Refinements of Jensen’s Inequality for Convex Functions on the Co-Ordinates in a Rectangle from the Plane

M. Adil Khan, T. Ali, A. Kılıçman, Q. Din

1. Introduction

A function \( \phi : [a, b] \rightarrow \mathbb{R} \) is said to be convex if

\[
\phi(\lambda x + (1 - \lambda)y) \leq \lambda \phi(x) + (1 - \lambda)\phi(y)
\]

holds for all \( x, y \in [a, b] \) and \( 0 \leq \lambda \leq 1 \). A function \( \phi \) is said to be strictly convex if the inequality in (1) is strict whenever \( x \neq y \) and \( 0 < \lambda < 1 \).

Let \( \phi : [a, b] \rightarrow \mathbb{R} \) be a convex function on \( [a, b] \). If \( x_i \in [a, b] \) and \( p_i > 0 \) such that \( P_n = \sum_{i=1}^{n} p_i \) then

\[
\phi \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \phi(x_i),
\]

is well known in the literature as Jensen’s inequality.

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the arithmetic-mean geometric-mean inequality, the Hölder and Minkowski inequalities, the Ky Fan inequality etc. can be obtained as particular cases of it.
In [7], the authors have investigated the following refinement of (2):

\[
\phi \left( \sum_{i=1}^{n} p_i x_i \right) \leq \min_{I} \left[ P_I \phi \left( \frac{\sum_{i \in I} p_i x_i}{P_I} \right) + \sum_{i \notin I} p_i \phi(x_i) \right]
\]

\[
\leq \frac{1}{2^n - n - 2} \left[ \sum_{i \in I_n} P_I \phi \left( \frac{\sum_{i \in I_n} p_i x_i}{P_I} \right) + (2^{n-1} - n) \sum_{i=1}^{n} p_i \phi(x_i) \right]
\]

\[
\leq \max_{I} \left[ P_I \phi \left( \frac{\sum_{i \in I_n} p_i x_i}{P_I} \right) + \sum_{i \notin I} p_i \phi(x_i) \right] \leq \sum_{i=1}^{n} p_i \phi(x_i),
\]

where \( \phi : C \to \mathbb{R} \) is a convex function defined on a convex set \( C \), \( x_i \in C \) and

\[ I = \{ l \subset I_n, I \neq I_n = \{ 1, \ldots, n \} \text{ s.t. } |I| \geq 2 \}, \quad i \in \{ 1, \ldots, n \}, \quad n \geq 3 \]

and \( P_I = \sum_{i \in I} p_i \) together with \( \sum_{i=1}^{n} p_i = 1 \).

In 2010 Dragomir obtained another refinement of Jensen’s inequality (see [15]):

\[
\phi \left( \frac{1}{n} \sum_{i=1}^{n} p_i x_i \right) \leq D(\phi, p, x, I) \leq \frac{1}{P_I} \sum_{i=1}^{n} p_i \phi(x_i),
\]

where

\[
D(\phi, p, x, I) = P_I \phi \left( \frac{1}{P_I} \sum_{i \in I_n} p_i x_i \right) + P_I \phi \left( \frac{1}{P_I} \sum_{i \notin I} p_i x_i \right)
\]

and

\[ \emptyset \neq I \subset I_n = \{ 1, \ldots, n \}, \quad \bar{I} = I_n \setminus I \neq \emptyset, \quad i \in \{ 1, \ldots, n \} \]

together with \( P_I = \sum_{i \in I} p_i, \ P_I = \sum_{i \notin I} p_i \) and \( x = (x_1, x_2, \ldots, x_n), \ p = (p_1, p_2, \ldots, p_n) \). Also in [6], the authors have proved a generalized refinement of (2) given as under:

\[
\phi \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \leq \frac{1}{n} \sum_{i=1}^{n} \phi \left( \sum_{j=1}^{l-1} \lambda_j x_{i+j} \right) \leq \frac{1}{n} \sum_{i=1}^{n} \phi(x_i),
\]

where \( \phi : [a, b] \to \mathbb{R} \) is a convex function, \( x := (x_1, \ldots, x_n) \in [a, b]^n \) such that \( x_{i+n} = x_i \) and \( \lambda := (\lambda_1, \ldots, \lambda_n) \) is a positive \( n \)-tuple together with \( \sum_{i=1}^{k} \lambda_i = 1 \) for some \( k, 2 \leq k \leq n \). More recently in 2015, the authors have given further generalizations of the results presented in [2, 3].

