In this paper, the concept of commutativity with respect to a closed subgroup of a product group, which is a generalization of multipliers under the usual sense, is introduced. As a consequence, we obtain characterization of operators on $L^2(G)$ which commute with left translation when $G$ is amenable.

1. Introduction

The subject of multipliers for $L^p(G)$ has been considered, in various forms, by a great number of authors. We may refer the reader e.g. to [7], [13] and [18]. It was shown by Wendel in [21] that $T$ is a left multiplier of $L^1(G)$ if and only if for some $\mu \in M(G)$, $T = \lambda_\mu$, here $\lambda_\mu$ is the operator of multiplication by $\mu$ on left. For $1 \leq p < \infty$, the bounded linear operators on $L^p(G)$ which commute with left translations was studied by Larsen [13].

For a locally compact group $G$, let $\text{Hom}(L^p(G), L^p(G))$ denote all bounded linear maps $T : L^p(G) \rightarrow L^p(G)$ commuting with the left translation operators $L_x$, and let $\text{Conv}(L^p(G), L^p(G))$ denote all bounded linear maps $T : L^p(G) \rightarrow L^p(G)$ commuting with the left convolution operators $\lambda_\phi, \phi \in L^1(G)$, where $\lambda_\phi(f) = \phi \ast f, f \in L^1(G)$. It is known that $\text{Conv}(L^\infty(G), L^\infty(G)) \subseteq \text{Hom}(L^\infty(G), L^\infty(G)), [15]$. We know that the bounded linear operators on $L^\infty(G)$ which commute with left convolution and left translations have been studied by Larsen [13].

Let $G$ be a locally compact group and let $\Gamma$ be a closed subgroup of $G \times G$. Let $T : L^p(G) \rightarrow L^{p'}(G), 1 \leq p, p' < \infty$, be a linear map. We say that $T$ is $\Gamma$-invariant (respectively $L^1(\Gamma)$-invariant) whenever $T(s f) = s T(f), (T(T^{(p)}_\phi f) = T^{(p')}_\phi T(f))$ for all $f \in L^p(G), (s, t) \in \Gamma$ and $\Phi \in L^1(\Gamma), [16]$.

Our first purpose in this paper is to study the relationship between these linear maps. We study when these concepts are equivalent. In the case $\Gamma = G \times \{e\}$, we say that $T$ commutes with the left translation, following [13]. In the case $\Gamma = \{(x, x); x \in G\}$, we say that $T$ commutes with conjugation. We want to shift our attention away from the study of multipliers of group algebras and begin a discussion on linear maps for group algebras which commute with translations and convolutions with respect to a closed subgroup of a product group. We shall give some indication of the relationship between these linear maps. Our second purpose in this paper is to characterize the amenability of a group with respect to the existence of multipliers maps.

\textit{Keywords.} Amenability, Banach algebra, bounded operator, locally compact group.

\textit{Mathematics Subject Classification.} Primary 43A10; Secondary 43A20

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\textbf{Abstract.} Let $G$ be a locally compact group and let $\Gamma$ be a closed subgroup of $G \times G$. In this paper, the concept of commutativity with respect to a closed subgroup of a product group, which is a generalization of multipliers under the usual sense, is introduced. As a consequence, we obtain characterization of operators on $L^2(G)$ which commute with left translation when $G$ is amenable.
2. Preliminaries and Notations

Throughout this paper, $G$ denotes a locally compact group with a fixed left Haar measure. Let $C_b(G)$ denote the Banach algebra of bounded continuous complex-valued functions on $G$ with the supremum norm, and let $C_0(G)$ be the closed subspace of $C_b(G)$ consisting of all functions in $C_b(G)$ which are vanishing at infinity. The Banach spaces $L^p(G), 1 \leq p \leq \infty$, are as defined in [12]. The convex subset of $L^1(G)$ consisting of all probability measures on $G$ will be denoted by $P^1(G)$. If $f$ is a complex-valued function defined locally almost everywhere on $G$, and $s, t, x \in G$ then

$$sf(x) := f(s^{-1}x), \quad f_t(x) := f(xt) \quad \text{and} \quad s_t f(x) := f(s^{-1}xt)$$

where they are defined.

