De-Haan Type Conditions for Max Domains of Attraction

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Abstract. Motivated by de Haan’s pioneering work, this paper gives some general sufficient and necessary conditions that functions are regularly varying and \(\Gamma\)-varying, respectively. Specially, these criteria can be employed to determine whether a given distribution belongs to one of the max-domains of attractions of extreme value distributions.

1. Introduction

For a distribution function (df) \(F\), if there exists some constants \(a_n > 0\) and \(b_n\) such that

\[
\lim_{n \to \infty} F^n(a_n x + b_n) = G(x)
\]

for all continuity points \(x\) of \(G\), where \(G\) is a non-degenerate df, then we say that \(F\) belongs to the max-domain of attraction of \(G\), abbreviated as \(F \in D(G)\). It is well-known that \(G\) must belong to one of the following three classes

- **Type I Gumbel**: \(\Lambda(x) = \exp(-\exp(-x))\), \(x \in \mathbb{R}\);
- **Type II Fréchet**: \(\Phi_\alpha(x) = \begin{cases} 0, & x < 0 \\ \exp(-x^{-\alpha}), & x \geq 0 \end{cases}\) for \(\alpha > 0\);
- **Type III Weibull**: \(\Psi_\alpha(x) = \begin{cases} \exp(-(x)^{-\alpha}), & x < 0 \\ 1, & x \geq 0 \end{cases}\) for \(\alpha > 0\).

Standard monographs of extreme value theory are de Haan [1], Leadbetter et al. [7], Resnick [10], Reiss [9], Embrechts et al. [3], Kotz and Nadarajah [6], de Haan and Ferreira [2], Falk et al. [4]. Other complementing references dealing with conditions for \(F \in D(G)\) are Geluk [5] and Peng et al. [8]. An interesting necessary and sufficient condition for \(F \in D(G)\) is the following one refered to as de Haan MDA condition, [cf. de Haan [1], pages 100-103, Theorem 2.6.1, Theorem 2.6.2 and its remark], namely

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Proposition 2.1. a) The following are equivalent:

we have
\[ x \in H \]

\[ \text{Theorem 1.1.} \]
2. The Class \( \Gamma \)
the corresponding results to MDA as
\[ \text{Motivated by the de Haan MDA criteria (Theorem 1.1), in this paper we present some necessary and sufficient conditions for } F \in D(G), \text{ which are in the spirit of de Haan [1]. Note that } F \in D(\Lambda) \text{ iff } \bar{F} \in \Gamma \text{-class and } F \in D(\Phi_a) \text{ iff } \bar{F} \in RV_{-a}. \text{ See [1] and [10]. Proposition 1.13 in [10] also shows that } F \in D(\Psi_a) \text{ iff } x_0 < \infty \text{ and } \bar{F}(x_0 - x^{-1}) \in RV_{-a}. \text{ The new criteria in this paper are formulated for general function } H \text{ while we get the corresponding results to MDA as } H = \bar{F}. \text{ The paper is organized as follows: In Section 2 we discuss } \Gamma^c(f) \text{-class, a variation of } \Gamma \text{-class given by [1] and [10]. Section 3 is for RV.} \]

2. The Class \( \Gamma^c(f) \)

An ultimate positive and measurable function \( H \) defined on an interval \((x_0, \infty)\) is in the class \( \Gamma^c(f) \) if it satisfies \( \lim_{z \to x_0} H(x) = 0 \) and

\[ \lim_{z \to x_0} \frac{H(x + zf(x))}{H(x)} = e^{-y}, \quad \forall y \in \mathbb{R}. \]

The function \( f \) is an auxiliary function and \( f \) satisfies \( f(x)/x \to 0 \) and \( f(x + yf(x))/f(x) \to 1, \forall y. \) Obviously, \( H \in \Gamma^c(f) \) iff \( 1/H \in \Gamma \) defined by de Haan [1]. Note that \( H \in \Gamma^c(f) \) if and only if for some (all) \( \alpha \in \mathbb{R}, \beta > 0, \) we have \( x^\alpha H(x) \in \Gamma(g) \), and then \( g(x) = f(x)/\beta. \) The following result is a result of de Haan.

Proposition 2.1. a) The following are equivalent:

(i) \( H \in \Gamma^c(f) \).