In [14], the concept of convex functions defined on the co-ordinates of the bidimensional interval of the plane of two variables was introduced:

**Definition 1.1.** Let us consider the bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). A function \( \phi : [a, b] \times [c, d] \to \mathbb{R} \) is called convex on the co-ordinates if the partial mappings \( \phi_y : [a, b] \to \mathbb{R} \) defined as \( \phi_y(t) := \phi(t, y) \) and \( \phi_x : [c, d] \to \mathbb{R} \) defined as \( \phi_x(s) := \phi(x, s) \), are convex for all \( x \in [a, b], y \in [c, d] \).

**Remark 1.2.** Note that every convex function \( \phi : [a, b] \times [c, d] \to \mathbb{R} \) is convex on the co-ordinates, but the converse is not generally true [14].

The following Jensen’s inequality for co-ordinate convex functions has been given in [4].
Theorem 1.3. (4) Let \( \phi : [a, b] \times [c, d] \rightarrow \mathbb{R} \) be a convex function on the co-ordinates on \([a, b] \times [c, d]\). If \( x \) is an \( n \)-tuple in \([a, b]\), \( y \) is an \( m \)-tuple in \([c, d]\), \( p \) is a non-negative \( n \)-tuple and \( \omega \) a non-negative \( m \)-tuple such that

\[
P_n = \sum_{i=1}^n p_i > 0 \text{ and } W_m = \sum_{j=1}^m \omega_j > 0, \text{ then}
\]

\[
\phi (x, y) \leq \frac{1}{2} \left( \frac{1}{P_n} \sum_{i=1}^n p_i \phi (x_i, \overline{y}) + \frac{1}{W_m} \sum_{j=1}^m \omega_j \phi (\overline{x}, y_j) \right) \leq \frac{1}{P_n W_m} \sum_{i=1}^n \sum_{j=1}^m p_i \omega_j \phi (x_i, y_j),
\]

(5)

where \( \overline{x} = \frac{1}{P_n} \sum_{i=1}^n p_i x_i \), \( \overline{y} = \frac{1}{W_m} \sum_{j=1}^m \omega_j y_j \).

For other refinements and generalizations of Jensen’s inequality and their applications see [1–6, 8–13, 17–23] and some of the references given in them.

In this article, we have generalized the results given in [6], [7] and [15] from convex functions defined on the subset of \( \mathbb{R} \) to convex functions defined on the co-ordinates on the bidimensional interval of the plane by constructing some new functionals depending on the function \( \phi \) and indexing sets, separating the discrete domain of it. Furthermore the result given in [6] is extended to co-ordinate convex functions.

2. Main Results

Terminologies and notations: Let \( \phi : [a, b] \times [c, d] \rightarrow \mathbb{R} \) be convex on the co-ordinates on \([a, b] \times [c, d]\).

If \( x_i \in [a, b], y_j \in [c, d], \) and \( p_i, \omega_j > 0, i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, m\} \) such that \( n, m \geq 3 \) with \( P_n = \sum_{i=1}^n p_i \) and \( W_m = \sum_{j=1}^m \omega_j \), and let \( \Omega_1 = \{ l^k : l^k \subset I_n = \{1, \ldots, n\}, |l^k| \geq 2, l^k \neq I_n \} \) and \( \Omega_2 = \{ j^l : j^l \subset J_m = \{1, \ldots, m\}, |j^l| \geq 2, j^l \neq I_m \} \), we assume \( \overline{l}^k = \{1, 2, \ldots, n\} \setminus l^k \) and \( \overline{j}^l = \{1, 2, \ldots, m\} \setminus j^l \). Define \( P_{\overline{l}} = \sum_{i \in \overline{l}} p_i \) and \( P_{\overline{j}} = \sum_{i \in \overline{j}} p_i \), and \( W_{\overline{j}} = \sum_{j \in \overline{j}} \omega_j \), \( W_{\overline{l}} = \sum_{i \in \overline{l}} \omega_i \). For the function \( \phi \) and the \( n, m \)-tuples \( x = (x_1, x_2, \ldots, x_n) \in [a, b]^n \), \( y = (y_1, y_2, \ldots, y_m) \in [c, d]^m \) and \( p = (p_1, p_2, \ldots, p_n), w = (w_1, w_2, \ldots, w_m) \), we define the following functionals:

