Path-Set Induced Closure Operators on Graphs

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Abstract. Given a simple graph, we associate with every set of paths of the same positive length a closure operator on the (vertex set of the) graph. These closure operators are then studied. In particular, it is shown that the connectedness with respect to them is a certain kind of path connectedness. Closure operators associated with sets of paths in some graphs with the vertex set $\mathbb{Z}^2$ are discussed which include the well known Marcus-Wyse and Khalimsky topologies used in digital topology. This demonstrates possible applications of the closure operators investigated in digital image analysis.

1. Introduction

Topological spaces in their usual meaning, i.e. those developed by Bourbaki [1], play a basic role in mathematical analysis (where even special topologies are employed, namely the Hausdorff ones). In many other branches of mathematics and in computer science, various generalizations of topological spaces are often used. In particular, closure operators obtained from the Kuratowski ones by omitting some axioms occur in numerous applications. For example, the so-called closure operators on categories studied in categorical topology are supposed to satisfy just the axioms of extensiveness and monotony (and a certain categorical axiom - the axiom of functionality) - see [4]. The closure operators usually employed in algebra are just the extensive, monotone and idempotent ones. If also the condition is satisfied that the closure of any set equals the union of the closures of all its finite subsets, we get the well-known algebraic closure operators (cf. [6]). The convex hull operators used in geometry (cf. [16]) are just the grounded, extensive, monotone and idempotent closure operators. Such closure operators occur also in logic - they are obtained by assigning, to a given set $A$ of formulas, the set of all formulas provable from $A$ - cf.[10]. The closure operators which are, in general, only grounded, extensive and monotone were studied by E.Čech in [2] and those which are moreover additive, i.e. the so-called pretopologies, were studied by the same author in [3]. Pretopologies were employed for solving problems related to digital image processing in [12-14].

In this note we will study closure operators on graphs which are only grounded, extensive and monotone in general (i.e. which are closure operators in the sense of [2]). These closure operators are induced by sets of paths of a given length. We will study properties of the induced closure operators, in particular the connectedness with respect to them. We will show that the connectedness is a path connectedness for certain special paths. To demonstrate possible applications of our results in digital image analysis, we will
discuss some closure operators associated with sets of paths in certain graphs with the vertex set \( \mathbb{Z}^2 \) and we will show that these closure operators include some known closure operators on \( \mathbb{Z}^2 \) used in digital topology, in particular the Marcus-Wyse and Khalimsky topologies.

2. Preliminaries

By a closure operator \( u \) on a set \( X \) we mean a map \( u: \exp X \to \exp X \) (where \( \exp X \) denotes the power set of \( X \)) which is

(i) grounded (i.e. \( u\emptyset = \emptyset \)),

(ii) extensive (i.e. \( A \subseteq X \implies A \subseteq uA \)), and

(iii) monotone (i.e. \( A \subseteq B \subseteq X \implies uA \subseteq uB \)).

The pair \((X, u)\) is then called a closure space and, for every subset \( A \subseteq X \), \( uA \) is called the closure of \( A \).

A closure operator \( u \) on \( X \) which is

(iv) additive (i.e. \( u(A \cup B) = uA \cup uB \) whenever \( A, B \subseteq X \)) and

(v) idempotent (i.e. \( uuA = uA \) whenever \( A \subseteq X \))

is called a Kuratowski closure operator or a topology and the pair \((X, u)\) is called a topological space.

Given a cardinal \( m \), a closure operator \( u \) on a set \( X \) and the closure space \((X, u)\) are called an \( S_m \)-closure operator and an \( S_m \)-space (briefly, an \( S_m \)-space), respectively, if the following condition is satisfied:

\[
A \subseteq X \implies uA = \bigcup\{uB : B \subseteq A, \text{card } B < m\}.
\]

The algebraic closure operators are obtained from the idempotent \( S_\infty \)-closure operators by omitting the requirement of groundedness. In [3], \( S_2 \)-closure operators and \( S_2 \)-spaces are called quasi-discrete. \( S_2 \)-topological spaces are usually called Alexandroff spaces - see [6]. Of course, any \( S_2 \)-closure operator is additive and is given by closures of the singleton subsets. It is also evident that any \( S_\alpha \)-closure operator is an \( S_\beta \)-closure operator whenever \( \alpha < \beta \). Since any closure operator on a set \( X \) is obviously an \( S_\infty \)-closure operator for each cardinal \( \alpha \) with \( \alpha > \text{card } X \), there exists a least cardinal \( \alpha \) such that \( u \) is an \( S_\alpha \)-closure operator. Such a cardinal is then an important invariant of the closure operator \( u \). Evidently, if \( \alpha \leq \aleph_0 \), then any additive \( S_\alpha \)-closure operator is an \( S_2 \)-closure operator.