(ii) There exist functions \( a, b, c \) such that

\[ H(x) = c(x) \exp \left( - \int_{a^*}^{x} \frac{a(z)}{b(z)} \, dz \right), \quad x \geq a^*, \]

and \( c(x) \to c > 0, a(x) \to 1 \) and \( b'(x) \to 0 \) as \( x \to x_0. \)
(iii) We have
\[
\lim_{x \to x_0} \frac{H(x) \int_0^x y H(z)dz\,dy}{\left(\int_0^x H(z)dz\right)^2} = 1.
\]

b) If \( H \in \Gamma^\circ(f) \), \( \alpha \in \mathbb{R} \), then
\[
\int_0^x z^\alpha H(z)dz \sim x^\alpha H(x)f(x) \in \Gamma^\circ(f).
\]
Conversely, if \( H \) is nonincreasing and if
\[
\int_0^x z^\alpha H(z)dz \in \Gamma^\circ(f),
\]
then \( H \in \Gamma^\circ(f) \).

Remark 2.2. If \( H \in \Gamma^\circ(f) \), the result implies that \( f(x) \sim b(x) \sim \int_0^x H(z)dz/H(x) \).

From Proposition 2.1, we can obtain the following corollary.

Corollary 2.3. Suppose that \( H \in \Gamma^\circ(f) \). Then

(i) for all \( \alpha > 0 \),
\[
\lim_{x \to x_0} \frac{H(x) \int_0^x \left(\int_0^y H(z)dz\right)^\alpha \, dy}{\left(\int_0^x H(z)dz\right)^{1+\alpha}} = \frac{1}{\alpha}.
\]

(ii) for all \( \alpha > \beta \), \( p \in \mathbb{R} \),
\[
\lim_{x \to x_0} \frac{x^p H^\beta(x) \int_0^x z^\alpha H(z)dz}{x^p H^\alpha(x) \int_0^x z^\beta H(z)dz} = \frac{\beta}{\alpha}.
\]

Proof. (i) From Proposition 2.1 b), it follows that
\[
\int_0^x H(z)dz \sim f(x)H(x) \in \Gamma^\circ(f)
\]
and then
\[
\left(\int_0^x H(z)dz\right)^\alpha \sim f^\alpha(x)H^\alpha(x) \in \Gamma^\circ(f/\alpha).
\]
It follows that
\[
\int_0^x \left(\int_0^y H(z)dz\right)^\alpha \, dy \sim \left(\int_0^x H(z)dz\right)^\alpha f(x)/\alpha.
\]
Now result (i) follows.

(ii) From Proposition 2.1 b), we have
\[
\int_0^x z^\alpha H^\alpha(z)dz \sim x^\alpha H^\alpha(x)f(x)/\alpha
\]
and
\[
\int_0^x z^\beta H^\beta(z)dz \sim x^\beta H^\beta(x)f(x)/\beta,
\]
then the result follows. \( \Box \)
Remark 2.4. Note that in Corollary 2.3 (ii), for all $\alpha > 0$, we have $\int_x^{x_0} z^\alpha H^\alpha(z)dz \in \Gamma^\alpha(f/\alpha)$.

The converse result is also true.

Theorem 2.5. Each of the following statements implies that $H \in \Gamma^\alpha(f)$:

(i) For some $\alpha > 0$,
$$\lim_{x \to x_0} \frac{H(x) \int_x^{x_0} H(z)dz \bigg|^\alpha \bigg|}{\bigg| \int_x^{x_0} H(z)dz \bigg|^{1+\alpha}} = \frac{1}{\alpha}.$$

(ii) For some $\alpha > \beta$, $p \in \mathbb{R}$,
$$\lim_{x \to x_0} \frac{x^\beta H^\beta(x) \int_x^{x_0} z^\alpha H^\alpha(z)dz}{x^\alpha H^\alpha(x) \int_x^{x_0} z^\beta H^\beta(z)dz} = \frac{\beta}{\alpha}.$$

(iii) If $H$ is nonincreasing and for some $p \in \mathbb{R}$, $\int_x^{x_0} z^p H(z)dz \in \Gamma^\alpha(f)$.