\[
F(\phi, p, x, l^k) = \frac{P_{\overline{l}}}{P_n} \phi \left( \frac{1}{P_{\overline{l}}} \sum_{i \in \overline{l}} p_i x_i, \overline{y} \right) + \frac{1}{P_n} \sum_{i \in \overline{l}} p_i \phi (x_i, \overline{y}),
\]

\[
F(\phi, w, y, j^l) = \frac{W_{\overline{j}}}{W_m} \phi \left( \overline{x}, \frac{1}{W_{\overline{j}}} \sum_{j \in \overline{j}} \omega_j y_j \right) + \frac{1}{W_m} \sum_{j \in \overline{j}} \omega_j \phi (\overline{x}, y_j),
\]

(6)

\[
D_1(l^k, j^l) = F(\phi, w, y, j^l) + F(\phi, p, x, l^k)
\]

(7)

\[
D_2(l^k, j^l) = \frac{1}{P_n} \sum_{i=1}^n p_i F(\phi, w, y, j^l, x_i) + \frac{1}{W_m} \sum_{j=1}^m \omega_j F(\phi, p, x, l^k, y_j)
\]

(8)
Remark 2.1. It is obvious that \( |\Omega_1| = 2^n - n - 2, |\Omega_2| = 2^m - m - 2 \), that is, \( k = 1, \ldots, 2^n - n - 2 \) and \( l = 1, \ldots, 2^m - m - 2 \) and throughout the paper we will denote \( 2^n - n - 2 \) by \( N \) and \( 2^m - m - 2 \) by \( M \).

The following lemma will be proved helpful in the further elaboration of the next refinement:

Lemma 2.2. Let \( \phi : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) be a function defined on \( \Delta \). If \( x_i \in [a, b], y_j \in [c, d], \) and \( p_i, w_j > 0, i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, m\}, n, m \geq 3, \) with \( P_n = \sum_{i=1}^{n} p_i \) and \( W_m = \sum_{j=1}^{m} w_j, \) then we have

\[
\frac{\sum_{l=1}^{M} \sum_{k=1}^{N} D_1(f^l, f^k)}{MN} = \frac{1}{N} \left[ \frac{\sum_{i=1}^{N} p_i x_i}{P_n} \right] + \left( \frac{2^{n-1} - n}{P_n} \right) \sum_{i=1}^{n} p_i \phi(x_i, y) \right] \\
+ \frac{1}{M} \left[ \sum_{l=1}^{M} \frac{W_l}{W_m} \phi(x, \frac{\sum_{j=1}^{m} w_j y_j}{W_f}) + \left( \frac{2^{m-1} - m}{W_m} \right) \sum_{j=1}^{m} w_j \phi(x, y_j) \right]
\]

and

\[
\frac{\sum_{l=1}^{M} \sum_{k=1}^{N} D_2(f^l, f^k)}{MN} = \frac{1}{W_m} \left[ \frac{1}{N} \left[ \sum_{i=1}^{N} \sum_{j=1}^{m} p_i \phi x_i, \frac{\sum_{j=1}^{m} w_j y_j}{W_f} \right] + \left( \frac{2^{n-1} - n}{P_n} \right) \sum_{i=1}^{n} \sum_{j=1}^{m} p_i \phi(x_i, y_j) \right] \\
+ \frac{1}{P_n} \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} p_i \phi x_i, \frac{\sum_{j=1}^{m} w_j y_j}{W_f} \right] + \left( \frac{2^{m-1} - m}{W_m} \right) \sum_{j=1}^{m} \sum_{i=1}^{n} p_i \phi(x_i, y_j) \right].
\]