Many concepts known for topological spaces (see e.g. [5]) may be naturally extended to closure spaces. Given a closure space \((X, u)\), a subset \( A \subseteq X \) is called closed if \( uA = A \), and it is called open if \( X - A \) is closed. A closure space \((X, u)\) is said to be a subspace of a closure space \((Y, v)\) if \( uA = vA \cap X \) for each subset \( A \subseteq X \) . We will speak briefly about a subspace \( X \) of \((Y, v)\). A closure space \((X, u)\) is said to be connected if \( \emptyset \) and \( X \) are the only subsets of \( X \) which are both closed and open. A subset \( X \subseteq Y \) is connected in a closure space \((Y, v)\) if the subspace \( X \) of \((Y, v)\) is connected. A maximal connected subset of a closure space is called a component of this space. All the basic properties of connected sets and components in topological spaces are preserved also in closure spaces. A closure space \((X, u)\) is said to be a \( T_0 \)-space if, for any points \( x, y \in X \), from \( x \in u(y) \) and \( y \in u(x) \) it follows that \( x = y \), and it is called a \( T_1 \)-space if each singleton subset of \( X \) is closed or open. Given closure spaces \((X, u)\) and \((Y, v)\), a map \( v : X \to Y \) is said to be a continuous map of \((X, u)\) into \((Y, v)\) if \( f(uA) \subseteq vf(A) \) for each subset \( A \subseteq X \).

3. N-Path Sets in Graphs and Associated Closure Operators

Given a natural number (i.e. a finite ordinal) \( m \), we write briefly \( i < m \) and \( i \leq m \) instead of \( 0 \leq i < m \) and \( 0 \leq i \leq m \), respectively, because \( i \) is considered to be a natural number, too (i.e. the linear order \( < \) and the corresponding partial order \( \leq \) are considered to be defined on just the set of natural numbers).

Let \( G = (V, E) \) be an (undirected simple) graph with \( V \neq \emptyset \) the vertex set and \( E \) the set of edges. We denote by \( \mathcal{P}(G) \) the set of all paths in \( G \), i.e. the set of all sequences \( (x_i : 1 \leq i \leq m) \) where \( m \) is a natural number, \( x_i \in V \) for every \( i \leq m \), and \( \{x_0, x_m\} \in E \) whenever \( i < m \). If \( \mathcal{B} \subseteq \mathcal{P}(G) \), we put \( \mathcal{B}^{-1} = \{(x_i : 1 \leq i \leq m) : \{x_{m-i}, i \leq m \} \in \mathcal{B}\} \).

Given a natural number \( n > 0 \), let \( \mathcal{B}_n \subseteq \mathcal{P}(G) \) be a set of paths of length \( n \). \( \mathcal{B}_n \) will be called an \( n \)-path set in \( G \). We put 

\[
\hat{\mathcal{B}}_n = \{(x_i : 1 \leq m) \in \mathcal{P}(G) : 0 < m \leq n \} \text{ and there exists } (y_i : i \leq n) \in \mathcal{B}_n \text{ such that } x_i = y_i \text{ for every } i \leq m \} (\text{so that } \mathcal{B}_n \subseteq \hat{\mathcal{B}}_n) \text{ and }
\]
Let \( B_n = \mathcal{B}_n \cup \mathcal{B}_n^{-1} \).

The elements of \( B_n \) will be called \( B_n \)-initial segments in \( G \).

For any subset \( X \subseteq V \) we put
\[
u_{B_n}(X) = X \cup \{x \in V; \text{there exists } (x_i; i \leq m) \in B_n \text{ with } \{x_i; i < m\} \subseteq X \text{ and } x_m = x\}.
\]

Clearly, \( \nu_{B_n} \) is an \( S_{n+1} \)-closure operator on \( V \). It will be said to be induced by \( B_n \).

Let \( G = (V, E) \) and \( H = (U, F) \) be graphs. Let \( B_n \) and \( C_n \) be \( n \)-path sets in \( G \) and \( H \), respectively (\( n > 0 \) a natural number). A map \( f : V \to U \) is said to be a \((B_n, C_n)\)-homomorphism from \( G \) to \( H \) if \((x_i; i \leq n) \in B_n \) implies \((f(x_i); i \leq n) \in C_n \). Note that, if \( n = 1 \) and \( B_n \) and \( C_n \) are the sets of all paths of length 1 in \( G \) and \( H \), respectively, we get the usual homomorphism of \( G \) into \( H \). The following statement is evident:

**Proposition 3.1.** Let \( G = (V, E) \) and \( H = (U, F) \) be graphs and let \( B_n \) and \( C_n \) be \( n \)-path sets in \( G \) and \( H \), respectively. If a map \( f : V \to U \) is a \((B_n, C_n)\)-homomorphism from \( G \) to \( H \), then it is a continuous map of \((V, u_{B_n})\) into \((U, u_{C_n})\).

Of course, closure operators induced by \( n \)-path sets are not additive in general. On the other hand, we have:

**Proposition 3.2.** Let \( G = (V, E) \) be a graph and let \( B_n \) be an \( n \)-path set in \( G \). Then the union of a system of closed subsets of \((V, u_{B_n})\) is a closed subset of \((V, u_{B_n})\).