Proof. (i) Define $A(x)$ and $R(x)$ as follows:
$$A(x) = \int_x^{x_0} H(z)dz,$$
$$R(x) = \frac{A^{1+\alpha}(x)}{\int_x^{x_0} A^\alpha(y)dy}.$$

In the case of (i), we have that $R(x)/H(x) \to a$. Taking the derivative of $R(x)$, we have
$$R'(x) = \frac{-(1+a)A'(x)H(x) \int_x^{x_0} A^\alpha(y)dy + A^{1+\alpha}(x)A^\alpha(x)}{\left(\int_x^{x_0} A^\alpha(y)dy\right)^2} \left(1 + \frac{R(x)}{H(x)}\right).$$

Note that $-R'(x)$ is positive for large values of $x$. Now we have
$$\frac{R'(x)}{R(x)} = -\frac{R(x)}{A(x)} \left(1 + a \frac{H(x)}{R(x)} - 1 \right) = -\frac{a(x)}{b(x)},$$

where $a(x) = a((1+a)H(x)/R(x) - 1)$ and $b(x) = aA(x)/R(x)$. Note that $a(x) \to 1$ and that
$$b(x) = a \int_x^{x_0} A^\alpha(y)dy / A^\alpha(x).$$

It is easy to see that
$$b'(x) = a \frac{-A^{2\alpha}(x) + aA^{\alpha-1}(x)H(x) \int_x^{x_0} A^\alpha(y)dy}{A^{2\alpha}(x)} = a \left(1 + \frac{H(x)}{R(x)}\right) \to 0.$$
Taking integrals, we have

\[ R(x) = C \exp \left( -\int_x^\infty \frac{a(z)}{b(z)} \, dz \right) \]

and we have (cf. Proposition 2.1, (ii)) that \( R \in \Gamma^*(b) \). Since \( H(x) \sim R(x)/\alpha \), we also get that \( H \in \Gamma^*(b) \).

(ii) To prove (ii) we define function \( r(x) \) and \( R(x) \) as follows:

\[ r(x) = \frac{x^{\beta/\alpha} H^\beta(x) \int_x^\infty z^{\mu/\alpha} H^\mu(z) \, dz}{x^{\mu/\alpha} H^\mu(x) \int_x^\infty z^{\beta/\alpha} H^\beta(z) \, dz} \]

and

\[ R(x) = \frac{\int_x^\infty z^{\mu/\alpha} H^\mu(z) \, dz}{\int_x^\infty z^{\beta/\alpha} H^\beta(z) \, dz}. \]

By assumption we have \( r(x) \to \beta/\alpha \) and \( R(x) \sim (\beta/\alpha) x^{\beta/\alpha} H^{\beta/\alpha}(x) \). Taking the derivative of \( R(x) \), we find

\[ R'(x) = \frac{- \left( \int_x^\infty z^{\mu/\alpha} H^\mu(z) \, dz \right) x^{\mu/\alpha} H^\mu(x) + \left( \int_x^\infty z^{\beta/\alpha} H^\beta(z) \, dz \right) x^{\beta/\alpha} H^\beta(x)}{\left( \int_x^\infty z^{\beta/\alpha} H^\beta(z) \, dz \right)^2} \]

\[ = \frac{x^{\mu/\alpha} H^\mu(x)}{\int_x^\infty z^{\beta/\alpha} H^\beta(z) \, dz} (r(x) - 1). \]

It follows that

\[ \frac{R'(x)}{R(x)} = \frac{1 - r(x)}{r(x)} \frac{x^{\mu/\alpha} H^\mu(x)}{\int_x^\infty z^{\beta/\alpha} H^\beta(z) \, dz} = \frac{a(x)}{b(x)}, \]

where

\[ a(x) = (1 - r(x)) \frac{x^{\beta/\alpha} H^\beta(x)}{R^{\beta/\alpha}(x)} \]

and

\[ b(x) = R^{-\alpha/\beta}(x) \int_x^\infty z^{\alpha/\beta} H^{\alpha/\beta}(z) \, dz = R^{-\beta/\alpha}(x) \int_0^\infty z^{\beta/\alpha} H^{\beta/\alpha}(z) \, dz. \]

First consider \( a(x) \). Using \( r(x) \to \beta/\alpha \) and \( R(x) \sim (\beta/\alpha) x^{\beta/\alpha} H^{\beta/\alpha}(x) \), we obtain that

\[ a(x) \to \delta = \left( 1 - \frac{\beta}{\alpha} \right)^{\alpha/\beta} > 0. \]

For \( b(x) \) we find

\[ -b'(x) = \frac{\beta}{\alpha - \beta} R^{-\alpha/\beta}(x) R'(x) \int_x^\infty z^{\beta/\alpha} H^{\beta/\alpha}(z) \, dz + R^{-\beta/\alpha}(x) x^{\beta/\alpha} H^{\beta/\alpha}(x) \]

\[ = I + II. \]