Let $G$ be a locally compact group, and let $\Gamma$ be any closed subgroup of the product group $G \times G$, with a fixed Haar measure denoted by $d(y, z)$ and modular function $\Delta$. We say that $G$ is $\Gamma$-amenable if there exists $m \in L^\infty(\Gamma)$ such that $m \geq 0, \|m\| = 1 \text{ and } m(h_1) = m(h)$ for each $h \in L^\infty(\Gamma)$ and $(s, t) \in \Gamma$. Li and Pier, [16], give a good account of the structure of $\Gamma$-amenability of a locally compact topological group $G$, see also [6]. Following Li and Pier, [16], we define

$$S_p h(x) = \int_{\Gamma} h(yxz^{-1})d\mu(y, z)$$

where $\mu \in M(\Gamma)$ ($M(\Gamma)$ is the Banach algebra of all bounded Borel measures on $\Gamma$) and $h \in L^\infty(\Gamma)$. For $1 \leq p < \infty, L^p(\Gamma)$ is a Banach left $L^1(\Gamma)$-module with module multiplication defined by

$$T_{\phi}^{(p)} f(x) = \int_{\Gamma} f(y^{-1}xz)\phi(y, z)\Delta(z)^{\frac{1}{p}}d(y, z)$$

where $f \in L^p(\Gamma)$ and $\Phi \in L^1(\Gamma)$. For $f \in L^p(\Gamma)$ and $\Phi \in L^1(\Gamma)$, we have $\|T_{\phi}^{(p)} f\|_p \leq \|f\|_p \|\Phi\|_1$ (see [16]).

We mainly follow [16] in our notation and refer to [19] for basic functional analysis and to [12] for basic harmonic analysis results. The duality action between Banach spaces is denoted by $\langle \cdot, \cdot \rangle$, thus for $h \in L^\infty(\Gamma)$ and $f \in L^1(\Gamma)$, we have $\langle h, f \rangle = \int f(x)h(x)dx$.

3. $\Gamma$-Invariant Operators

We know that an affine continuous mapping $T$ from $L^1(G)$ into $L^1(G)$ commutes with left translation if and only if $T(\phi * f) = \phi * T(f)$ for each $\phi \in L^1(G)$ and $f \in L^p(G)$, [14]. Recently, convolution operators of hypergroup algebras have been studied by Pavel in [18]. The following theorem shows that a bounded linear operator $T$ from $L^1(G)$ to itself is $\Gamma$-invariant if and only if $T$ is $L^1(\Gamma)$-invariant.

**Theorem 3.1.** Let $G$ be a locally compact group, let $p \geq 1$ be a real number, and let $T : L^p(G) \to L^p(G)$ be a continuous linear operator. Then the following properties are equivalent:

(i) $T$ is $\Gamma$-invariant, i.e. $T(s_t f) = s_t T(f)$, for every $(s, t) \in \Gamma$ and $f \in L^p(G)$;

