Some Identities Relating to Degenerate Bernoulli Polynomials

Taekyun Kim\textsuperscript{a}, Dae San Kim\textsuperscript{b}, Hyuck-In Kwon\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
\textsuperscript{b}Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
\textsuperscript{c}Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

Abstract. Recently, Carlitz degenerate Bernoulli numbers and polynomials have been studied by several authors (see \cite{3, 4}). In this paper, we consider new degenerate Bernoulli numbers and polynomials, different from Carlitz degenerate Bernoulli numbers and polynomials, and give some formulae and identities related to these numbers and polynomials.

1. Introduction

The ordinary Bernoulli numbers are defined by

\begin{equation}
B_0 = 1, \quad (B + 1)^n - B_n = \begin{cases} 
1, & \text{if } n = 1, \\
0, & \text{if } n > 1,
\end{cases}
\end{equation}

with the usual convention about replacing $B^n$ by $B_n$.

The Bernoulli polynomials are defined by

\begin{equation}
B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad \text{(see } \cite{1, 20}).
\end{equation}

From (1) and (2), we note that

\begin{equation}
\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt} = \left( \frac{t}{e^t - 1} \sum_{n=0}^{d-1} e^{n} \right) e^{xt}.
\end{equation}
Thus, by (3), we get
\[ B_n(x) = d^{n-1} \sum_{a=0}^{d-1} B_n \left( \frac{a + x}{d} \right) \]
(4)
where \( n \in \mathbb{N} \cup \{0\} \) and \( d \in \mathbb{N} \).

Let \( \chi \) be a Dirichlet character with conductor \( d \in \mathbb{N} \). The generalized Bernoulli numbers are defined by
\[ B_{n,\chi} = d^{n-1} \sum_{a=0}^{d-1} \chi(a) B_n \left( \frac{a}{d} \right), \quad (n \geq 0), \quad \text{(see [12, 18, 20]).} \]
(5)

Carlitz introduced the degenerate Bernoulli polynomials given by the generating function
\[ \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}}} = \sum_{n=0}^{\infty} \beta_n(x | \lambda) \frac{t^n}{n!}, \quad \text{(see [3, 4]).} \]
(6)

When \( x = 0, \beta_n(\lambda) = \beta_n(0 | \lambda) \) are called the degenerate Bernoulli numbers. From (6), we note that
\[ \lim_{\lambda \to 0} \beta_n(x | \lambda) = B_n(x), \quad (n \geq 0). \]
(7)

In this paper, we consider new degenerate Bernoulli numbers and polynomials, different from Carlitz degenerate Bernoulli numbers and polynomials, and give some formulae and identities related to these numbers and polynomials.

2. Degenerate Bernoulli Polynomials

Let us consider the new degenerate Bernoulli polynomials as follows:
\[ \frac{1}{(1 + \lambda t)^{\frac{1}{\lambda}}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \]
(8)

When \( x = 0, \beta_{n,\lambda} = \beta_{n,\lambda}(0) \) are called the degenerate Bernoulli numbers. Note that \( \lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x) \).

From (8), we have
\[ \frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \]

We observe that
\[ \frac{1}{\lambda} \log(1 + \lambda t) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n + 1} t^{n+1}. \]
(10)

Thus, by (8), (9) and (10), we get
\[ \beta_{n,\lambda}(1) - \beta_{n,\lambda} = \begin{cases} 0 & \text{if } n = 0, \\ (-\lambda)^{n-1} (n-1)! & \text{if } n \geq 1, \quad \beta_{0,\lambda} = 1. \end{cases} \]
(11)
From (8), we note that
\[
\log (1 + \lambda t)^{\frac{1}{t}} (1 + \lambda t)^{\frac{1}{t}} = \left(1 + \lambda t^{\frac{1}{t}} - 1\right) \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \frac{t^m}{m!}
\]
\[
= t \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{1}{l+1} \beta_{n-l,\lambda}(x) \frac{n!}{(l+1)!} t^l.
\]
where
\[
(x | \lambda)_n = x(x - \lambda) \cdots (x - \lambda(n-1)).
\]