Proof. Since, from (6) we know that

\[
D_1(f^l, f^k) = F(\phi, w, y, f^l) + F(\phi, p, x, f^k).
\]
Therefore, we have

\[
\frac{\sum_{j=1}^{M} \sum_{k=1}^{N} D_1(t^k, f^j)}{MN} = \frac{1}{MN} \left[ \sum_{i=1}^{M} \sum_{k=1}^{N} \left[ F(\phi, p, x, t^k) + F(\phi, w, y, f^j) \right] \right]
\]

\[
= \frac{1}{MN} \sum_{j=1}^{M} \sum_{k=1}^{N} \left\{ \frac{P_n}{P_m} \phi \left( \frac{1}{P_n} \sum_{j \in J^i} p_j x_j, g \right) + \frac{1}{P_n} \sum_{j \in J^i} p_j \phi (x_j, g) \right\}
\]

\[
+ \frac{W_p}{W_m} \phi \left( x, \frac{1}{W_p} \sum_{j \in J^i} w_j y_j \right) + \frac{1}{W_m} \sum_{j \in J^i} w_j \phi (x, y_j) \right\}
\]

\[
= \frac{1}{N} \left[ \sum_{k=1}^{N} P_n P_m \phi \left( \frac{a_k}{P_n}, g \right) + \frac{(2^{n-1} - n)}{P_n} \sum_{j=1}^{n} p_j \phi (x_j, g) \right]
\]

\[
+ \frac{1}{M} \left[ \sum_{i=1}^{M} W_p W_m \phi \left( \frac{\sum_{j \in J^i} w_j y_j}{W_p} \right) + \frac{(2^{m-1} - m)}{W_m} \sum_{j=1}^{m} w_j \phi (x, y_j) \right].
\]

Here it is obvious that

\[
\sum_{k=1}^{N} p_j \phi (x_j, y_j) = (2^{n-1} - (n - 1)) \sum_{j=1}^{n} p_j \phi (x_j, y_j),
\]

since every \( p_i \phi (x_i, y_i) \) appears as many times as there is a subset \( I_k \subset I_n, |I_k| \geq 2 \), and that doesn’t contain the index \( i \). Similarly we can prove the second part of the lemma. \( \square \)

The following refinement of Theorem 1.3 holds:

**Theorem 2.3.** Suppose that \( \phi : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R} \) is convex on the co-ordinates on \( \Delta \). If \( x_i \in [a, b], y_j \in [c, d], \) and \( p_i, w_j > 0, i \in [1, 2, \ldots, n], j \in [1, 2, \ldots, m], n, m \geq 3 \), with \( P_n = \sum_{i=1}^{n} p_i \) and \( W_m = \sum_{j=1}^{m} w_j \), then for any \( t^k \in \Omega_1 \) and \( f^j \in \Omega_2 \) we have

\[
\phi (x, g) \leq \frac{1}{2} \min_{k=1, \ldots, N} D_1(t^k, f^j) \leq \frac{1}{2} \frac{1}{MN} \sum_{i=1}^{M} \sum_{k=1}^{N} D_1(t^k, f^j) \leq \frac{1}{2} \max_{k=1, \ldots, N} D_1(t^k, f^j)
\]

\[
\leq \frac{1}{2} \left[ \frac{1}{P_n} \sum_{i=1}^{n} p_i \phi (x_i, g) + \frac{1}{W_m} \sum_{j=1}^{m} w_j \phi (x, y_j) \right]
\]

\[
\leq \frac{1}{2} \min_{k=1, \ldots, N} D_2(t^k, f^j) \leq \frac{1}{2} \frac{1}{MN} \sum_{i=1}^{M} \sum_{k=1}^{N} D_2(t^k, f^j) \leq \frac{1}{2} \max_{k=1, \ldots, N} D_2(t^k, f^j)
\]

\[
\leq \frac{1}{P_n W_m} \sum_{i=1}^{n} \sum_{j=1}^{m} p_i w_j \phi (x_i, y_j),
\]

(9)

where \( x = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \ g = \frac{1}{W_m} \sum_{j=1}^{m} w_j y_j, 1 \leq k \leq N, 1 \leq l \leq M. \)
Proof. One-dimensional Jensen’s inequality gives us
\[ \phi(x, g) \leq \frac{1}{W_m} \sum_{j=1}^{m} w_j \phi(x_j, y_j) \quad \text{and} \quad \phi(x, y) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \phi(x_i, y_j). \]

