Abstract. The goal of this paper is to define the spaces \( V_{\lambda,\sigma}^{0}(p) \) and \( V_{\lambda,\sigma}^{0}(p) \) by using de la Vallée Poussin and invariant mean. Furthermore, we characterize certain matrices in \( V_{\lambda,\sigma}^{0} \) which will up a gap in the existing literature.

1. Introduction and Background

Let \( w \) denote the set of all real and complex sequences \( x = (x_{k}) \). By \( l_{\infty} \) and \( c \), we denote the Banach spaces of bounded and convergent sequences \( x = (x_{k}) \) normed by \( ||x|| = \sup_{k}|x_{k}| \), respectively. A linear functional \( L \) on \( l_{\infty} \) is said to be a Banach limit \([1]\) if it has the following properties:

1. \( L(x) \geq 0 \) if \( n \geq 0 \) (i.e. \( x_{n} \geq 0 \) for all \( n \)),
2. \( L(e) = 1 \) where \( e = (1, 1, ...) \),
3. \( L(Dx) = L(x) \), where the shift operator \( D \) is defined by \( D(x_{n}) = \{x_{n+1}\} \).

Let \( B \) be the set of all Banach limits on \( l_{\infty} \). A sequence \( x \in \ell_{\infty} \) is said to be almost convergent if all Banach limits of \( x \) coincide. Let \( \hat{c} \) denote the space of almost convergent sequences. Lorentz [3] has shown that

\[
\hat{c} = \left\{ x \in \ell_{\infty} : \lim_{m} d_{m,n}(x) \text{ exists uniformly in } n \right\},
\]

where

\[
d_{m,n}(x) = \frac{x_{n} + x_{n+1} + x_{n+2} + \cdots + x_{n+m}}{m + 1}.
\]

If \( p_{k} \) is real and \( p_{k} > 0 \), we define (see, Maddox [4])

\[
c_{0}(p) = \left\{ x : \lim_{k \to \infty} |x_{k}|^{p_{k}} = 0 \right\}
\]

and

\[
c(p) = \left\{ x : \lim_{k \to \infty} |x_{k} - l|^{p_{k}} = 0, \text{ for some } l \right\}
\]
If \( p_m \) is real such that \( p_m > 0 \) and \( \sup p_m < \infty \), we define (see, Nanda [16])

\[
\hat{c}_0(p) = \left\{ x : \lim_{m \to \infty} \left\| d_{m,n}(x) \right\| = 0, \text{ uniformly in } n \right\}
\]

and

\[
\hat{c}(p) = \left\{ x : \lim_{m \to \infty} \left\| d_{m,n}(x - l) \right\| = 0, \text{ for some } l, \text{ uniformly in } n \right\}.
\]

Shafer [26] defined the \( \sigma \)-convergence as follows: Let \( \sigma \) be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional \( \phi \) on \( L_\infty \) is said to be an invariant mean or a \( \sigma \)-mean provided that

(i) \( \phi(x) \geq 0 \) when the sequence \( x = (x_k) \) is such that \( x_k \geq 0 \) for all \( k \),

(ii) \( \phi(c) = 1 \), where \( c = (1,1,1,\ldots) \), and

(iii) \( \phi(x) = \phi(x_{\sigma(k)}) \) for all \( x \in L_\infty \).

We denote by \( V_\sigma \) the space of \( \sigma \)-convergent sequences. It is known that \( x \in V_\sigma \) if and only if

\[
\frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} \to \text{ a limit}
\]

as \( m \to \infty \), uniformly in \( n \). Here \( \sigma^k(n) \) denotes the \( k \)-th iterate of the mapping \( \sigma \) at \( n \).

A \( \sigma \)-mean extends the limit functional on \( c \) in the sense that \( \phi(x) = \lim x \) for all \( x \in c \) if and only if \( \sigma \) has no finite orbits, that is to say, if and only if, for all \( n > 0, k \geq 1, \sigma^k(n) \neq n \).