**Proposition 3.3.** Let \( G = (V, E) \) be a graph and let \( B_n \) be an \( n \)-path set in \( G \). If \( u_{B_n} \) is idempotent, then \((V, u_{B_n})\) is an Alexandroff topological space.

**Definition 3.4.** An \( n \)-path set \( B_n \) in a graph is said to be terse provided that the following condition holds:

If \((x_i; i \leq n)\) \((y_i; i \leq n) \in B_n \) are paths with \( \{x_0, x_1\} = \{y_0, y_1\} \), then \((x_i; i \leq n) = (y_i; i \leq n)\).

Thus, for example, a 1-path set in a graph \( G = (V, E) \) is terse if, for every edge \( \{x, y\} \in E \), it contains at most one of the paths \((x, y)\) and \((y, x)\).

**Theorem 3.5.** Let \( B_n \) be a terse \( n \)-path set in a graph \( G = (V, E) \). Then \((v, u_{B_n})\) is a \( T_0 \)-space.

**Theorem 3.6.** Let \( B_n \) be a terse \( n \)-path set in a graph \( G = (V, E) \). Then \( B_n \) is a minimal element (with respect to the set inclusion) of the set of all \( n \)-path sets \( \mathcal{D}_n \) in \( G \) satisfying \( u_{D_n} = u_{B_n} \).
Given a natural number \( n \)

**Proof.** Let \( D_n \) be an \( n \)-path set in \( G \) with \( u_{D_n} = u_{B_n} \) and let \( D_n \subseteq B_n \). Let \( x_i \in \{0,1,2,3\} \). Then \( (x_0, x_1) \in B_n \) hence \( x_1 \in u_{B_n}[x_0] \). Consequently, \( x_1 \in u_{B_n}[x_0] \) so that \( (x_0, x_1) \in B_n \). Therefore, there is \( (y_i \leq i \leq n) \in D_n \) such that \( y_0 = x_0 \) and \( y_1 = x_1 \). Since \( (y_i \leq i \leq n) \in B_n \) and \( B_n \) is terse, we have \( (x_i \leq i \leq n) = (y_i \leq i \leq n) \). Consequently, \( D_n = B_n \). □

**Definition 3.7.** An \( n \)-path set \( B_n \) in a graph is said to be strongly terse provided that the following condition holds:

If \( (x_i \leq i \leq n) \), \( (y_i \leq i \leq n) \in B_n \) are paths with \( x_0, x_1 = \{y_0, y_1\} \) for some \( i_0, i_1 \leq n \), then \( (x_i \leq i \leq n) = (y_i \leq i \leq n) \).

**Theorem 3.8.** If \( B_n \) is a strongly terse \( n \)-path set in a graph \( G \), then the correspondence \( B_n \leftrightarrow u_{B_n} \) is one-to-one.

**Proof.** Given a natural number \( n \) \( > 0 \), for any closure operator \( u \) on \( G \) we define an \( n \)-path set in \( G \) as follows:

\[ B_n(u) = \{(x_i \leq i \leq n) \in \mathbb{P}(G) \mid (x_i \leq i \leq n) \text{ has the property that, for each } j, 0 \leq j \leq n, \text{ and each proper subset } A_j \subseteq \{x_i \leq j \}, \text{ we have } x_j \in u[x_j \leq j] - uA_j \}. \]

Let \( B_n \) be a strongly terse \( n \)-path set in \( G \) and let \( (x_i \leq i \leq n) \in B_n \). If \( i_0, 0 \leq i_0 \leq n \), is an arbitrary natural number, then \( x_{i_0} \in u_{B_n}[x_i \leq i_0] \). Let \( A \subseteq \{x_i \leq i_0 \} \) be an arbitrary subset and let \( x_{i_0} \in u_{B_n}[A] \). Then there exists a path \( (y_j \leq j \leq n) \in B_n \) and a natural number \( j_0 \), \( 0 \leq j_0 \leq n \), such that \( x_{i_0} = y_{j_0} \) and \( (y_j \leq j \leq n) \subseteq A \), hence \( (y_j \leq j \leq n) \subseteq \{x_i \leq j \} \leq \{x_i \leq i_0 \} \). Thus, \( (y_j \leq j \leq n) \subseteq \{x_i \leq j \} \leq \{x_i \leq i_0 \} \) and, consequently, \( (x_i \leq i \leq n) = (y_i \leq i \leq n) \). Hence, \( i_0 = j_0 \) and, therefore, \( A = \{x_i \leq i \} \) is a proper subset. We have shown that \( (x_i \leq i \leq n) \in B_n \) whenever \( A \subseteq \{x_i \leq j \} \leq \{x_i \leq i_0 \} \) is a proper subset. It follows that \( (x_i \leq i \leq n) \in B_n(u_{B_n}) \) which results in \( B_n \subseteq B_n(u_{B_n}) \).