It is known that Daehee numbers are given by the generating function
\[
\frac{\log(1 + t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}.
\]

Now, we observe that
\[
\log (1 + \lambda t)^{\frac{1}{t}} (1 + \lambda t)^{\frac{1}{t}} = \frac{\log(1 + \lambda t)}{\lambda t} \left(1 + \lambda t^{\frac{1}{t}} \right)
\]
\[
= t \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{n!}{l!} D_l \lambda^l (x | \lambda)_{n-l} \frac{t^n}{n!}.
\]
Thus, by (12) and (13), we get
\[
\sum_{l=0}^{n} \frac{n!}{l!} D_l \lambda^l (x | \lambda)_{n-l} \beta_{n-l,\lambda}(x) = \sum_{l=0}^{n} \frac{n!}{l!} D_l \lambda^l (x | \lambda)_{n-l}.
\]

By (8), we easily get
\[
\beta_{n,\lambda}(x) = \sum_{l=0}^{n} \frac{n!}{l!} \beta_{l,\lambda} (x | \lambda)_{n-l}, \quad (n \geq 0).
\]

Therefore, by (14) and (15), we obtain the following theorem.

**Theorem 2.1.** For \(n \geq 0\), we have
\[
\sum_{l=0}^{n} \frac{n!}{l!} D_l \lambda^l (x | \lambda)_{n-l} \beta_{n-l,\lambda}(x) = \sum_{l=0}^{n} \frac{n!}{l!} D_l \lambda^l (x | \lambda)_{n-l},
\]
and
\[
\beta_{n,\lambda}(x) = \sum_{l=0}^{n} \frac{n!}{l!} \beta_{l,\lambda} (x | \lambda)_{n-l}.
\]

Moreover,
\[
\beta_{n,\lambda}(1) - \beta_{n,\lambda} = \begin{cases} 0, & \text{if } n = 0, \\ (-\lambda)^{n-1} (n-1)! & \text{if } n \geq 1, \end{cases} \quad \beta_{0,\lambda} = 1.
\]
By (8), we get
\[
\frac{\log (1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} = \frac{\log (1 + \lambda t)^{\frac{1}{\lambda}} - \sum_{a=0}^{d-1} (1 + \lambda t)^{\frac{a}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} = \frac{1}{d} \frac{\log (1 + \lambda t)^{\frac{1}{\lambda}} - \sum_{a=0}^{d-1} (1 + \lambda t)^{\frac{a}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}
\]
\[
= \frac{1}{d} \sum_{a=0}^{d-1} \frac{\beta_{n,\lambda}(\frac{a + x}{d}) d^{n} n!}{m!}
\]
\[
= \sum_{a=0}^{d-1} \left\{ d^{n-1} \sum_{a=0}^{d-1} \beta_{n,\lambda}(\frac{a + x}{d}) \right\} \frac{t^{m}}{m!}
\]

Thus, by (16), we obtain the following theorem.

**Theorem 2.2.** For \( n \geq 0 \), we have
\[
\beta_{n,\lambda}(x) = d^{n-1} \sum_{a=0}^{d-1} \beta_{n,\lambda}(\frac{a + x}{d}).
\]

It is not difficult to show that
\[
\frac{\log (1 + \lambda t)^{\frac{1}{\lambda}}}{\lambda} \sum_{l=0}^{n-1} (1 + \lambda t)^{\frac{l}{\lambda}} = \frac{\log (1 + \lambda t)^{\frac{1}{\lambda}} - \sum_{a=0}^{d-1} (1 + \lambda t)^{\frac{a}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} = \sum_{m=0}^{\infty} \left\{ \beta_{m,\lambda}(n) - \beta_{m,\lambda} \right\} \frac{t^{m}}{m!}
\]
\[
= \sum_{m=0}^{\infty} \left( \beta_{m+1,\lambda}(n) - \beta_{m+1,\lambda} \right) \frac{t^{m}}{m+1}
\]