(ii) $T$ is $L^1(\Gamma)$-invariant, i.e. $T(T_{\phi}^{(p)} f) = T_{\phi}^{(p)} T(f)$, for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose $T(s_t f) = s_t T(f)$, for every $(s, t) \in \Gamma$ and $f \in L^p(G)$. Let $\Phi \in L^1(\Gamma)$. Write $\Phi = \Phi_1 + \Phi_2 + i(\Phi_2 - \Phi_1)$, where $\Phi_1, \Phi_2$ are respectively the real and imaginary parts of $\Phi$, and for $i = 1, 2$, $\Phi_+$ and $\Phi_-$ are respectively the positive and negative variations of $\Phi_i$. It suffices to show that $T(T_{\phi}^{(p)} f) = T_{\phi}^{(p)} T(f)$ for every $f \in L^p(G)$ and $\Phi \in P^1(\Gamma)$. Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{\|\Phi\|_1}$. By Theorem 19.18 in [12], there exists a compact subset $K$ in $\Gamma$ such that $\int_{\Gamma \setminus K} \Phi(x, y)d(x, y) < \delta$. Using the continuity of the mappings $(y, z) \mapsto s_t^* \Delta(z)$$^{\frac{1}{2}}$
and \((y,z) \mapsto yT(f)_z \Delta(z)\) from \(\Gamma\) to \(U^\prime(G)\), by Theorem 20.4 in [12], we can find an open, relatively compact neighbourhood \(U_y \times V_z\) of \((y,z) \in \Gamma\) such that
\[\|y f_z \Delta(z) - y f_z \Delta(t)\|_p < \delta, \quad \|y T(f)_z \Delta(z) - y T(f)_t \Delta(t)\|_p < \delta\]
for every \((s,t) \in U_y \times V_z\). Let \((y_1,z_1), \ldots, (y_n,z_n)\) in \(\Gamma\) be such that \((y_1,z_1) = (e,e)\) and \(K \subseteq \bigcup_{i=2}^n U_{y_i} \times V_{z_i}\). We put \(E_1 = \Gamma \setminus K\) and define inductively \(E_i = (U_{y_i} \times V_{z_i}) \cap (\Gamma \setminus \bigcup_{j=1}^{i-1} E_j)\) for \(i = 2, \ldots, n\). So we have
\[\|y f_z \Delta(z) - y f_z \Delta(t)\|_p < \delta, \quad \|y T(f)_z \Delta(z) - y T(f)_t \Delta(t)\|_p < \delta\]
whenever \((y,z) \in E_i\) for \(i = 2, \ldots, n\). Write \(c_i = \int_{E_i} \Phi(y,z)d(y,z)\), where \(i = 1, \ldots, n\). Since \(\Gamma = \bigcup_{i=1}^n E_i\) is a finite union of pairwise disjoint subsets of \(\Gamma\), we have
\[1 = \int_{\Gamma} \Phi(x,y)d(x,y) = \sum_{i=1}^n \int_{E_i} \Phi(x,y)d(x,y) = \sum_{i=1}^n c_i.\]
Let \(q\) be the Holder conjugate of \(p\). For every \(h \in C_c(G)\) (the space of complex valued continuous functions on \(G\) with compact support), we have
\[
\left| \int_G h(x) \int_{E_1} \left(y f_z(x) \Delta(z) - f(x) \Phi(y,z) d(y,z) dx \right) \right| \leq 2 \|h\|_q \|f\|_p \delta \leq \frac{\epsilon}{4|T|} \|h\|_q
\]
and also
\[
\frac{\epsilon}{2|T|} \|h\|_q \geq \frac{\epsilon}{4|T|} \|h\|_q + \sum_{i=2}^n \int_{E_i} \Phi(y,z) \|y f_z \Delta(z) - y f_z \Delta(t)\|_p \|\Phi\|_q d(y,z)
\]
\[
\geq \sum_{i=1}^n \int_{E_i} \Phi(y,z) \int_G \left|y f_z(x) \Delta(z) - y f_z(x) \Delta(t)\right| h(x) d(x,y)
\]
\[
\geq \left| \int_G h(x) \sum_{i=1}^n \int_{E_i} \left(y f_z(x) \Delta(z) - y f_z(x) \Delta(t)\right) \Phi(y,z) d(y,z) dx \right|
\]
\[
= \left| \left< T^{(p)}_{\Phi} f - \sum_{i=1}^n c_{iy} f_z \Delta(z_i) \right, h \right> \right|
\]
Since this holds for all \(h \in C_c(G)\), by Theorem 12.13 in [12], we conclude that
\[
\left< T^{(p)}_{\Phi} f - \sum_{i=1}^n c_{iy} f_z \Delta(z_i) \right, h \right> \leq \frac{\epsilon}{2|T|}.
\]
It follows that \(\left\| T^{(p)}_{\Phi} f - \sum_{i=1}^n c_{iy} T(f)_z \Delta(z_i) \right\|_p \leq \frac{\epsilon}{2}\). Similarly, one can show that
\[
\left\| T^{(p)}_{\Phi} T(f) - \sum_{i=1}^n c_{iy} T(f)_z \Delta(z_i) \right\|_p \leq \frac{\epsilon}{2}.
\]
Therefore \(\|T^{(p)}_{\Phi} f - T^{(p)}_{\Phi} T(f)\|_p \leq \epsilon\). As \(\epsilon > 0\) is chosen arbitrary, we have \(T^{(p)}_{\Phi} f = T^{(p)}_{\Phi} T(f)\). Thus (i) implies (ii).
Hence using (2) and (3), we have 

\[ \|p(s_1) \Delta (z) \|^2 - s T(f)_1 \|_p < \frac{\epsilon}{2 \|T\|} \] 

whenever \((y, z) \in U \times V\), see Theorem 20.4 in [12]. Choose \( \Phi \in L^1(\Gamma) \) with \( \text{supp} \Phi \subseteq U \times V \). For every \( h \in C_c(\mathbb{G}) \), we have 

\[ \frac{\epsilon}{2 \|T\|} \| h \|_p \geq \int_T \left( \|p(s_1) \Delta (z) \|^2 - s T(f)_1 \|_p \| h \|_p \right) \Phi(y, z) \, d(y, z) \]

\[ \geq \int_G \int_T \| h(x) \|_p \| p(s_1) \Delta (z) \|^2 - s T(f)_1 \|_p \Phi(y, z) \, d(y, z) \, dx \]

\[ \geq \left( t_{\Phi}^{(p)} p s_1 - s T(f)_1, h \right). \]

By Theorem 12.13 in [12], 

\[ \left\| t_{\Phi}^{(p)} p s_1 - s T(f)_1 \right\|_p \leq \frac{\epsilon}{2 \|T\|}. \tag{1} \]