In case \( \sigma \) is the translation mapping \( n \to n + 1 \), a \( \sigma \)-mean reduces to the unique Banach limit and \( V_\sigma \) reduces to \( \hat{c} \).

2. (\( \sigma, \lambda \))-Convergence

We define the following:

Let \( \lambda = (\lambda_m) \) be a non-decreasing sequence of positive numbers tending to \( \infty \) such that

\[
\lambda_{m+1} \leq \lambda_m + 1, \quad \lambda_1 = 1.
\]

A sequence \( x = (x_k) \) of real numbers is said to be \( (\sigma, \lambda) \)-convergent to a number \( L \) if and only if \( x \in V_\lambda^\sigma \), where

\[
V_\lambda^\sigma = \left\{ x \in L_\infty : \lim_{m \to \infty} t_{m,n}(x) = L \text{ uniformly in } n; L = (\sigma, \lambda) - \lim x \right\},
\]

\[
t_{m,n}(x) = \frac{1}{\lambda_m} \sum_{l \in I_m} x_{\sigma^l(n)},
\]

and \( I_m = [m - \lambda_m + 1, m] \) (see, [24]). Note that \( c \subset V_\lambda^\sigma \subset L_\infty \). For \( \sigma(n) = n + 1 \), \( V_\lambda^\sigma \) is reduced to the space \( V_\lambda \) of almost \( \lambda \)-convergent sequences and if we take \( \sigma(n) = n + 1 \) and \( \lambda = n \), \( V_\lambda^\sigma \) reduce to \( \hat{c} \) (see, [16]).

It is quite natural to expect that the sequence \( V_\sigma^\lambda \) and \( V_{\sigma_0}^\lambda \) can be extended to \( V_{\sigma_0}^\lambda(p) \) and \( V_{\sigma_0}^\lambda(p) \) just as \( \hat{c} \) and \( \hat{c}_0 \) were extended to \( \hat{c}(p) \) and \( \hat{c}_0(p) \) respectively.

The main object of this paper is to study \( V_{\sigma_0}^\lambda(p) \) and \( V_{\sigma_0}^\lambda(p) \) (the definitions are given below) and characterize certain matrices in \( V_{\sigma_0}^\lambda(p) \).

If \( p_m \) is real such that \( p_m > 0 \) and \( \sup p_m < \infty \), we define

\[
V_{\sigma_0}^\lambda(p) = \left\{ x : \lim_{m \to \infty} \left\| t_{m,n}(x) \right\|^p = 0, \text{ uniformly in } n \right\}
\]
Lemma 3.2. \( V^\lambda_\sigma(p) = \left\{ x : \lim_{m \to \infty} |t_{m,n}(x - le)|^{p_n} = 0, \text{ for some } l, \text{ uniformly in } n \right\}. \)

In particular, if \( p_m = p > 0 \) for all \( m \), we have \( V^\lambda_{c_0}(p) = V^\lambda_\sigma(p) \) and \( V^\lambda_{c_0}(p) = V^\lambda_{c_0}(p) \). In Theorem 4, we prove that \( V^\lambda_{c_0}(p) \) and \( V^\lambda_\sigma(p) \) are complete linear topological spaces. Theorem 7 characterizes the matrices in the class \((c_0(p), V^\lambda_{c_0}(p)).\) In Theorem 8 we determine the matrix in the class \((c(p), V^\lambda_{c})).\) Matrix transformations between sequence spaces have also been discussed by Savas and Mursaleen ([23]), Mursaleen ([7–15]), Savas ([17–22, 25]) and many others.