Thus we get
\[
\frac{\log (1 + \lambda t)^{\frac{1}{\lambda}}}{\lambda} \sum_{l=0}^{n-1} (1 + \lambda t)^{\frac{l}{\lambda}} = \sum_{k=0}^{\infty} \left( \frac{k}{k!} D_{l} \sum_{l=0}^{n-1} (l \mid \lambda)_{m} \frac{t^{m}}{m!} \right)
\]
\[
= \sum_{k=0}^{\infty} \left( \frac{k}{k!} D_{l} \sum_{l=0}^{n-1} (l \mid \lambda)_{m} \frac{t^{m}}{m!} \right)
\]
From (17) and (18), we have
\[
\frac{\beta_{k+1,\lambda}(n) - \beta_{k+1,\lambda}}{k+1} = \sum_{l=0}^{n-1} \left( \sum_{i=0}^{k} \binom{k}{i} D_i \lambda^i (l | \lambda)_{k-i} \right).
\] (19)

Therefore, by (19), we obtain the following theorem.

**Theorem 2.3.** For \( n \geq 1 \) and \( k \geq 0 \), we have
\[
\frac{1}{k+1} \{ \beta_{k+1,\lambda}(n) - \beta_{k+1,\lambda} \} = \sum_{l=0}^{n-1} \left( \sum_{i=0}^{k} \binom{k}{i} D_i \lambda^i (l | \lambda)_{k-i} \right).
\]

Replacing \( t \) by \( \frac{1}{\lambda} \log (1 + \lambda t) \) in (3), we get
\[
\log (1 + \lambda t) = \frac{1}{(1 + \lambda t) \log (1 + \lambda t)} - 1 = \sum_{n=0}^{\infty} B_n(x) \lambda^{-n} \frac{1}{n!} (\log (1 + \lambda t))^n
\]
\[
\quad = \sum_{m=0}^{\infty} B_m(x) \lambda^{-m} \sum_{n=m}^{\infty} S_1(n, m) \frac{\lambda^n}{n!}
\]
\[
\quad = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} B_m(x) \lambda^{-m} S_1(n, m) \right) \frac{\lambda^n}{n!}
\]
where \( S_1(n, m) \) is the Stirling number of the first kind.

On the other hand,
\[
\log (1 + \lambda t) = \frac{1}{(1 + \lambda t) \log (1 + \lambda t)} - 1 = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{\lambda^n}{n!}.
\] (20)

Therefore, by (20) and (21), we obtain the following theorem.

**Theorem 2.4.** For \( n \geq 0 \), we have
\[
\beta_{n,\lambda}(x) = \sum_{m=0}^{n} B_m(x) \lambda^{-m} S_1(n, m).
\]

Replacing \( t \) by \( \frac{1}{\lambda} \left( e^{\lambda t} - 1 \right) \) in (5), we have
\[
\left( e^{\lambda t} - 1 \right) = \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \frac{1}{m!} \left( e^{\lambda t} - 1 \right)^m
\]
\[
\quad = \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n}{n!}
\]
\[
\quad = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \beta_{m,\lambda}(x) \lambda^{-m} S_2(n, m) \right) \frac{\lambda^n}{n!},
\]
where \( S_2(n, m) \) is the Stirling number of the second kind.

Thus, by (22), we obtain the following theorem.
Theorem 2.5. For $n \geq 0$, we have
\[ B_n(x) = \sum_{m=0}^{n} \beta_{m,\lambda}(x) \lambda^{n-m} S_2(n, m). \]

For $d \in \mathbb{N}$, let $\chi$ be a Dirichlet character with conductor $d$. Then, we define the generalized degenerate Bernoulli numbers attached to $\chi$:
\[ \log (1 + \lambda t)^\frac{1}{d} \sum_{a=0}^{d-1} \chi(a) (1 + \lambda t)^\frac{a}{d} = \sum_{n=0}^{\infty} \beta_{n,\lambda,d} \frac{t^n}{n!}. \] (23)

From (8) and (23), we have
\[ \sum_{n=0}^{\infty} \beta_{n,\lambda,d} \frac{t^n}{n!} = \log (1 + \lambda t)^\frac{1}{d} \sum_{a=0}^{d-1} \chi(a) (1 + \lambda t)^\frac{a}{d} \]
\[ = \frac{1}{d} \sum_{a=0}^{d-1} \chi(a) \sum_{n=0}^{\infty} \beta_{n,\lambda} \left( \frac{a}{d} \right) \frac{t^n}{n!} \]
\[ = \sum_{n=0}^{\infty} \left( d^{n+1} \sum_{a=0}^{d-1} \chi(a) \beta_{n,\lambda} \left( \frac{a}{d} \right) \right) \frac{t^n}{n!}. \] (24)

Therefore, by (24), we obtain the following theorem.