By Jensen’s inequality, we get
\[
F(\phi, p, x_i, l^i, y_j) = \frac{P_i}{P_n} \phi \left( \frac{1}{P_i} \sum_{i \in p} p_i x_i, y_j \right) + \frac{1}{P_n} \sum_{i \in p} p_i \phi(x_i, y_j)
\]
\[
\leq \frac{P_i}{P_n} \phi \left( \frac{1}{P_i} \sum_{i \in p} p_i x_i, y_j \right) + \frac{1}{P_n} \sum_{i \in p} p_i \phi(x_i, y_j)
\]
\[
= \phi \left( \frac{1}{P_i} \sum_{i \in p} p_i x_i, y_j \right) \quad \text{for every } i \in P.
\]

As the function $\phi$ is convex on the first co-ordinate, so we have
\[
F(\phi, p, x_i, l^i, y_j) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \phi(x_i, y_j).
\] (10)

Now, from (10) and (11), we have
\[
\phi(x, y) \leq F(\phi, p, x_i, l^i, y_j) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \phi(x_i, y_j).
\] (12)

Similarly, we can write
\[
\phi(x, g) \leq F(\phi, w, y, l^i, x_i) \leq \frac{1}{W_m} \sum_{j=1}^{m} w_j \phi(x_i, y_j).
\] (13)

Multiplying (12) and (13) respectively by $w_j$ and $p_i$ and summing over $i$ and $j$, we obtain
\[
\frac{1}{W_m} \sum_{j=1}^{m} w_j \phi(x, y) \leq \frac{1}{W_m} \sum_{j=1}^{m} w_j F(\phi, p, x_i, l^i, y_j) \leq \frac{1}{P_n W_m} \sum_{i=1}^{n} \sum_{j=1}^{m} p_i w_j \phi(x_i, y_j),
\] (14)

and
\[
\frac{1}{P_n} \sum_{i=1}^{n} p_i \phi(x, g) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i F(\phi, w, y, l^i, x_i) \leq \frac{1}{P_n W_m} \sum_{i=1}^{n} \sum_{j=1}^{m} p_i w_j \phi(x_i, y_j).\\\] (15)
Adding (14) and (15), one has the following
\[
\frac{1}{2} \left[ \frac{1}{P_n} \sum_{j=1}^{n} p_j \phi(x_j, y) + \frac{1}{W_m} \sum_{j=1}^{m} w_j \phi(x, y) \right] \leq \frac{1}{2} \left[ \frac{1}{P_n} \sum_{j=1}^{n} p_j F(\phi, w, y, j, x_i) + \frac{1}{W_m} \sum_{j=1}^{m} w_j F(\phi, p, x, l_k, y) \right]
\]
\[
\leq \frac{1}{P_n W_m} \sum_{j=1}^{n} \sum_{i=1}^{m} p_j w_j \phi(x_i, y_j).
\] (16)

Now, setting \( x_i = \phi \) and \( y_j = \phi \) in (12), (13) and adding we have
\[
\phi(\phi, \phi) \leq \frac{1}{2} \left[ F(\phi, w, y, f', x_i) + F(\phi, p, x, l_k) \right] \leq \frac{1}{2} \left[ \frac{1}{P_n} \sum_{j=1}^{n} p_j \phi(x_j, y) + \frac{1}{W_m} \sum_{j=1}^{m} w_j \phi(x, y) \right].
\]

Combining (16) and (17) we obtain
\[
\phi(\phi, \phi) \leq \frac{1}{2} \left[ F(\phi, w, y, f', x_i) + F(\phi, p, x, l_k) \right] \leq \frac{1}{2} \left[ \frac{1}{P_n} \sum_{j=1}^{n} p_j \phi(x_j, y) + \frac{1}{W_m} \sum_{j=1}^{m} w_j \phi(x, y) \right]
\]
\[
\leq \frac{1}{P_n W_m} \sum_{j=1}^{n} \sum_{i=1}^{m} p_j w_j \phi(x_i, y_j).
\]

