\end{equation}
for all $y \in X$ and $t > 0$.

Proof. It follows from (7) that
\begin{equation}
\mu_{f(y) - f(x)}(t) \geq \mu'_{\phi(0)}(y) \left( \frac{2\alpha k^2 - 1)(\alpha - k^3)t}{k^3 + \alpha} \right)
\end{equation}
for all $y \in X$ and $t > 0$. Applying the triangle inequality and (16), we have
\begin{equation}
\mu_{f(y) - f(x)}(t) \geq \mu'_{\phi(0)}(y) \left( \frac{2\alpha k^2 - 1)(\alpha - k^3)t}{k^3 + \alpha} \right)
\end{equation}
for all $y \in X$ and $m, n \in \mathbb{Z}^+$ with $n > m \geq 0$. Then the sequence $\{k^m f(y)\}$ is a Cauchy sequence in a complete RN-space $(Y, \mu, T_M)$ and so it converges to some point $C(y) \in Y$. We can define a mapping $C : X \to Y$ by
\[ C(y) = \lim_{n \to \infty} \frac{k^3 f(y)}{k^3} \]
for all $y \in X$. Then the mapping $C$ satisfies (1) and (15). The remaining assertion goes through in the similar method to the corresponding part of Theorem 2.1. This complete the proof. \(\square\)

Corollary 2.3. Let $p \in \mathbb{R}$ be positive real number with $p \neq 3$ and a fixed unit point of $z_0 \in Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ and satisfying
\begin{equation}
\mu_{\Lambda f(x,y)}(t) \geq \mu'_{\phi(0)}(\|x\|, \|y\|, z_0) \end{equation}
for all $x, y \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that
\begin{equation}
\mu_{f(y) - C(y)}(t) \geq \mu'_{\phi(0)}( \|y\|, z_0) \left( \frac{2\alpha k^2 - 1)(\alpha - k^3)t}{k^3 + \alpha} \right)
\end{equation}
for all $x, y \in X$ and $t > 0$.

Proof. Let $\phi : X^2 \to Z$ be defined by $\phi(x, y) = (\|x\|^p + \|y\|^p)z_0$. Then, by Theorem 2.1, we obtain the desired result, where $\alpha = k^3$. \(\square\)

Remark. (1) An example to illustrate that the functional equation (1) is not stable for $p = 3$ in Corollary 2.3 (see [13]).
(2) In Corollary 2.3, if we assume that
\[ \phi(x, y) = \|x\|^p \|y\|^p z_0 \]
or
\[ \phi(x, y) = (\|x\|^p \|y\|^p + \|x\|^{p+q} + \|y\|^{p+q})z_0, \]
then we have the product stability of Ulam-Gavuta-Rassias and the mixed product-sum stability of Rassias, respectively.
Example 2.4. Let \((X, \| \cdot \|)\) be a Banach normed space and
\[
\mu_x(t) = \frac{t}{t + \|x\|}
\]
for all \(x \in X\) and \(t > 0\). Then \((X, \mu, \min)\) is a complete RN-space. Also, let
\[
\mu_{\phi(x,y)}(t) = \frac{t}{t + \|\phi(x,y)\|}
\]
for all \(x, y \in X\) and \(t > 0\). Then \((X, \mu', \min)\) is a RN-space.

Define a mapping \(f : X \to X\) by \(f(x) = x^3 + \|x\|z_0\), where \(z_0\) is a unit point in \(X\). By a simple calculation, we have
\[
\|\Delta f(x, y)\| = \|f(x + ky) - kf(x + y) + k(k - 1)f(y)\|
\leq 2k(k^2 - 1)\|y\| - 2k(k^3 - 1)(\|x\| + \|y\|).
\]
Then it follows that
\[
\mu_{\Delta f(x,y)}(t) \geq \mu_{\phi(x,y)}(t)
\]
for all \(x, y \in X\) and \(t > 0\), where \(\phi(x, y) = 2k(k^2 - 1)(\|x\| + \|y\|)\). Also, we obtain
\[
\mu_{\phi(0,k^3)}(k^3 - \alpha)t = \frac{k^3(k^3 - \alpha)t}{t + k(k^3 - 1)\|y\|}
\]
where \(0 < \alpha < k^3\), and
\[
\lim_{n \to \infty} \mu_{\phi(0,k^3)}(k^3 - \alpha)t = 1.
\]
Thus all the conditions of Theorem 2.1 hold. Therefore, there exists a unique cubic mapping \(C : X \to X\) such that
\[
\mu_{f(y) - C(y)}(t) \geq \frac{(k^3 - \alpha)t}{(k^3 - \alpha)t + (k^3 + \alpha)||y||}
\]
and \(C(y) = y^3\) for all \(y \in X\) and \(t > 0\).