Conversely, let \( (x_i \leq i \leq n) \in B_n(u_{B_n}) \). As \( x_1 \in u_{B_n}[x_0] \) and \( B_n \) is strongly terse, there is a unique path \( (y_i \leq i \leq n) \in B_n \) with \( x_0 = y_0 \) and \( x_1 = y_1 \). Let \( i_0, 1 \leq i_0 \leq n \), be a natural number such that \( x_i = y_i \) for all \( i < i_0 \). Since \( x_{i_0} \in u_{B_n}[x_i \leq i_0] \), there is a path \( (z_j \leq j \leq n) \in B_n \) and a natural number \( j_0 \), \( 0 \leq j_0 \leq n \), such that \( x_{i_0} = z_{i_0} \) and \( (z_j \leq j \leq n) \subseteq \{x_i \leq j \} \leq \{x_i \leq i_0 \} \) and, consequently, \( (x_i \leq i \leq n) = (y_i \leq i \leq n) \). We have shown that \( (x_i \leq i \leq n) \in B_n \), which results in \( B_n(u_{B_n}) \subseteq B_n \). Therefore, \( B_n(u_{B_n}) = B_n \) and the proof is complete. □

We will need the following

**Lemma 3.9.** Let \( B_n \) be a strongly terse \( n \)-path set in a graph \( G \), let \( p, q \) be natural numbers with \( 0 \leq p < q \leq n \), and let \( (x_i \leq i \leq p), (y_i \leq i \leq q) \in B_n \)-initial segments in \( G \) such that \( (x_i \leq i \leq p) \subseteq \{y_i \leq i \leq q\} \). There exists a unique path \( (z_i \leq i \leq n) \in B_n \) such that \( x_i = z_i \) for all \( i \leq p \) and \( y_i = z_i \) for all \( i \leq q \).

**Proof.** As \( (x_i \leq i \leq p) \) is a \( B_n \)-initial segment, there is a path \( (t_i \leq i \leq n) \in B_n \) such that \( 1) \ t_i = x_i \) for all \( i \leq p \) or \( 2) \ t_i = l_{p-i} \) for all \( i \leq p \). Similarly, as \( (y_i \leq i \leq q) \) is a \( B_n \)-initial segment, there is a path \( (z_i \leq i \leq n) \in B_n \) such that \( 1) \ y_i = z_i \) for all \( i \leq q \) or \( 2) \ y_i = z_{q-i} \) for all \( i \leq q \). Admit that 1) and 2') are valid. Then \( t_0 = x_0 = y_0 = z_0 \) and \( t_1 = x_1 \in \{y_i \leq i \leq q\} \subseteq \{z_i \leq i \leq n\} \), hence \( (t_i \leq i \leq n) = (z_i \leq i \leq n) \). We have \( z_0 = y_0 = z_q \), which is a contradiction. Admit that 2) and 1') are valid. Then \( t_0 = x_0 \in \{y_i \leq i \leq q\} \subseteq \{z_i \leq i \leq n\} \) and \( t_1 = x_{p-1} \in \{y_i \leq i \leq q\} \subseteq \{z_i \leq i \leq n\} \), hence \( (t_i \leq i \leq n) = (z_i \leq i \leq n) \). We have \( t_p = x_0 = y_0 = z_q = l_p \), which is a contradiction because \( p < q \). Consequently, only 1) and 1') are valid simultaneously. Then \( t_0 = x_0 = y_0 = z_0 \) and \( t_1 = x_1 \in \{y_i \leq i \leq q\} \subseteq \{z_i \leq i \leq n\} \), hence \( (t_i \leq i \leq n) = (z_i \leq i \leq n) \). The existence of \( (z_i \leq i \leq n) \) is proved. As \( B_n \) is (strongly) terse, \( (z_i \leq i \leq n) \) is clearly unique. □

**Definition 3.10.** An \( n \)-path set \( B_n \) in a graph \( G = (V, E) \) is said to be plain provided that the following condition is satisfied:

If \( (x_i \leq i \leq n), (y_i \leq i \leq n) \in B_n \) and there are natural numbers \( i_0, i_1, i_0', i_1' \leq n, i_0 \neq i_1, \) such that \( x_{i_0} = y_{i_0} \) and \( x_{i_1} = y_{i_1} \), then \( (x_i \leq i \leq n) = (y_i \leq i \leq n) \).

Clearly, every plain \( n \)-path set is strongly terse and every 1-path set is plain.
**Theorem 3.11.** Let $B_n$ be a plain $n$-path set in a graph $G = (V,E)$, let $x \in V$ and let $A, B \subseteq V$ be minimal (with respect to set inclusion) subsets with the properties $x \in u_{B_n}A - A$ and $x \in u_{B_n}B - B$, respectively. Then $A \cap B = \emptyset$ or $A = B$.