Interchanging the roles of \( s_f \) and \( s T(f)_1 \), we see at once that 

\[ \left\| t_{\Phi}^{(p)} p s_1 - s T(f)_1 \right\|_p \leq \frac{\epsilon}{2}. \tag{2} \]

On the other hand, \( t_{\Phi}^{(p)} p s_1 = \Delta(s_1)^{1/2} \Delta_r(s_1, t_1)^{1/2} T_{\Phi^{(p)} s_1} f \) and also 

\[ t_{\Phi}^{(p)} p s_1 - T(f)_1 = \Delta(s_1)^{1/2} \Delta_r(s_1, t_1)^{1/2} T_{\Phi^{(p)} s_1} f. \]

Now (1) gives 

\[ \left\| \Delta(s_1)^{1/2} \Delta_r(s_1, t_1)^{1/2} T_{\Phi^{(p)} s_1} f - T(f)_1 \right\|_p \leq \frac{\epsilon}{2} \]

and so 

\[ \left\| T_{\Phi}^{(p)} p s_1 - T(f)_1 \right\|_p \leq \frac{\epsilon}{2}. \tag{3} \]

Hence using (2) and (3), we have \( \| T(s f)_1 - T(f)_1 \|_p \leq \epsilon \). As \( \epsilon > 0 \) is chosen arbitrary, we have \( T(s f)_1 = s T(f)_1 \). \( \square \)

In the Theorem 3.1, we discussed linear operators for the pair \((L^p(\mathbb{G}), L^p(\mathbb{G}))\) when \( p = p' \). We cannot verify if the claim in Theorem 3.1 remains true if \( T : L^p(\mathbb{G}) \to L^p(\mathbb{G}) \) for \( p \neq p' \). For a compact abelian group, the bounded linear maps from \( L^p(\mathbb{G}) \to L^p(\mathbb{G}) \) which commutes with translations have been studied by Larsen, see Theorem 5.2.4 in [13]. In the following proposition, we study the case that \( p \) is not necessarily equal to \( p' \) for unimodular groups.

**Proposition 3.2.** Let \( G \) be a unimodular locally compact group and \( 1 \leq p, p' < \infty \). Suppose that \( T : L^{p'}(\mathbb{G}) \to L^{p'}(\mathbb{G}) \) is a continuous linear map. Then the following properties are equivalent:

(i) \( T \) is \( \Gamma \)-invariant;

(ii) \( T(t_{\Phi}^{(p')} f) = t_{\Phi}^{(p')} T(f) \) for every \( f \in L^p(\mathbb{G}) \) and \( \Phi \in L^1(\Gamma) \).

**Proof.** To prove this proposition, one may rewrite the proof of Theorem 3.1 where \( \Delta \equiv 1 \). \( \square \)
Let $G$ be a locally compact group and $1 < p < \infty$. The collection of all continuous linear maps $T : L^p(G) \to L^p(G)$ which is $\Gamma$-invariant, will be denoted by $\mathcal{M}_\Gamma(L^p(G))$. If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $T \to T^*$ is an isometric algebra isomorphism of $\mathcal{M}_\Gamma(L^p(G))$ onto $\mathcal{M}_\Gamma(L^q(G))$. By Theorem 3.1, the correspondence between $T$ and $T^*$ defines an isometric algebra isomorphism from $\mathcal{M}_\Gamma(L^p(G))$ onto $\mathcal{M}_\Gamma(L^q(G))$ (see Subsection 6.2) where here denotes the space of continuous linear operator $T : L^p(G) \to L^q(G)$ such that $TT^*(f) = \delta_\psi^*(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(G)$. Clearly $\mathcal{M}_\Gamma(L^p(G))$ is a unital Banach subalgebra of $\mathcal{B}(L^p(G))$. It is also the case that $\mathcal{M}_\Gamma(L^p(G))$ is complete in the strong operator topology.