A linear topological space \( X \) is called paranormed space if there exists a subadditive function \( g : X \to \mathbb{R}^+ \) such that \( g(0) = 0 \) and \( g(x) = g(-x) \) and the multiplication is continuous, that is, \( \lambda_n \to \lambda \) and \( g(x_n - x) \to 0 \) imply that \( g(\lambda_n x_n - \lambda x) \to 0 \) for \( \lambda \)'s \( \in \mathbb{C} \) and \( x \in X. \)

Suppose that \( M = \max(1, \sup p_m = H). \) Since \( p_m/M \leq 1, \) we have for all \( m \) and \( n, \)

\[ |t_{mn}(x + y)|^{p_n/M} \leq |t_{mn}(x)|^{p_n/M} + |t_{mn}(y)|^{p_n/M} \]

and for all \( \lambda \in \mathbb{C} \)

\[ |\lambda|^{p_n/M} \leq \max(1, |\lambda|). \]

By using (1) and (2) we can see that \( V^\lambda_{c_0}(p) \) and \( V^\lambda_{c}(p) \) are linear spaces.

3. Main Results

We first establish a number of lemmas about \( V^\lambda_{c_0}(p) \) and \( V^\lambda_{c}(p). \)

Lemma 3.1. \( V^\lambda_{c_0}(p) \) is a linear topological space paranormed by \( g \) where

\[ g(x) = \sup_{m,n} |t_{m,n}(x)|^{p_n/M}. \]

Proof. One can easily see that \( g(0) = 0 \) and \( g(x) = g(-x). \) The subadditivity of \( g \) follows from (1). It remains to show that the scalar multiplication is continuous. It follows from (2) that for \( \mu \in \mathbb{C} \) and \( x \in V^\lambda_{c_0}(p) \)

\[ g(\mu x) \leq \max(1, |\mu|) g(x). \]

Therefore \( \mu \to 0, x \to 0 \Rightarrow \mu x \to 0 \) and if \( \mu \) is fixed, \( x \to 0 \Rightarrow \mu x \to 0. \) If \( x \in V^\lambda_{c_0}(p) \) is fixed, given \( \varepsilon > 0, \) there exists \( m_0 \) such that

\[ \sup_{m > m_0} |\mu t_{m,n}(x)|^{p_n/M} < \varepsilon/2, \]

for all \( n \) and we can choose \( \delta > 0 \) such that for \( |\mu| < \delta, \) we have

\[ \sup_{m \leq m_0} |\mu t_{m,n}(x)|^{p_n/M} < \varepsilon/2, \]

for all \( n. \) Thus from (3) and (4) we get

\[ |\mu| < \delta \Rightarrow g(\mu x) \leq \varepsilon. \]

This completes the proof. \( \square \)

Lemma 3.2. \( V^\lambda_{c}(p) \) is a linear topological space paranormed by \( g, \) if \( \inf p_m > 0. \)
Proof. It is enough to show that for fixed \( x \in V^\lambda_0(p), \mu \to 0 \Rightarrow \mu x \to 0 \). Let \( \inf p_m = p' > 0 \), then we have
\[
\varrho(\mu x) \leq \max(\|\mu\|, |\mu'|) \varrho(x).
\]
The result follows from the above inequality. \( \square \)

**Lemma 3.3.** \( V^\lambda_{\alpha_0}(p) \) and \( V^0_{\alpha_0}(p)(\inf p_m > 0) \) are complete with respect to their paranorm topologies.