**Proof.** Suppose that $A \cap B \neq \emptyset$. There exist a path $(x_i | i \leq n) \in B_n$ and a natural number $i_0$, $0 < i_0 \leq n$, such that $x = x_{i_0}$ and $x_i \in A$ for each $i < i_0$. Similarly, there exist a path $(y_j | j \leq n) \in B_n$ and a natural number $i_1$, $0 < i_1 \leq n$, such that $x = y_{i_1}$ and $y_i \in B$ for each $i < i_1$. Since $A$ and $B$ are minimal, we have $A = \{x_i | i < i_0\}$ and $B = \{y_j | j < i_1\}$. Let $y \in A \cap B$ be a point. Then $y \neq x$ and we have $\{x,y\} \subseteq \{x_i | i \leq n\} \cap \{y_j | j \leq n\}$. Consequently, $(x_i | i \leq n) = (y_j | j \leq n)$. It follows that $A = B$ because $i_0 = i_1$. \(\square\)

4. Connectedness

**Definition 4.1.** Let $B_n$ be an $n$-path set in a graph $G = (V,E)$. A sequence $C = (x_i | i \leq m)$, $m > 0$, of vertices of $V$ is called a $B_n$-walk in $G$ if there is an increasing sequence $(j_k | k \leq p)$ of natural numbers with $j_0 = 0$ and $j_p = m$ such that $j_k - j_{k-1} \leq n$ and $(x_i | j_k \leq i \leq j_k + 1)$ is a $B_n$-walk for every $k$ with $k \leq p$. The sequence $(j_k | k \leq p)$ is said to be a binding sequence of $C$.

If the members of $C$ are pairwise different, then $C$ is called a $B_n$-path in $G$.

A $B_n$-walk $C$ is said to be closed if $x_0 = x_m$, and it is said to be a $B_n$-cycle if, for every pair $i_0,i_1$ of different natural numbers with $i_0,i_1 \leq m$, $x_{i_0} = x_{i_1}$ is equivalent to $\{i_0,i_1\} = \{0,m\}$.

Of course, every $B_n$-walk $(B_n$-path$)$ in a graph $G = (V,E)$ is a walk (path) in $G$ and both concepts coincide if $B_n = B_1 = E'$ where $E'$ is an arbitrary set of paths of length 1 such that, (1) for every $(x,y) \in E$, $(x,y) \in E'$ and $(y,y) \in E'$ and (2) for every $(x,y) \in E'$, $(x,y) \in E$.

Observe that, if $(x_0,x_1,\ldots,x_m)$ is a $B_n$-walk in $G$, then $(x_m,x_{m-1},\ldots,x_0)$ is a $B_n$-walk in $G$, too. Further, if $(x_i | i \leq m)$ and $(y_j | j \leq p)$ are $B_n$-walks in $G$ with $x_m = y_0$, then, putting $z_i = x_i$ for all $i \leq m$ and $z_i = y_{i-m}$ for all $i$ with $m \leq i \leq m + p$, we get a $B_n$-walk $(z_i | i \leq m + p)$ in $G$.

**Definition 4.2.** Let $B_n$ be an $n$-path set in a graph $G = (V,E)$. A subset of $V$ is said to be $B_n$-connected if it is connected in $(V,u_{B_n})$. A maximal $B_n$-connected subset of $V$ is called a $B_n$-component of $G$.

Of course, every initial segment of $G$ is $B_n$-connected.

**Theorem 4.3.** Let $B_n$ be an $n$-path set in a graph $G = (V,E)$. A subset $A \subseteq V$ is $B_n$-connected if and only if any two different vertices from $A$ may be joined by a $B_n$-walk in $G$ contained in $A$.

**Proof.** If $A = \emptyset$, then the statement is trivial. Let $A \neq \emptyset$. If any two vertices from $A$ can be joined by a $B_n$-walk in $G$ contained in $A$, then $A$ is clearly $B_n$-connected. Conversely, let $A$ be $B_n$-connected and suppose that there are vertices $x,y \in A$ which cannot be joined by a $B_n$-walk in $G$ contained in $A$. Let $B$ be the set of all vertices from $A$ which can be joined with $x$ by a $B_n$-it walk in $G$ contained in $A$. Let $z \in u_{B_n}B \cap A$ be a vertex and assume that $z \notin B$. Then there are a path $(x_i | i \leq n) \in B_n$ and a natural number $i_0$, $0 < i_0 \leq n$, such that $z = x_{i_0}$ and $\{x_i | i < i_0\} \subseteq B$. Thus, $x$ and $x_0$ can be joined by a $B_n$-walk in $G$ contained in $A$, and also $x_0$ and $z$ can be joined by a $B_n$-walk in $G$ contained in $A$ - namely by the $B_n$-initial segment $(x_i | i \leq i_0) \in B_n$. It follows that $x$ and $z$ can be joined by a $B_n$-walk in $G$ contained in $A$, which is a contradiction. Therefore, $z \in B$, i.e. $u_{B_n}B \cap A = B$. Consequently, $B$ is closed in the subspace $A$ of $(V,u_{B_n})$.