2.2 The fixed point method

Recall that a mapping \(d : X^2 \to [0, +\infty]\) is called a generalized metric on a nonempty set \(X\) if
1. \(d(x, y) = 0\) if and only if \(x = y\);
2. \(d(x, y) = d(y, x)\);
3. \(d(x, z) \leq d(x, y) + d(y, z)\) for all \(x, y, z \in X\).

A set \(X\) with the generalized metric \(d\) is called a generalized metric space.

The following fixed point theorem proved by Diaz and Margolis [8] plays an important role in proving our theorem:

Theorem 2.5. ([8]) Suppose that \((\Omega, d)\) is a complete generalized metric space and \(J : \Omega \to \Omega\) is a strictly contractive mapping with Lipshitz constant \(L < 1\). Then, for each \(x \in \Omega\), either \(d(f^n x, f^{n+1} x) = \infty\) for all nonnegative integers \(n \geq 0\) or there exists a natural number \(n_0\) such that
1. \(d(f^n x, f^{n+1} x) < \infty\) for all \(n \geq n_0\);
2. the sequence \(\{f^n x\}\) is convergent to a fixed point \(y^*\) of \(J\);
3. \(y^*\) is the unique fixed point of \(J\) in the set \(\Lambda = \{y \in \Omega : d(f^n x, y) < \infty\}\);
4. \(d(y, y^*) \leq \frac{1}{1-L}d(y, f y)\) for all \(y \in \Lambda\).
Theorem 2.6. Let $\psi : X^2 \to D^*$ \((\psi(x, y)\) is denoted by $\psi_{x,y}\) be an even function such that, for some $0 < \alpha < k^3$,

$$\psi_{x,y}(t) \leq \psi_{kx,ky}(\alpha t)$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies

$$\mu_{f(cx, cy)}(t) \geq \psi_{x,y}(t)$$

for all $x, y \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(y) - C(y)}(t) \geq 2\alpha \frac{k^4(k^2 - 1)t}{k^3 + \alpha}$$

for all $y \in X$ and $t > 0$.

Proof. It follows from (20) and the similar methods in the proof of Theorem 2.1 that

$$\mu_{g(y)}(-f(y))(t) \geq \psi_{0,y}\left(\frac{2k^2(k^2 - 1)t}{k^3 + \alpha}\right)$$

for all $y \in X$ and $t > 0$. Let $\Omega$ be a set of all mappings from $X$ into $Y$ and introduce a generalized metric on $\Omega$ as follows:

$$d(g, h) = \inf\{c \in [0, \infty) : \mu_{\gamma(y) - h(y)}(ct) \geq \psi_{0,y}(t), \forall y \in X\}.$$  

where, as usual, $\inf\emptyset = -\infty$. It is easy to show that $(\Omega, d)$ is a generalized complete metric space ([6]). Now, let us consider the mapping $J : \Omega \to \Omega$ defined by

$$Jg(y) = \frac{g(ky)}{k^3}$$

for all $g \in \Omega$ and $y \in X$. Let $g, h$ in $\Omega$ and $c \in [0, \infty)$ be an arbitrary constant with $d(g, h) < c$. Then we have

$$\mu_{g(y) - h(y)}(ct) \geq \psi_{0,y}(t)$$

for all $y \in X$ and $t > 0$, whence

$$\mu_{g(y) - h(y)}\left(\frac{\alpha}{k^3}ct\right) \geq \psi_{0,y}(t)$$

for all $y \in X$ and $t > 0$ and so

$$d(Jg, Jh) \leq \frac{\alpha c}{k^3} \leq \frac{\alpha}{k^3}d(g, h)$$

for all $g, h \in \Omega$. Then $J$ is a strictly contractive self-mapping on $\Omega$ with the Lipschitz constant $\frac{\alpha}{k^3} < 1$. It follows from (22) that

$$d(f, Jf) \leq \frac{k^3 + \alpha}{2k^2(k^2 - 1)}.$$  

Due to Theorem 2.5, there exists a mapping $C : X \to Y$, which is a unique fixed point of $J$ in the set $\Omega_1 = \{g \in \Omega : d(f, g) < \infty\}$ such that

$$C(y) = \lim_{n \to \infty} \frac{f(k^n y)}{k^{3n}}.$$
for all \( y \in X \) since \( \lim_{n \to \infty} d(l^n f, C) = 0 \). Again, it follows from Theorem 2.5 that
\[
d(f, C) \leq \frac{1}{1-L}d(f, f) \leq \frac{k^3 + \alpha}{2k^4(k^2 - 1)(1 - L)}.
\]