**Theorem 3.3.** Let $G$ be a locally compact group, and suppose $T : L^\infty(G) \to L^\infty(G)$ is a weak$^*$-weak$^*$ continuous linear operator. Then the following properties are equivalent:

(i) $T$ is $\Gamma$-invariant, i.e. $T(h_t) = z T(h_t)$ for every $(s, t) \in \Gamma$ and $h \in L^\infty(G)$;

(ii) $T(S_\phi h) = S_\phi T(h)$ for every $h \in L^\infty(G)$ and $\Phi \in L^1(\Gamma)$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose $T(h_t) = z T(h_t)$ for every $(s, t) \in \Gamma$ and $h \in L^\infty(G)$. Then for $F \in L^\infty(G)^*$, $h \in L^\infty(G)$ the pairing $(\langle T(F), h \rangle) = (F, T(h))$ defines the adjoint of $T$ as a linear operator $T^* : L^\infty(G)^* \to L^\infty(G)^*$. Moreover, suppose $[h_a] \subseteq L^\infty(G)$ converges in the weak$^*$ sense to $h \in L^\infty(G)$, that is, $\lim_a |h_a(f) - \langle f, h \rangle| = 0$ for each $f \in L^1(G)$. Then $T(h_a) \to T(h)$ in the weak$^*$ topology. For $f \in L^1(G)$,

$$\langle T^*(f), h_a \rangle = \langle T(h_a), f \rangle \to \langle T(h), f \rangle = \langle T^*(f), h \rangle.$$

This shows that $T^*(f)$ is weak$^*$ continuous. By Theorem 3.10 in [19], $T^*(f) \in L^1(G)$. Since $T$ is $\Gamma$-invariant, we see for each $f \in L^1(G)$, $h \in L^\infty(G)$ and each $(s, t) \in \Gamma$ that

$$\langle T^*(f), h \rangle = \langle \lambda f, T(h) \rangle = \int_G f(s^{-1}xt) T(h)(x) dx = \Delta(t)^{-1} \langle \lambda, T(h_{t^{-1}}) \rangle, f \rangle,$$

$$= \Delta(t)^{-1} \langle T(h_{t^{-1}}), f \rangle = \langle T^*(f), h_{t^{-1}} \rangle = \langle T^*(f), h \rangle.$$

Hence $T|_{L^1(G)}$ is $\Gamma$-invariant. Moreover $T|_{L^1(G)}$ is continuous on $L^1(G)$. Indeed, let $f_n, f_0 \in L^1(G)$ be such that $\lim_n \parallel f_n - f_0 \parallel = 0$. Then for each $h \in L^\infty(G)$ we have

$$\parallel T^*(f) - f_0, h \parallel \leq \parallel T^*(f) - T^*(f_0), h \parallel + \parallel T^*(f_0) - f_0, h \parallel \leq \parallel f - f_0 \parallel \parallel T(h) \parallel + \parallel f_0 \parallel \parallel h \parallel.$$

Consequently $(T^*(f) - f_0, h) = 0$ for each $h \in L^\infty(G)$, and hence $T^*(f) = f_0$. Thus $T^*$ is a closed operator and so, by the Closed Graph Theorem, it is continuous. It follows from the preceding result, Theorem 3.1, that $T^*(T^*(f)) = T^*(f)$ for all $\Phi \in L^1(\Gamma)$ and $f \in L^1(G)$. Now let $h \in L^\infty(G), \Phi \in L^1(\Gamma)$ and $f \in L^1(G)$ be given. Elementary calculations again reveal that

$$\langle S_\phi T(h), f \rangle = \langle T(h), T^*(f) \rangle = \langle h, T^*(T^*(f)) \rangle = \langle h, T^*(f) \rangle = \langle S_\phi h, T^*(f) \rangle = \langle S_\phi h, T(f) \rangle.$$

We conclude that $T(S_\phi h) = S_\phi T(h)$.

(ii) $\Rightarrow$ (i). Let $U \times V$ be a compact neighbourhood of $(e, e)$ and fixed. Let $(U_{\alpha} \times V_{\alpha})$ be a net of compact neighbourhoods of $(e, e)$ contained in $U \times V$ ordered by set inclusion $(U_{\alpha} \times V_{\alpha} \subseteq U_{\beta} \times V_{\beta}$ if and only if $U_{\beta} \subseteq U_{\alpha} \times V_{\alpha}$, with $\bigcap U_{\alpha} \times V_{\alpha} = \{e, e\}$), which forms a directed set. Let $\{\Phi_{\alpha}\}$ be a choice of measures in $\mathcal{P}^1(\Gamma)$ such that $\Phi_{\alpha}(G \setminus U_{\alpha} \times V_{\alpha}) = 0$ for all $\alpha$. Now assume $h \in L^\infty(G)$ and $(s, t) \in \Gamma$. Let $f \in L^1(G)$ and $\epsilon > 0$. There exists a neighbourhood $U_{\alpha_0} \times V_{\alpha_0}$ of $(e, e)$ such that $\parallel f - f_{\alpha}, \Delta(z) - f(z) \parallel < \epsilon$ whenever $(y, z) \in U_{\alpha_0} \times V_{\alpha_0}$, see Theorem 20.4 in [12]. For every $\alpha \geq \alpha_0$, we have