Further, let $z \in u_{B_n}(A - B) \cap A$ be a vertex and assume that $z \in B$. Then $z \notin A - B$, thus there are a path $(x_i | i < n) \in B_n$ and a natural number $i_0$, $0 < i_0 < i < n$, such that $z = x_{i_0}$ and $\{x_i | i < i_0\} \subseteq A - B$. Since $x$ can be joined with $z$ by a $B_n$-walk in $G$ contained in $(A$ (because we have assumed that $z \in B$) and $z$ can be joined with $x_0$ by a $B_n$-walk in $G$ contained in $A$ - namely by the initial segment $(x_{i-1} | i < i_0) \in B_n$, also $x$ and $x_0$ can be joined by a $B_n$-walk in $G$ contained in $A$. This is a contradiction with $x_0 \notin B$. Thus, $z \notin B$, i.e. $u_{B_n}(A - B) \cap A = A - B$. Consequently, $A - B$ is closed in the subspace $A$ of $(V,u_{B_n})$. Hence, $A$ is the union of the nonempty disjoint sets $B$ and $A - B$ closed in the subspace $A$ of $(V,u_{B_n})$. But this is a contradiction because $A$ is $B_n$-connected. Therefore, any two points of $A$ can be joined by a $B_n$-walk in $G$ contained in $A$. \(\square\)
Note that, if two vertices of a graph $G$ can be joined by a $B_n$-path in $G$, they need not be joined by a $B_n$-path in $G$.

**Theorem 4.4.** If $B_n$ is a plain $n$-path set in $G$, then every $B_n$-path in $G$ has exactly one binding sequence.

**Proof.** Let $C = (x_i | i \leq m)$ be a $B_n$-path in $G$ and let $(y_j | j \leq p)$ and $(l_k | k \leq q)$ be binding sequences of $C$. Clearly, $j_0 = l_0 = 0$. Let $k < p$ be a natural number such that $k \leq q$ and $l_k = l_q$. Then $l_k = j_q = m = l_q$, hence $k < q$. Assume that $j_{k+1} < l_{k+1}$. As both $(x_i | j_i \leq i \leq j_{k+1})$ and $(x_i | l_i \leq i \leq l_{k+1})$ are $B_n$-initial segments, there exists a path $(y_i | i \leq n) \in B_n$ with $y_i = x_{j_{k+1}}$ for all $i \leq l_{k+1} - l_k$ by Lemma 3.9. Since $j_{k+1} < l_{k+1} \leq m$, we have $j_{k+1} < m$. Therefore, we have $l_i = j_{k+1}$ for all $i \leq l_{k+1} - l_k$. Let $D_n$ be a $B_n$-path in $G$ with $D_n = B_n$, and, on the other hand, $z_0 = x_{j_{k+1}}$ or $z_0 = x_{j_{k+1}}$. Since $l_{k+1} = l_{k+1}$, we get $j_{l_{k+1}} = l_{l_{k+1}}$. The proof is complete. \[\square\]

The previous theorem enables us to define:

**Definition 4.5.** Let $B_n$ be a plain $n$-path set in a graph $G$, let $C$ be a $B_n$-path in $G$ and let $(j_i | k \leq p)$ be the binding sequence of $C$. Then $C$ is called a $B_n$-arc in $G$ provided that, for every $B_n$-initial segment $D \in B_n$ with $D \subseteq C$, there is a natural number $k < p$ such that $D$ is a subset of the $B_n$-initial segment $(x_i | j_i \leq i \leq j_{k+1})$. We say that a subset $B_n$ is a $B_n$-arc in $G$ joining vertices $x$ and $y$. If $D$ is a $B_n$-walk in $G$ joining $x$ and $y$ such that $D \subseteq C$, then $D = C$.

**Theorem 4.6.** Let $B_n$ be a plain $n$-path set in $G$ and $C$ be a $B_n$-arc in $G$ joining vertices $x$ and $y$. If $D$ is a $B_n$-walk in $G$ joining $x$ and $y$ such that $D \subseteq C$, then $D = C$.

**Proof.** Let $C = (x_i | i \leq m) \in B_n$ be a $B_n$-walk in $G$ with $y_0 = x$ and $y_1 = y$ such that $|y_i; i \leq s| \subseteq |x_i; i \leq m|$. Let $(j_i | k \leq p)$ be the binding sequence of $C$ and for each $k < p$ put $C_k = (x_i | j_i \leq i \leq j_{k+1})$. Let $(l_i | k \leq q)$ be the binding sequence of $(y_i | i \leq s)$ and for each $k < q$ put $D_k = (y_i | l_i \leq i \leq l_{k+1})$. Clearly, $x_i = x_0 = x = y_0 = y_1$. Let $k < q$ be a natural number with $k < p, j_k = k_k$, and $x_i = y_i$ for all $i \leq j_k$ (such $k$ exists because we may put $k = 0$). As $x_k = y_k \neq x_{j_k} = y_j$, we have $k < p$. Since $D_k \subseteq C$, there is a $B_n$-initial segment $C_k$ with $D_k \subseteq C_k$. As $y_k = y_k = x_{j_k} \subseteq D_k \cap C_k$, we have $k_0 = k$. (Of course, we also have $y_k = x_{j_k} \subseteq D_k \cap C_k$, but $D_k \cap C_k$ has $y_k \notin y_k \in D_k \cap C_k$ because $y_k \notin y_k \in D_k \cap C_k$.) Hence $D_k \subseteq C_k$. Clearly, we have $D_k \subseteq D_{k+1} \cup D_k$. This is a contradiction because $y_k \notin D_k \cap D_{k+1}$. Therefore, we get $j_k = k$, and, consequently, $x_i = x$ for all $i \leq j_k$. Thus, according to the mathematical induction principle, whenever $k < q$, we have $k = p, j_k = l_k$ and $x_i = y_i$ for all $i \leq j_k$. As $x_{j_k+1} = y_{j_k+1} = y$, we get $j_{l_{k+1}} = j_{l_{k+1}}$, i.e. $p = q$. Hence $(x_i | i \leq m) = (y_i | i \leq s)$. \[\square\]