**Proof.** Let \( \{x^i\} \) be Cauchy sequence in \( V^\lambda_{\alpha_0}(p) \). Then \( \{x^i_k\} \) for each \( k \), is Cauchy in \( \mathbb{C} \) and hence \( x^i_k \to x^0_k \) for each \( k \). Put \( x^0 = \{x^0_k\} \). Given \( \varepsilon > 0 \), there exists \( N_0 \) such that for \( i, j > N_0 \),
\[
|t_{m,n}(x^i - x^j)|^{p_m/M} < \varepsilon/5
\]
for all \( m \) and \( n \). Taking limit as \( j \to \infty \) we get
\[
|t_{m,n}(x^i - x^0)|^{p_m/M} < \varepsilon/5,
\]
for all \( m \) and \( n \). Therefore \( (x^i - x^0) \) and by linearity \( x^0 \in V^\lambda_{\alpha_0}(p) \). If \( \{x^i\} \) be Cauchy sequence in \( V^\lambda_{\alpha_0}(p) \) then there exists \( x^0 \) such that \( x^i \to x^0 \). We now show that \( x^0 \in V^\lambda_{\alpha_0}(p) \). Since \( x^i \in V^\lambda_{\alpha_0}(p) \) there exists \( l \in \mathbb{C} \) such that
\[
|t_{m,n}(x^i - l e)|^{p_m/M} < \varepsilon/5,
\]
for all \( m \) and \( n \). From that \( (5), (7) \) and \( (1) \) it follows that
\[
|t_{m,n}(l e - l e)|^{p_m/M} < 3/5\varepsilon.
\]
Thus \( \{l e\} \) is Cauchy in \( \mathbb{C} \) and therefore there exists \( l \in \mathbb{C} \) such that
\[
|t_{m,n}(l e - l e)|^{p_m/M} < 3/5\varepsilon.
\]
Now by \( (1), (6), (7) \) and \( (8) \) we get
\[
|t_{m,n}(x^0 - l e)|^{p_m/M} < \varepsilon.
\]
This completes the proof. \( \square \)

Combining the above lemmas we have

**Theorem 3.4.** \( V^0_{\alpha_0}(p) \) and \( V^0_{\alpha'}(p)(\inf p_m > 0) \) are complete linear topological spaces paranormed by \( \varrho \) as defined in Lemma 1.

In general \( \varrho \) is not a norm. If \( p_m = p \) for all \( m \) then clearly \( \varrho \) is a norm.

The following proposition give inclusion relations among the spaces \( V^\lambda_{\alpha_0}(p) \) and \( V^\lambda_{\alpha'}(p) \). These are routine verifications and therefore we omit the proofs.

**Proposition 3.5.** If \( 0 < p_m \leq q_m < \infty \), then
\[
(i) \quad V^\lambda_{\alpha_0}(p) \subset V^\lambda_{\alpha_0}(q)
\]
\[
(ii) \quad V^\lambda_{\alpha'}(p) \subset V^\lambda_{\alpha'}(q).
\]
For $r > 0$, a nonempty subset $U$ of a linear space is said to be absolutely $r$-convex if $x, y \in U$ and $|x|^r + |y|^r \leq 1$ together imply that $ax + \mu y \in U$. A linear topological space $X$ is said to be $r$-convex (see Maddox and Roles [5]) if every neighbourhood of 0 in $X$ contains as absolutely $r$-convex neighbourhood of 0 in $X$. We have:

**Proposition 3.6.** $V_{\alpha}^1(p)$ and $V_{\sigma}^1(p)$ are 1-convex.

**Proof.** If $0 < \delta < 1$, then

$$U = \{x : g(x) \leq \delta\}$$

is an absolutely 1-convex set, for let $a, b \in U$ and $|\alpha| + |\mu| \leq 1$, then

$$g(\alpha a + \mu b) \leq (|\alpha| + |\mu|)^{1/m} \delta \leq \delta.$$ 

This completes the proof. \(\square\)

Let $X$ and $Y$ be two nonempty subsets of the space $w$ of complex sequences. Let $A = (a_{nk})$, $(n, k = 1, 2, ...)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each $n$. Throughout $\sum_k$ denotes summation over $k$ from $k = 1$ to $k = \infty$. If $x = (x_k) \in X \Rightarrow Ax = (A_n(x)) \in Y$ we say that $A$ defines a matrix transformation from $X$ to $Y$ and we denote it by $A : X \rightarrow Y$. By $(X, Y)$ we mean the class of matrices $A$ such that $A : X \rightarrow Y$.