$$\parallel (S_{\Phi_{\alpha}} h_{t^{-1}} - S_{\Phi_{\alpha_0}} h_{t^{-1}}, f \parallel \leq \int_G \int_G |h_{t^{-1}}(x)| [f_\alpha(y)(x) \Delta(z) - f(x) \Phi_{\alpha}(y, z)] dx dy,$$

$$\leq \int_G \parallel h_{t^{-1}} \parallel \parallel f_\alpha \Delta(z) - f(x) \Phi_{\alpha_0}(y, z) \parallel dy \leq \epsilon \parallel h \parallel.$$
We conclude that $S_{\varphi_i} h_i$ converges to $\cdot h_i$ in the weak’ topology. Similarly, one can show that $S_{\varphi_j} h_j \to S(h)$ in the weak’ topology. By Proposition 2.2 in [16] and its proof, $S_{\varphi_i} h_i = S_{\varphi_j} h_j$ for every $\alpha$. Thus

$$
\langle T(h), f \rangle = \lim_{\alpha} (S_{\varphi_i} T(h), f) = \lim_{\alpha} (T(S_{\varphi_i} h_i), f) = \lim_{\alpha} (T(h), f)
$$

for every $f \in L^1(G)$. This proves that $T$ is $\Gamma$-invariant. □

In the following example, we define an operator $T$ which satisfies the equivalent conditions given in Theorem 3.3, but it is not continuous with respect to weak’ topology.

**Example 3.4.** Consider $G = \mathbb{Z}$, the additive group of the integers, and let

$$X = \{ h \in \ell^\infty(\mathbb{Z}); \lim_{|n| \to \infty} h(n) \in \mathbb{R} \}.
$$

Let $\Gamma = \mathbb{Z} \times \{0\}$ and let $T : X \to X$ be given by $T(h) = \lim_{|n| \to \infty} h(n)1$. Then $\mu T(h) = T(\mu h)$ for all $\mu \in \mathbb{Z}$ and $h \in X$. An extension of the Hahn-Banach theorem assures the existence of a continuous linear mapping on all of $\ell^\infty(\mathbb{Z})$ to itself which is $\Gamma$-invariant and coincides with $T$ on $X$. We again denote this extension by $T$. Suppose that $T$ is a weak’-weak’ continuous operator on $\ell^\infty(\mathbb{Z})$. A similar argument to the Theorem 3.3 can be used to show that $T^*(\ell^1(\mathbb{Z})) \subseteq \ell^1(\mathbb{Z})$ and $T^*$ is $\Gamma$-invariant. So $T^*$ restricted to $\ell^1(\mathbb{Z})$ is a multiplier from $\ell^1(\mathbb{Z})$ into $\ell^1(\mathbb{Z})$. Consequently, by Wendel’s theorem [21], there exists $\mu \in M(\mathbb{Z})$ such that $T^*(f) = \mu * f$ for every $f \in \ell^1(\mathbb{Z})$. It is not hard to see that $\mu = 0$, which is a contradiction. We conclude that $T : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is $\Gamma$-invariant and cannot be weak’-weak’ continuous.

The following example shows that the hypothesis of weak’-weak’ continuity in Theorem 3.3 is essential.

**Example 3.5.** Let $G$ be a nondiscrete, compact abelian group. By Proposition 2.3 in [17], there exists $m \in L^\infty(G)^\ast$ such that $\langle m, h \rangle = \langle m, h \rangle$ for every element $(e, t)$ of the closed subgroup $\Gamma = \{e\} \times G$ and $h \in L^\infty(G)$, and also $\langle m, S_{\varphi_0} h_0 \rangle \neq \langle m, h_0 \rangle$ for some $h_0 \in L^\infty(G)$ but $\Phi_0 \in P^\Gamma(\Gamma)$. Define $T : L^\infty(G) \to L^\infty(G)$ by $T(h) = \langle m, h \rangle 1$. It is evident that this operator is $\Gamma$-invariant and $S_{\varphi_0} T(h_0) \neq T(S_{\varphi_0} h_0)$.

**Proposition 3.6.** Let $G$ be a unimodular locally compact group, and let $p \geq 1$ be a real number. Suppose that $T : L^p(G) \to L^\infty(G)$ is a linear map. Then the following properties are equivalent:

(i) $T$ is $\Gamma$-invariant;

(ii) $T(B_{\varphi} f) = S_{\varphi} T(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$, where for $\Phi \in L^1(\Gamma)$, $\overline{\Phi}$ is defined by $\overline{\Phi}(y, z) = \Phi(y^{-1}, z^{-1}) \Delta_\Gamma(y^{-1}, z^{-1})$ (see [16]).