The previous statement is plain. Hence, $B_n$-archs are minimal (with respect to set inclusion) $B_n$-walks joining a given pair of vertices.

**Definition 4.7.** Let $B_n$ be a plain $n$-path set in $G$. A closed $B_n$-walk $C = (x_i | i \leq m)$ in $G$ is said to be simple if its binding sequence $(j_i | k \leq p)$ has the property that $p \geq 3$ and $(x_{j_{k+1}} | m) \cup | l \leq k \leq p |$ is an arc in $G$ whenever $k < p$.

It is evident that every simple closed $B_n$-walk in $G$ is a $B_n$-cycle in $G$.

Let $B_n$ be an $n$-path set in $G = (V, E), U \subseteq V$ be a subset, and $H = (U, F)$ be the induced subgraph of $G$. We denote by $B_n / U$ the subset of $B_n$ consisting of the paths belonging to $B_n$ that are paths in $H$. Thus, $B_n / U$ is an $n$-path set in $H$. We say that a subset $J \subseteq V$ separates $G$ into two components if, putting $U = V - J$, the induced subgraph $(U, F)$ of $G$ has exactly two $B_n / U$-components.
Definition 4.8. A simple closed $B_n$-path $(x_i)_{i \leq m}$ in $G$ is said to be a Jordan curve if $|x_i|_{i \leq m}$ separates $G$ into two components.

The following statement is evident:

Theorem 4.9. A simple closed $B_n$-path $(x_i)_{i \leq m}$ in $G$ is a Jordan curve if and only if there are $B_n$-connected subsets $X_1, X_2 \subseteq V$ with $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = V - \{x_i \mid i \leq m\}$ such that a subset $A \subseteq X_1 \cup X_2$ is not $B_n$-connected whenever $A \supset X_1$ or $A \supset X_2$.

5. Closure Operators Induced by $N$-Path Sets in Graphs with the Vertex Set $\mathbb{Z}^2$

It is well known that closure operators which are more general than the Kuratowski ones have useful applications in computer science. By Theorem 4.3, $B_n$-connectedness in graphs is a certain type of path connectedness. This enables us to apply closure operators induced by $n$-path sets in digital topology, which is a theory that has arisen for the study of geometric and topological properties of digital images (cf. [11]). One of the problems of digital topology is to find structures for the digital plane which is a theory that has arisen for the study of geometric and topological properties of digital images. A basic criterion of such a convenience is the validity of a digital analogue of the Jordan curve theorem (which says that every simple closed curve in the real plane separates this plane into precisely two connected components). The thing is that digital Jordan curves represent borders of the Jordan curve theorem (which says that every simple closed curve in the real plane separates this plane into precisely two connected components). The thing is that digital Jordan curves represent borders of objects on digital pictures. Though the classical approach to the problem is based on using graph theory (the 4-adjacency and 8-adjacency relations) for structuring the digital plane $\mathbb{Z}^2$ convenient for the study of digital images. A basic criterion of such a convenience is the validity of a digital analogue of the Jordan curve theorem (which says that every simple closed curve in the real plane separates this plane into precisely two connected components). The thing is that digital Jordan curves represent borders of objects on digital pictures. Though the classical approach to the problem is based on using graph theory (the 4-adjacency and 8-adjacency relations) for structuring the digital plane, a new approach based on using topological structures is being developed since the beginning of the 90’s of the last century - see [8]. There are two well-known topologies on $\mathbb{Z}^2$ which are employed in digital topology, the so-called Marcus-Wyse topology [9] and Khalimsky topology [7]. Recall that the Marcus-Wyse and Khalimsky topologies are the Alexandroff topologies $s$ and $t$, respectively, on $\mathbb{Z}^2$ with the closures of singleton subsets given as follows:

For any $z = (x, y) \in \mathbb{Z}^2$,

$$s[z] = \begin{cases} [z] \cup A_4(z) & \text{if } x + y \text{ is even}, \\ [z] & \text{otherwise.} \end{cases}$$

and

$$t[z] = \begin{cases} [z] \cup A_4(z) & \text{if } x, y \text{ are even,} \\ ([x + i, y] ; i \in \{-1, 0, 1\}) & \text{if } x \text{ is odd and } y \text{ is even,} \\ ([x, y + j] ; j \in \{-1, 0, 1\}) & \text{if } x \text{ is odd and } y \text{ is even,} \\ [z] & \text{otherwise.} \end{cases}$$

We will show that both of these topologies and also the closure operators discussed in [12] may be obtained as closure operators induced by $n$-path sets in certain graphs with the vertex set $\mathbb{Z}^2$.