We now characterize the matrices in the class $(c_0(p), V_{\alpha}^1(p))$. We write

$$t_{m,n}(Ax) = \sum_k a(n, k, m) x_k$$

where

$$a(n, k, m) = \frac{1}{\lambda_{nk}} \sum_{\mu_{nk}} a_{\mu(n,k)}.\]$$

**Theorem 3.7.** $A \in (c_0(p), V_{\alpha}^1(p)$ if and only if

\[(i) \text{ there exists an integer } B > 1 \text{ such that }\]

$$C_n = \sup_m \left\{ \sum_k |a(n, k, m)|^{1/p} \right\}^{p^w} < \infty, \quad (\forall n)$$

\[(ii) \lim_{m \rightarrow \infty} |a(n, k, m)|^{p^w} = 0 \text{ uniformly in } n.\]

**Proof.** Necessity. Suppose that $A \in (c_0(p), V_{\alpha}^1(p))$. Define $\varepsilon_k = \{\delta_{nk}\}$ where $\delta_{nk} = 0 (n \neq k), = 1 (n = k).$ Since $\varepsilon_k \in c_0(p), (ii)$ must hold. Fix $n \in \mathbb{Z}^+$. Put $f_{m,n}(x) = \left|t_{m,n}(Ax)\right|^{p^w}$. Now $\{f_{m,n}\}_{m,n}$ is a sequence of continuous linear functionals such that $\lim_{m} f_{m,n}(x)$ exists. Therefore by uniform boundedness principle for $0 < \delta < 1$, there exists $S_{\delta}[0] \subset c_0(p)$ and a constant $K$ such that $f_{m,n}(x) \leq K$ for each $m$ and $x \in S_{\delta}[0]$. Define for each $r$:

$$y_{k}^{(r)} = \begin{cases} \delta_{k}^{p^w} \sigma(a(n, k, m)), & 0 \leq k \leq r; \\ 0, & r < k. \end{cases}$$

Now $y_{k}^{(r)} \in S_{\delta}[0]$ and

$$\left\{ \sum_{k=1}^{r} |a(n, k, m)|^{1/p} \right\}^{p^w} \leq K.$$
for each \( m \) and each \( m \) where \( B = \delta^{-k} \). Therefore (i) holds and this proves this necessity.

Sufficiency. Suppose that the conditions (i) and (ii) hold and that \( x \in c_0(p) \). Fix \( n \in \mathbb{Z}^+ \). Given \( \varepsilon > 0 \), there exists \( k_0 \) such that for \( k \) and \( m \) both larger than \( k_0 \),

\[
B^{1/p_n} |x_k| < (\varepsilon/C_n)^{1/p_n}.
\]

We have, for \( C = \max(1, 2^{l-1}) \) the inequality (see Maddox 7, p. 346)

\[
\left| t_{m,n}(A(x)) \right|^{p_n} \leq C(S_1 + S_2),
\]

where

\[
S_1 = \left| \sum_{k \leq k_0} a(n, k, m)x_k \right|^{p_n}
\]

and

\[
S_2 = \left| \sum_{k > k_0} a(n, k, m)x_k \right|^{p_n}.
\]

Since (ii) holds there exists \( m_0 \in \mathbb{Z}^+ \) such that for \( m > m_0 \), \( |a(n, k, m)| < \varepsilon^{1/p_n} \). Therefore for such \( m \),

\[
S_1 \leq \left( \sum_{k \leq k_0} |a(n, k, m)x_k| \right)^{p_n} < \varepsilon \left( \sum_{k \leq k_0} |x_k| \right)^{p_n}
\]

\[
< \varepsilon \max \left[ 1, \left( \sum_{k \leq k_0} |x_k| \right)^M \right].
\]

Again for \( m > m_0 \)

\[
S_2^{1/p_n} \leq \sum_{k > k_0} |a(n, k, m)x_k| < \varepsilon^{1/p_n},
\]

and consequently

\[
S_2 \leq \varepsilon, (\forall m > m_0).
\]