Proof. (i) $\Rightarrow$ (ii). Let $T$ be a linear map from $L^p(G)$ into $L^\infty(G)$ which is $\Gamma$-invariant. Let $T^* : L^\infty(G)^\ast \to L^1(G)$ denote the adjoint of $T$, where $q$ is the Holder conjugate of $p$. Since $T$ is $\Gamma$-invariant, it is easy to see that $T^*(f) = \cdot T(f)$ for every $f \in L^1(G)$ and $(s, t) \in \Gamma$. An application of the Closed Graph Theorem shows that $T^*|_{L^1(G)}$ is a continuous linear map. By Proposition 3.2, $T^*(T_{\varphi} f) = T^*_{\varphi} T^*(f)$ whenever $f \in L^1(G)$ and $\Phi \in L^1(\Gamma)$. An argument similar to the proof of Theorem 3.3 shows that $T(T_{\varphi} f) = S_{\varphi} T(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$.

(ii) $\Rightarrow$ (i). Let $T(B_{\varphi} f) = S_{\varphi} T(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$. Then for each $f \in L^1(G)$, $\Phi \in L^1(\Gamma)$
and \( g \in C_c(G) \), we have
\[
\langle T(T^{(i)}_\phi f), g \rangle = \langle T^{(i)}_\phi f, T(g) \rangle = \int \int f(y^{-1}xz)\phi(y, z)T(g)(x)d(y, z)dx
\]
\[
= \int \int f(x)\phi(y, z)T(g)(yxz^{-1})d(y, z)dx = \langle S_g T(g), f \rangle
\]
\[
= \langle T(T^{(p)}_{\phi g}), f \rangle = \langle T^{(p)}_{\phi g}, T^*(f) \rangle
\]
\[
= \int \int T^*(f)(yxz^{-1})\bar{\phi}(y, z)g(x)d(y, z)dx
\]
\[
= \int \int T^*(f)(y^{-1}xz)\Phi(y, z)g(x)d(y, z)dx
\]
\[
= \langle T^{(p)}_\Phi T^*(f), g \rangle
\]

Hence \( T^*(T^{(i)}_\phi f) = T^{(p)}_\phi T^*(f) \) for every \( f \in L^1(G) \) and \( \Phi \in L^1(\Gamma) \). By Proposition 3.2, \( T^*|_{L^1(G)} \) is \( \Gamma \)-invariant. Clearly \( T \) is \( \Gamma \)-invariant and this completes our proof. \( \square \)

It is a standard device to embed \( L^\infty(G) \) into \( B(L^1(G), L^\infty(G)) \) by transformation \( T \), so that \( T(h)(f) = f \ast h \). So \( T \) allows us to consider the strong operator topology on \( L^\infty(G) \) that we shall denote by \( \tau_s \). It is known that the norm topology on \( L^\infty(G) \) is stronger than the \( \tau_s \)-topology, see Proposition 4 in [3].

**Proposition 3.7.** Let \( G \) be a compact group, and let \( p \geq 1 \) be a real number. Suppose that \( T : L^\infty(G) \to L^p(G) \) is a \( \tau_s \)-continuous linear map. Then the following properties are equivalent:

(i) \( T \) is \( \Gamma \)-invariant;

(ii) \( T(S_{gh}) = T^{(p)}_{\phi g} T(h) \) for every \( h \in L^\infty(G) \) and \( \Phi \in L^1(\Gamma) \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( q \) be the Holder conjugate of \( p \). We first show that for each \( f \in L^q(G) \), \( T^*(f) \in L^1(G) \). Indeed, if \( \{h_n\} \) is a net in \( L^\infty(G) \) and \( h_n \to h \) in the \( \tau_s \)-topology, then
\[
\langle T^*(f), h_n \rangle = \langle f, T(h_n) \rangle \to \langle f, T(h) \rangle = \langle T^*(f), h \rangle,
\]

since \( T \) is \( \tau_s \)-continuous. Since \( L^1(G) \) is the dual of \( (L^\infty(G), \tau_s) \) (see Corollary 2 in [3]), so \( T^*(f) \in L^1(G) \). Now, suppose that \( T \) is \( \Gamma \)-invariant. It is easy to see that \( T^* \) is continuous. Moreover, \( T^*(s_t f) = s_t T^*(f) \), for every \( (s, t) \in \Gamma \) and \( f \in L^q(G) \), since as usual we have for every \( f \in L^q(G) \), \( (s, t) \in \Gamma \) and \( h \in L^\infty(G) \) that
\[
\langle T^*(f), h \rangle = \langle s_t f, T(h) \rangle = \langle f, s_t^{-1} T(h) \rangle = \langle f, T(s_t^{-1} h) \rangle = \langle T^*(f), s_t^{-1} h \rangle.
\]

By Proposition 3.2, \( T^*(T^{(p)}_{\phi g} f) = T^{(i)}_{\phi g} T^*(f) \) for every \( f \in L^q(G) \) and \( \Phi \in L^1(\Gamma) \). It is not hard to see that \( T(S_{gh}) = T^{(p)}_{\phi g} T(h) \) for every \( h \in L^\infty(G) \) and \( \Phi \in L^1(\Gamma) \).