First consider \( I \). Using the expression for \( R'(x) \) and then using \( R(x) \sim (\beta/\alpha) x^{\beta/\alpha} H^{\beta/\alpha}(x) \), we have

\[ I = \frac{\beta}{\alpha - \beta} R^{-\alpha/\beta}(x) x^{\beta/\alpha} H^{\beta/\alpha}(x) (r(x) - 1) \]

\[ \to \frac{\beta}{\alpha - \beta} \left( \frac{\beta}{\alpha} \right)^{-\alpha/\beta} \left( \frac{\beta}{\alpha} - 1 \right) = - \left( \frac{\beta}{\alpha} \right)^{-\beta/\alpha}. \]
For II we find
\[ H \rightarrow \left( \frac{b}{a} \right)^{-\beta/(\alpha-\beta)}. \]

It follows that \( b'(x) \rightarrow 0 \). Taking integrals, we get that
\[ R(x) = C \exp \left( -\int_{x}^{\infty} \frac{a(z)}{b(z)} \, dz \right) \]
and using Proposition 2.1 (ii), we find that \( R \in \Gamma^{b/\delta} \). From here it follows that \( x^{\beta}H(x) \in \Gamma^{\alpha} \), and hence also that \( H \in \Gamma^{\alpha} \).

(iii) This is Proposition 2.1 b). \( \square \)

3. The Class \( \text{RV}_{-\alpha} \)

An ultimate positive and measurable function \( H \) is in the class \( \text{RV}_{-\gamma}, \gamma > 0 \), if it satisfies
\[ \lim_{x \to \infty} \frac{H(xy)}{H(x)} = y^{-\gamma}, \]
see, de Haan [1] and Resnick [10].

If \( \gamma > 1 \), Karamata’s Theorem shows that \( H \in \text{RV}_{-\gamma} \) implies
\[ \int_{x}^{\infty} H(y) \, dy \sim \frac{xH(x)}{y-1} \in \text{RV}_{1-\gamma}. \]

Conversely, if \( \int_{x}^{\infty} H(y) \, dy \in \text{RV}_{1-\gamma} \), and if \( H \) is nonincreasing, then \( H \in \text{RV}_{-\gamma} \).

Similar to Corollary 2.3, we have the following result.

**Corollary 3.1.** Suppose that \( H \in \text{RV}_{-\gamma} \). Then

(i) Suppose that \( \gamma > 1 \) and \( \alpha \) so that \( \alpha(\gamma - 1) > 1 \). We have
\[ \lim_{x \to \infty} \frac{H(x) \int_{x}^{\infty} \left( \int_{y}^{\infty} H(z) \, dz \right)^{\alpha}}{\left( \int_{x}^{\infty} H(z) \, dz \right)^{1+\alpha}} = \frac{\gamma - 1}{\alpha(\gamma - 1) - 1}. \]

(ii) Suppose \( \alpha, q \) are so that \( q\gamma > 1, \alpha(q\gamma - 1) > 1 \). We have
\[ \lim_{x \to \infty} \frac{H^q(x) \int_{x}^{\infty} \left( \int_{y}^{\infty} H^q(z) \, dz \right)^{\alpha}}{\left( \int_{x}^{\infty} H^q(z) \, dz \right)^{1+\alpha}} = \frac{q\gamma - 1}{\alpha(q\gamma - 1) - 1}. \]

(iii) For all \( \alpha > \beta > 0, \alpha(\gamma - p) > 1, \beta(\gamma - p) > 1, p \in \mathbb{R} \), \( x^\beta H^\beta \int_{x}^{\infty} z^{\beta a} H^a(z) \, dz \)
\[ \lim_{x \to \infty} \frac{x^\beta H^\beta}{x^\alpha H^a(x) \int_{x}^{\infty} z^{\beta a} H^a(z) \, dz} = \frac{\beta(\gamma - p) - 1}{\alpha(\gamma - p) - 1}. \]
Proof. (i) This is a simple consequence of Karamata’s Theorem.
(ii) This follows from (i) since $H^\beta \in RV_{-\gamma}$.
(iii) If $H \in RV_{-\gamma}$, we have $z^{\alpha\gamma}H(x) \in RV_{-\alpha(\gamma-\delta)}$ and
\[
\int_x^\infty z^{\alpha\gamma}H(z)dz \sim \frac{x^{\alpha\gamma+1}H(x)}{\alpha(\gamma-\delta) - 1}.
\]
In a similar way we have
\[
\int_x^\infty z^{\beta\gamma}H(z)dz \sim \frac{x^{\beta\gamma+1}H(x)}{\beta(\gamma-\delta) - 1}.
\]
The result follows. \qed

We also have the converse results. In the next result we assume that $H(x) = \bar{f}(x)$, the tail distribution.