The statement in the theorem follows by taking the min and max of \( D_1(l_k, f') \) and \( D_2(l_k, f') \) over the indices \( k \) and \( l \) with \( 1 \leq k \leq N, 1 \leq l \leq M \) and together with Lemma 2.2 and using the fact that
\[
\min_{k=1,...,N} D_1(l_k, f') \leq \frac{\sum_{k=1}^{M} \sum_{k=1}^{N} D_1(l_k, f')}{MN} \leq \max_{k=1,...,M} D_1(l_k, f') \quad (17)
\]

and
\[
\min_{k=1,...,N} D_2(l_k, f') \leq \frac{\sum_{k=1}^{M} \sum_{k=1}^{N} D_2(l_k, f')} {MN} \leq \max_{k=1,...,M} D_2(l_k, f'). \quad (18)
\]

This completes the desired proof. \( \square \)

**Remark 2.4.** For \( l_k = [\mu] \), \( \mu \in [1, ..., n] \) and \( f' = [v] \), \( v \in [1, ..., m] \), the above functionals take the form given below
\[
F(\phi, p, x, l_k, y_j) = F(\phi, p, x, [\mu], y_j) = \frac{1}{P_n} \sum_{j=1}^{n} p_j \phi(x_j, y_j),
\]
\[
F(\phi, w, y, f', x_i) = F(\phi, w, y, [v], x_i) = \frac{1}{W_m} \sum_{j=1}^{m} w_j \phi(x_i, y_j),
\]
\[
D_1(\mu, [v]) = F(\phi, w, y, [v]) + F(\phi, p, x, [\mu]) = \frac{1}{W_m} \sum_{j=1}^{m} w_j \phi(x_i, y_j) + \frac{1}{P_n} \sum_{j=1}^{n} p_j \phi(x_j, y_j),
\]
\[
D_2(\mu, [v]) = \frac{2}{P_n W_m} \sum_{i=1}^{n} \sum_{j=1}^{m} p_j w_j \phi(x_i, y_j).
\]
and the refinement given in Theorem 2.3 shrinks to the result given in Theorem 1.3.

In the next theorem, subsets of equivalent cardinality are observed.

**Theorem 2.5.** Suppose that \( \phi : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is convex on the co-ordinates on \( \Delta \). If \( x_i \in [a, b], y_j \in [c, d], \) and \( p_i, w_j > 0, i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, m\}, n, m \geq 3 \), with \( P_n = \sum_{i=1}^{n} p_i \) and \( W_m = \sum_{j=1}^{m} w_j \), then for any \( I^1 \in \Omega_1 \) and \( J^1 \in \Omega_2 \) such that \(|I^1| = s \geq 2\) and \(|J^1| = r \geq 2\) we have

\[
\phi(x, y) \leq \frac{1}{2} \min_{|I^1|=s} \sum_{i=1}^{n} p_i \phi(x_i, y) + \frac{1}{2} \max_{|J^1|=r} \sum_{j=1}^{m} w_j \phi(x, y_j),
\]

Proof. The statement in the theorem follows by taking the min and max of the functionals given in (1) and (2), after choosing every subset \( I^1 \in \Omega_1 \) and \( J^1 \in \Omega_2 \), such that \(|I^1| = s\) and \(|J^1| = r\), with \( 2 \leq s < n \) and \( 2 \leq r < m \). We use the facts mentioned in (17) and (18), where \( \binom{n}{s} \) and \( \binom{m}{r} \) represent the number of subsets \( I^1 \subset I_n \) and \( J^1 \subset J_m \), \(|I^1| = s\), \(|J^1| = r\). Note that

\[
\sum_{I^1 \subset I_n, |I^1| = s} \sum_{i \in I^1} p_i \phi(x_i, y_j) = \left( \binom{n}{s} - \binom{n-1}{s-1} \right) \sum_{i=1}^{n} p_i \phi(x_i, y_j),
\]

since every \( p_i \phi(x_i, y_j) \) in the double sum appears as many times as there are subsets \( I^1 \subset I_n, |I^1| = s \geq 2 \) such that \( i \notin I^1 \). The subsets \( I^1 \subset I_n \), with \(|I^1| = s\) and \( I^1 \in I_n \) is constructed by adding \( s - 1 \) elements from the \( n - 1 \) available once. Algebraically,

\[
\left( \frac{n}{s} - \frac{n-1}{s-1} \right) \sum_{i=1}^{n} p_i \phi(x_i, y_j) = \left( \frac{n-1}{s} \right) \sum_{i=1}^{n} p_i \phi(x_i, y_j).
\]