Then we conclude
\[
\mu_{f(y)-C(y)}(t) \geq \psi_{0,y}(\frac{2k(k^2 - 1)(k^3 - \alpha)t}{k^3 + \alpha})
\]
for all \( y \in X \) and \( t > 0 \), where \( L = \frac{\alpha}{\beta} \). The remaining assertion goes through in the similar method to the corresponding part of Theorem 2.1. This completes the proof.

**Theorem 2.7.** Let \( \psi : X^2 \to Z \) be an even function such that, for some \( 0 < k^3 < \alpha \),
\[
\psi_{x,y}(t) \geq \psi_{x,y}(at)
\]
for all \( x, y \in X \) and \( t > 0 \). If \( f : X \to Y \) is a mapping with \( f(0) = 0 \) which satisfies (20), then there exists a unique cubic mapping \( C : X \to Y \) such that
\[
\mu_{f(y)-C(y)}(t) \geq \psi_{0,y}(\frac{2k(k^2 - 1)(\alpha - k^3)t}{k^3 + \alpha})
\]
for all \( y \in X \) and \( t > 0 \).

**Proof.** It follows from (20) that
\[
\mu_{f(y)-f(y)}(t) \geq \psi_{0,y}(\frac{2ak(k^2 - 1)t}{k^3 + \alpha})
\]
for all \( y \in X \) and \( t > 0 \). Let \( \Omega \) and \( d \) be as in the proof of Theorem 2.6. Then \( (\Omega, d) \) is a generalized complete metric space ([6]) and we consider the mapping \( J : \Omega \to \Omega \) defined by
\[
Jg(y) = k^3 g(y)
\]
for all \( g \in \Omega \) and \( y \in X \). So, we have
\[
d(f, fh) \leq \frac{k^3}{\alpha}d(g, h)
\]
for all \( g, h \in \Omega \). Then \( J \) is a strictly contractive self-mapping on \( \Omega \) with the Lipschitz constant \( \frac{k^3}{\alpha} < 1 \). It follows from (25) that
\[
d(f, f) \leq \frac{k^3 + \alpha}{a(2k(k^2 - 1))}
\]
Due to Theorem 2.5, there exists a mapping \( C : X \to Y \), which is a unique fixed point of \( J \) in the set \( \Omega_1 = \{ g \in \Omega : d(f, g) < \infty \} \), such that
\[
C(y) = \lim_{n \to \infty} k^3n f(y)
\]
for all \( y \in X \) since \( \lim_{n \to \infty} d(l^n f, C) = 0 \). Again, it follows from Theorem 2.5 that
\[
d(f, C) \leq \frac{1}{1-L}d(f, f) \leq \frac{k^3 + \alpha}{2k(k^2 - 1)(\alpha - k^3)}
\]
for all \( y \in X \) and \( t > 0 \), where \( L = \frac{\alpha}{\beta} \). The remaining assertion goes through in the similar method to the corresponding part of Theorem 2.2. This completes the proof. \( \square \)
Now, we present a corollary that is an application of Theorem 2.6 in the classical case.

**Corollary 2.8.** Let X be a real normed space, $\theta \geq 0$ and $p$ be a real number with $0 < p < 1$. Assume that $f : X \to X$ is a mapping with $f(0) = 0$ which satisfies

$$
\mu_{f(x,y)}(t) \geq \frac{t}{1 + \theta(|x|^p + |y|^p)}
$$

for all $x, y \in X$ and $t > 0$. Then the limit $C(x) = \lim_{n\to\infty} \frac{f^n(x)}{2^n}$ exists for all $y \in X$ and $C : X \to X$ is a unique cubic mapping such that

$$
\mu_{f(y)-C(y)}(t) \geq \frac{2k(k^2 - 1)(k^3 - k^p)t}{2k(k^2 - 1)(k^3 - k^p)t + \theta(k^3 + k^p)|y|^p}
$$

for all $y \in X$ and $t > 0$.

**Proof.** Let $\psi_{x,y}(t) = \frac{t}{1 + \theta(|x|^p + |y|^p)}$ for all $x, y \in X$ and $t > 0$ and $a = k^p$. Then we obtain the desired result. \qed

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**References**