While the concept of $B_n$-connectedness is useless if $n = 1$ (because then it coincides with the usual graph connectedness), a different situation occurs if $n > 1$.

Example 5.1. Put $B_1 = \{(x_i, y_i) \mid i \leq 1\}; (x_i, y_i) \in \mathbb{Z}^2$ for every $i \leq 1$, $|x_0 - x_1| + |y_0 - y_1| = 1$, $x_0 + y_0$ even) and let $G = (\mathbb{Z}^2, E)$ be an arbitrary graph (with the vertex set $\mathbb{Z}^2$) such that $B_1 \subseteq \mathcal{P}(G)$. Then $B_1$ is a plain 1-path set in $G$. A portion of the set $B_1$ is shown in the following figure where the 1-paths from $B_1$ are represented by arrows oriented from first to last vertex.
Clearly, \((\mathbb{Z}^2, u_{B_n})\) is a connected Alexandroff \(T_1\)-space in which the points \((x, y) \in \mathbb{Z}^2\) with \(x + y\) even are open while those with \(x + y\) odd are closed. The closure operator \(u_{B_n}\) coincides with the Marcus-Wyse topology.

**Example 5.2.** For an arbitrary natural number \(n\), let \(B_n\) be the set of all sequences \(((x_i, y_i) | i \leq n)\) such that \((x_i, y_i) \in \mathbb{Z}^2\) for every \(i \leq n\) and one of the following eight conditions is satisfied:

1. \(x_0 = x_1 = \ldots = x_n\) and there is \(k \in \mathbb{Z}\) such that \(y_i = 2kn + i\) for all \(i \leq n\),
2. \(x_0 = x_1 = \ldots = x_n\) and there is \(k \in \mathbb{Z}\) such that \(y_i = 2kn - i\) for all \(i \leq n\),
3. \(y_0 = y_1 = \ldots = y_n\) and there is \(k \in \mathbb{Z}\) such that \(x_i = 2kn + i\) for all \(i \leq n\),
4. \(y_0 = y_1 = \ldots = y_n\) and there is \(k \in \mathbb{Z}\) such that \(x_i = 2kn - i\) for all \(i \leq n\),
5. there is \(k \in \mathbb{Z}\) such that \(x_i = 2kn + i\) for all \(i \leq n\) and there is \(l \in \mathbb{Z}\) such that \(y_i = 2ln + i\) for all \(i \leq n\),
6. there is \(k \in \mathbb{Z}\) such that \(x_i = 2kn + i\) for all \(i \leq n\) and there is \(l \in \mathbb{Z}\) such that \(y_i = 2ln - i\) for all \(i \leq n\),
7. there is \(k \in \mathbb{Z}\) such that \(x_i = 2kn - i\) for all \(i \leq n\) and there is \(l \in \mathbb{Z}\) such that \(y_i = 2ln + i\) for all \(i \leq n\),
8. there is \(k \in \mathbb{Z}\) such that \(x_i = 2kn - i\) for all \(i \leq n\) and there is \(l \in \mathbb{Z}\) such that \(y_i = 2ln - i\) for all \(i \leq n\).

Let \(G = (\mathbb{Z}^2, E)\) be an arbitrary graph (with the vertex set \(\mathbb{Z}^2\)) such that \(B_n \subseteq \mathcal{P}(G)\). Then \(B_n\) is a plain \(n\)-path set in \(G\). A portion of the set \(B_n\) is demonstrated in the following figure. The paths belonging to \(B_n\) are represented by arrows oriented from first to last term. Between any pair of neighboring parallel horizontal or vertical arrows (having the same orientation), there are \(n - 1\) more parallel arrows with the same orientation that are not displayed.

Clearly, \((\mathbb{Z}^2, u_{B_n})\) is a connected Alexandroff \(T_0\)-space. The closure operators \(u_{B_n}\) coincide with the closure operators on \(\mathbb{Z}^2\) studied in [12] where a Jordan curve theorem was proved for these operators thus showing their convenience for applications in digital topology. In particular, \(u_{B_n}\) coincides with the Khalimsky topology on \(\mathbb{Z}^2\) (for which a Jordan curve theorem was proved in [7]). Note that for \(n > 1\), the \(B_n\)-connectedness in \(G_n\) cannot be obtained as the usual graph connectedness. In particular, for \(n = 2\), the above portion of the graph \(G_n\) looks, in full detail, as follows:
References