Hence the sufficiency follows from (9) and (10). This completes the proof. \( \square \)

We now have

**Theorem 3.8.** \( A \in (c(p), V^\alpha) \) if and only if

(i) there exists some integer \( B > 1 \) such that

\[
D_n = \sup_m \sum_k |a(n, k, m)| B^{-1/p_n} \in \mathcal{C}, (\forall n);
\]

(ii) there exists \( \alpha \in C \) such that \( \lim_{m \to \infty} a(n, k, m) = \alpha_k \) uniformly in \( n \);

(iii) there exists \( \alpha \in C \) such that \( \lim_{m \to \infty} \sum_k a(n, k, m) = \alpha \) uniformly in \( n \).
Proof. Necessity. Let \( A \in (c(p), V_0^p) \). Since \( e_l \) and \( e \) are in \( c(p) \), (ii) and (iii) must hold. Fix \( n \in \mathbb{Z}^+ \). Put \( \sigma_{m,n}(x) = t_{m,n}(Ax) \). Since \((c(p), V_0^p) \subset (c_0(p), V_0^p)\) \((\sigma_{m,n})_m \) is a sequence of continuous linear functionals on \( c_0(p) \), such that \( \lim \sigma_{m,n}(x) \) exists uniformly in \( n \). Therefore as in the necessity part of Theorem 7 the result follows from uniform boundedness principle.

Sufficiency. Suppose that conditions (i) – (iii) hold and \( x \in c(p) \). Then there exists \( l \) such that \( |x_k \to l|^p \to 0 \). Hence given \( 0 < \epsilon < 1 \), \( \exists k_0 : \forall k < k_0 \)

\[
|x_k \to l|^p \leq \frac{\epsilon}{B(2D_n + 1)} < 1
\]

and therefore for \( k < k_0 \)

\[
B^{1/p} |x_k \to l| < B^{M/p} |x_k \to l| < (\epsilon/2D_n + 1)^{M/p} < \epsilon/2D_n + 1.
\]

By (i) and (ii) we have

\[
\sum_k |a(n,k,m) - a_k| B^{-1/p} < 2D_n.
\]

Hence

\[
\sum_{k \geq k_0} |(a(n,k,m) - a_k) (x_k - l)| < \epsilon.
\]

(11)

Also,

\[
\lim \sum_{k \geq k_0} |(a(n,k,m) - a_k) (x_k - l)| = 0,
\]

(12)

uniformly in \( n \). Therefore by (11) and (12) we get

\[
\lim \sum_k a(n,k,m)x_k = l \alpha + \sum_k a_k (x_k - l)
\]

(13)

uniformly in \( n \). This completes the proof. \( \square \)

**Corollary 3.9.** \( A \in (c_0(p), V_0^p) \) if and only if conditions (i) and (ii) of Theorem 7 hold.

We write \((c(p), V_0^p, P)\) to denote the subset of \((c(p), V_0^p)\) such that \( Ax \) is \((\alpha, \lambda)-\) convergent to the limit of \( x \) in \( c(p) \). We now consider the class \((c(p), V_0^p, P)\).

**Theorem 3.10.** \( A \in (c(p), V_0^p, P) \) if and only if (i) the condition of Theorem 8 holds; (ii) \( \lim a(n,k,m) = 0 \) uniformly in \( n \); (iii) \( \sum_k a(n,k,m) = 1 \), uniformly in \( n \).

**Proof.** Let \( A \in (c(p), V_0^p, P) \). Then the conditions hold by Theorem 3. Let the conditions (i) – (iii) hold. Then by Theorem 8 \( A \in (c(p), V_0^p) \) and (13) reduces to

\[
\lim \sum_k a(n,k,m)x_k = l
\]

uniformly in \( n \). This proves the theorem. \( \square \)

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References