Theorem 3.2. (i) Suppose that all integrals exist and that
\[
\lim_{x \to \infty} \frac{H(x) \int_x^\infty \left( \int_y^\infty H(z)dz \right)^\alpha dy}{\left( \int_x^\infty H(z)dz \right)^{\alpha + 1}} = \delta,
\]
where $\alpha \delta > 1$. Then $H \in RV_{-\gamma}$ where $\gamma = \delta/(\alpha \delta - 1) + 1$.

(ii) Suppose that all integrals exist and that for some $\alpha > \beta > 0$, $p \in \mathbb{R}$ we have
\[
\lim_{x \to \infty} \frac{x^{\alpha\gamma}H^\beta(x) \int_x^\infty z^{\alpha\gamma}H^\alpha(z)dz}{x^{\beta\gamma}H^\alpha(x) \int_x^\infty z^{\beta\gamma}H^\beta(z)dz} = \delta,
\]
where $\delta < \beta/\alpha$. Then $H \in RV_{-\gamma}$, where $\gamma = p + (1-\delta)/(\beta - \delta \alpha)$.

Proof. (i) Define $A(x)$ and $R(x)$, $h(x)$ as follows:
\[
A(x) = \int_x^\infty H(z)dz, \quad R(x) = \frac{\int_x^\infty A^\alpha(y)dy}{A^\alpha(x)}, \quad h(x) = \frac{H(x)R(x)}{A(x)}.
\]
In the case of (i), we have that $h(x) \to \delta$. Taking the derivative of $R(x)$, we have
\[
R'(x) = -\frac{A^\alpha(x) + \alpha \int_x^\infty A^\alpha(y)dy A^{\alpha-1}(x)H(x)}{A^{2\alpha}(x)}
= -1 + a h(x).
\]
Since $h(x) \to \delta$, we have $R'(x) \to -1 + a \delta$ and then we obtain that $R(x)/x \to a \delta - 1$. It follows that
\[
\frac{x}{R(x)} \to \frac{1}{a \delta - 1}
\]
and then
\[
\frac{xH(x)}{A(x)} \to \frac{\delta}{a \delta - 1}.
\]
Karamata’s theorem shows that $H \in RV_{-\gamma}$ with $\gamma = \delta/(a \delta - 1) + 1$. Note that $\delta = (\gamma - 1)/(\alpha(\gamma - 1) - 1)$.

(ii) We proceed as in the proof of Theorem 2.5. Define functions $r(x)$ and $R(x)$ as follows:
\[
r(x) = \frac{x^{\beta\gamma}H^\beta(x) \int_x^\infty z^{\beta\gamma}H^\beta(z)dz}{x^{\alpha\gamma}H^\alpha(x) \int_x^\infty z^{\alpha\gamma}H^\alpha(z)dz}
\]
Using $\alpha/\beta$ where $\delta = \delta \alpha/(\alpha - \beta) = 1$, we have

$$D \alpha/\beta = \delta \alpha/(\alpha - \beta) (\delta - 1) + \delta \delta \alpha/(\alpha - \beta) = d.$$
It follows that
\[ \frac{b(x)}{x} \to \delta^{-\alpha/(\alpha-\beta)} \frac{\delta \alpha - \beta}{\alpha - \beta}, \]
and as a consequence also that
\[ \frac{xR'(x)}{R(x)} = \frac{xa(x)}{b(x)} \to \frac{(1-\delta)(\alpha - \beta)}{\delta \alpha - \beta}. \]

It follows that \( R \in RV_{-\gamma} \), where
\[ \gamma = \frac{(1-\delta)(\alpha - \beta)}{\beta - \delta \alpha}. \]

Since \( R(x) \sim \delta x^{\alpha-\beta} H^{\alpha-\beta}(x) \), we obtain that \( H \in RV_{-\gamma} \), where
\[ \gamma = p + \frac{1-\delta}{\beta - \delta \alpha}. \]

Note that the last expression implies that
\[ \delta = \frac{\beta(\gamma - p) - 1}{\alpha(\gamma - p) - 1}. \]

The proof is complete. \( \square \)

**Remark 3.3.** As we mentioned that \( \overline{H} \in D(\Psi_\alpha) \) iff \( x_0 < \infty \) and \( H(x_0 - x^{-1}) \in RV_{-\alpha} \). By Corollary 3.1 and Theorem 3.2, we can derive corresponding results for \( \overline{H} \in D(\Psi_\alpha) \).

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**References**