Similar arguments can be given for the subsets \( J^1 \subset J_m \), with \(|J^1| = r \geq 2\) and one has

\[
\sum_{J^1 \subset J_m, |J^1| = r} \sum_{j \in J^1} w_j \phi(x, y_j) = \left( \frac{m}{r} - \frac{m-1}{r-1} \right) \sum_{j=1}^{m} w_j \phi(x, y_j).
\]

Every partition of \( I_n = \{1, \ldots, n\} \) and \( J_m = \{1, \ldots, m\} \) gives the statement obtained in the next theorem.

**Theorem 2.6.** Suppose that \( \phi : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is convex on the co-ordinates on \( \Delta \). If \( x_i \in [a, b], y_j \in [c, d], \) and \( p_i, w_j > 0, i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, m\}, n, m \geq 4 \), with \( P_n = \sum_{i=1}^{n} p_i \) and \( W_m = \sum_{j=1}^{m} w_j \). For every integers
Subtracting 1

Proof.

Taking the max of the right hand side in (20) for $I^k \subset I_n$, $|I^k| \geq 2$ and $J^l \subset J_m$, $|J^l| \geq 2$, the proof is making through. □

Corollary 2.8. Under the conditions of Theorem 2.3, we obtain:

$$\frac{1}{P_n W_m} \sum_{i=1}^{n} \sum_{j=1}^{m} p_i w_j \phi(x_i, y_j) - \phi(x, y) \leq \max_{f, f'} \left[ \frac{1}{P_n W_m} \sum_{i=1}^{n} \sum_{j=1}^{m} p_i w_j \phi(x_i, y_j) - \frac{1}{2} D_1(I^k, J^l) \right] \geq 0. \tag{21}$$
Proof. The proof is similar to that of corollary 2.7 only use $D_2(l^k, f')$ instead of $D_1(l^k, f')$. □

Now we give another refinement of the Jensen’s inequality for the convex function defined on the co-ordinates of the bidimensional interval in the plane:

**Theorem 2.9.** Let $\phi : [a, b] \times [c, d] \to \mathbb{R}$ be a co-ordinate convex function on $[a, b] \times [c, d]$. If $x_i \in [a, b]$, $y_j \in [c, d]$ such that $x_{i+m} = x_i$, $y_{j+m} = y_j$ and $p_i, w_j > 0$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, with $\sum_{i=1}^{k} p_i = 1$ and $\sum_{j=1}^{l} w_j = 1$, for some $k$ and $l$, $2 \leq k \leq n$ and $2 \leq l \leq m$, then we have

$$
\phi(x, y) \leq \frac{1}{2} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi \left( \frac{\sum_{r=0}^{k-1} p_{r+1} x_{i+r}, \ y_j}{} \right) + \frac{1}{m} \sum_{j=1}^{m} \phi \left( x_i, \frac{\sum_{r=0}^{l-1} w_{r+1} y_{j+r}}{} \right) \right] \leq \frac{1}{2} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi(x_i, y) + \frac{1}{m} \sum_{j=1}^{m} \phi(x, y_j) \right]
$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{m} \sum_{j=1}^{m} y_j$.

Proof. Since $\phi_{y_j} : [a, b] \to \mathbb{R}$ is convex, so by Jensen’s inequality, we have

$$
\phi_{y_j} \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) = \phi_{y_j} \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \leq \phi_{y_j} \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{r=0}^{k-1} p_{r+1} x_{i+r} \right)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \phi \left( \sum_{r=0}^{k-1} p_{r+1} x_{i+r}, \ y_j \right) \leq \frac{1}{n} \sum_{i=1}^{n} \phi \left( \sum_{r=0}^{k-1} p_{r+1} x_{i+r}, \ y_j \right)
$$

therefore,

$$
\phi \left( \frac{1}{n} \sum_{i=1}^{n} x_i, \ y_j \right) \leq \frac{1}{n} \sum_{i=1}^{n} \phi \left( \sum_{r=0}^{k-1} p_{r+1} x_{i+r}, \ y_j \right).
$$