Theorem 2.6. For $n \geq 0$, $d \in \mathbb{N}$, we have
\[ \beta_{n,\lambda,d} = d^{n+1} \sum_{a=0}^{d-1} \chi(a) \beta_{n,\lambda} \left( \frac{a}{d} \right). \]

3. Further Remark

Let $p$ be a fixed prime number. Throughout this section, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is normalized as $\|p\|_p = \frac{1}{p}$. Let us assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. In Section 2, we introduced the degenerate Bernoulli polynomials given by the generating function
\[ \log (1 + \lambda t)^\frac{1}{d} \sum_{a=0}^{d-1} \left( a \right) \frac{t^n}{n!}. \]

Let $d$ be a positive integer with $(d, p) = 1$. Then we set
\[ X = \lim_{N \to \infty} \left( \mathbb{Z}_p / dp^N \mathbb{Z}_p \right); \]
\[ a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\}; \]
\[ X^* = \bigcup_{0 \leq a < dp^N / p^a} \left( a + dp^N \mathbb{Z}_p \right). \]

We shall usually take $0 \leq a < dp^N$ when we write $a + dp^N \mathbb{Z}_p$. Now, we will use Theorem 2.2 to prove a $p$-adic distribution result.
Theorem 3.1. For $k \geq 0$, let $\mu_{k,\beta}$ be defined by
\[
\mu_{k,\beta}^{(\lambda)}(a + dp^N Z_p) = \left(dp^N\right)^{k-1} \beta_{k,\frac{a}{dp^N}},
\]
(25)
Then $\mu_{k,\beta}^{(\lambda)}$ extends to a $C_p$-valued distribution on compact open sets $U \subset X$.

Proof. It suffices to show that
\[
\sum_{i=0}^{p-1} \mu_{k,\beta}^{(\lambda)}(a + idp^N + dp^{N+1} Z_p) = \left(dp^{N+1}\right)^{k-1} \sum_{i=0}^{p-1} \beta_{k,\frac{a + idp^N}{dp^{N+1}}}.
\]
\[
= \left(dp^N\right)^{k-1} \sum_{i=0}^{p-1} \beta_{k,\frac{a}{dp^N}} \left(\frac{dp^N}{dp^{N+1}} + i\right)\]
\[
= \left(dp^N\right)^{k-1} \beta_{k,\frac{a}{dp^N}} \left(\frac{dp^{N+1}}{dp^N}\right)
\]
\[
= \mu_{k,\beta}^{(\lambda)}(a + dp^N Z_p).
\]

The locally constant function $\chi$ can be integrated against the distribution $\mu_{k,\beta}$ defined by (25), and the result is
\[
\int_X \chi(x) d\mu_{k,\beta}^{(\lambda)}(x) = \lim_{N \to \infty} \sum_{x=0}^{dp^{N-1}} \chi(x) \mu_{k,\beta}^{(\lambda)}(x + dp^N Z_p)
\]
\[
= \lim_{N \to \infty} \left(dp^N\right)^{k-1} \sum_{x=0}^{dp^{N-1}} \chi(x) \beta_{k,\frac{dp^N}{dp^{N+1}}} \left(\frac{x}{dp^N}\right)
\]
\[
= \beta_{k,\lambda,\lambda}.
\]
From (26), we have
\[
\int_X \chi(x) d\mu_{k,\beta}^{(\lambda)}(x) = \beta_{k,\lambda,\lambda}, \quad (k \geq 0).
\]

References