On the other hand, since $\phi_{x_i} : [a, b] \to \mathbb{R}$ is convex, so again by Jensen’s inequality and simple calculations one can get

$$
\frac{1}{n} \sum_{i=1}^{n} \phi \left( \sum_{r=0}^{k-1} p_{r+1} x_{i+r}, \ y_j \right) \leq \frac{1}{n} \sum_{i=1}^{n} \phi \left( x_i, \ y_j \right)
$$

the combination of (22) and (23) yields

$$
\phi \left( \frac{1}{n} \sum_{i=1}^{n} x_i, \ y_j \right) \leq \frac{1}{n} \sum_{i=1}^{n} \phi \left( \sum_{r=0}^{k-1} p_{r+1} x_{i+r}, \ y_j \right) \leq \frac{1}{n} \sum_{i=1}^{n} \phi \left( x_i, \ y_j \right).
$$

Similarly, the convexity of $\phi_{x_i} : [c, d] \to \mathbb{R}$ implies the following

$$
\phi \left( x_i, \frac{1}{m} \sum_{j=1}^{m} y_j \right) \leq \frac{1}{m} \sum_{j=1}^{m} \phi \left( x_i, \sum_{r=0}^{l-1} w_{r+1} y_{j+r} \right) \leq \frac{1}{m} \sum_{j=1}^{m} \phi \left( x_i, \ y_j \right).
$$
Multiplying (24) and (25) by $\frac{1}{n}$ and $\frac{1}{m}$ respectively and summing over $j, i$ and then adding the obtained results, one has the following

$$
\frac{1}{2} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi(x_i, y_j) + \frac{1}{m} \sum_{j=1}^{m} \phi(x, y_j) \right] \leq \frac{1}{2} \left[ \frac{1}{mn} \sum_{j=1}^{m} \sum_{i=1}^{n} \phi \left( \sum_{r=0}^{k-1} \alpha_{r+1} x_{i+r}, y_j \right) + \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \phi \left( \sum_{r=0}^{l-1} \alpha_{r+1} y_{j+r} \right) \right]
$$

$$
\leq \frac{1}{mn} \sum_{j=1}^{m} \sum_{i=1}^{n} \phi(x_i, y_j). \quad (26)
$$

Furthermore by setting $x_i \to \bar{x}$ and $y_j \to \bar{y}$ in (24) and (25) respectively we get

$$
\phi \left( \frac{1}{n} \sum_{i=1}^{n} x_i, \bar{y} \right) \leq \frac{1}{n} \sum_{i=1}^{n} \phi \left( \sum_{r=0}^{k-1} \alpha_{r+1} x_{i+r}, \bar{y} \right) \leq \frac{1}{n} \sum_{i=1}^{n} \phi \left( x_i, \bar{y} \right) \quad (27)
$$

and

$$
\phi \left( \bar{x}, \frac{1}{m} \sum_{j=1}^{m} y_j \right) \leq \frac{1}{m} \sum_{j=1}^{m} \phi \left( \bar{x}, \sum_{r=0}^{l-1} \alpha_{r+1} y_{j+r} \right) \leq \frac{1}{m} \sum_{j=1}^{m} \phi \left( \bar{x}, y_j \right). \quad (28)
$$

Now adding them, we obtain

$$
\phi(\bar{x}, \bar{y}) \leq \frac{1}{2} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi \left( \sum_{r=0}^{k-1} \alpha_{r+1} x_{i+r}, \bar{y} \right) + \frac{1}{m} \sum_{j=1}^{m} \phi \left( \bar{x}, \sum_{r=0}^{l-1} \alpha_{r+1} y_{j+r} \right) \right] \leq \frac{1}{2} \left[ \frac{1}{n} \sum_{i=1}^{n} \phi(x_i, \bar{y}) + \frac{1}{m} \sum_{j=1}^{m} \phi(\bar{x}, y_j) \right].
$$

Hence, we have the desired result. \qed

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