(ii) \( \Rightarrow \) (i). Since \( T(S_{gh}) = T^{(p)}_{\phi g} T(h) \) for every \( h \in L^\infty(G) \) and \( \Phi \in L^1(\Gamma) \), we have \( T^*(T^{(p)}_{\phi g} f) = T^{(i)}_{\phi g} T^*(f) \) for every \( f \in L^q(G) \) and \( \Phi \in L^1(\Gamma) \). By Proposition 3.2, \( T^* \) is \( \Gamma \)-invariant and so \( T \) is \( \Gamma \)-invariant. \( \square \)
4. Amenability and Translation Operators

For \( T \in \mathcal{M}_f(L^\infty(G)) \), we are able to speak of the translate \( _T t \), which is that continuous linear operator which associates the element \( _T t(h) = T(h_t) \in L^\infty(G) \) to each \( h \in L^\infty(G) \). Recall that the weak operator topology on \( \mathcal{B}(L^\infty(G)) \) is the locally convex topology defined by the family of seminorms

\[ \varphi = |p_{h,\varphi}; p_{h,\varphi}(T) = |(T(h), \varphi)|, h \in L^\infty(G) \text{ and } \varphi \in L^1(G). \]

\( T \) is said to be \textit{weakly almost periodic} if the set \( \{ _T t; (s, t) \in \Gamma \} \) of translates of \( T \) is relatively compact with respect to weak operator topology on the set \( \mathcal{B}(L^\infty(G)) \) of bounded linear operators from \( L^\infty(G) \) to \( L^\infty(G) \), [8].

\textbf{Theorem 4.1.} Let \( G \) be a locally compact group, and let \( \Gamma = G \times \{ \varepsilon \} \). Then the following properties are equivalent:

(i) \( G \) is amenable;

(ii) There is a non-zero weakly almost periodic linear operator \( T \) in \( \mathcal{M}_f(L^\infty(G)) \).

\textit{Proof.} (i) \( \Rightarrow \) (ii). Amenability of \( G \) is equivalent to the \( \Gamma \)-amenability; hence, for an invariant mean \( m \), the mapping \( T : h \mapsto \langle m, h \rangle I \), is a rank one operator and hence weakly compact. The set \( \{ _T t; s \in G \} \) is just a singleton \( \{ T \} \) and so is compact in any topologies. Clearly \( T \) is invariant.

(ii) \( \Rightarrow \) (i). Let \( T \in \mathcal{M}_f(L^\infty(G)) \) be a non-zero weakly almost periodic operator of \( L^\infty(G) \) to itself. It is known that \( \{ T \} \in \mathcal{M}_f(L^\infty(G)), [10] \). For \( h \in L^\infty(G) \), the map \( T \mapsto T(h) \) from \( \mathcal{M}_f(L^\infty(G)) \) into \( L^\infty(G) \) is continuous when \( L^\infty(G) \) is equipped with the weak topology. Thus \( \{ T(h); s \in G \} \) is relatively weakly compact. Hence \( \{ T(h); s \in G \} \) is relatively weakly compact, see Theorem 5.35 in [1]. Since this holds for all \( h \in L^\infty(G) \), we conclude that \( T \) is a weakly almost periodic operator on \( L^\infty(G) \) (see Exercise VI 9.2 in [5]). Since \( T \neq 0 \), \( |T| \neq 0 \). If \( h \geq 0 \), then \( |T(h)| \leq ||h|||T|(1) \), and it follows that \( |T|(1) > 0 \). We conclude that, \( \frac{1}{|T|} \) is a weakly almost periodic operator on \( L^\infty(G) \). Without loss of generality we may assume that \( T \) is a positive operator and \( T(1) = 1 \). Let \( WAP(L^\infty(G)) \) denote the space of weakly almost periodic functions on \( G \) i.e. the set of all \( f \in L^\infty(G) \) such that \( |f|/f; y \in \Gamma \) is relatively compact in the weak topology of \( L^\infty(G) \). Recall that an application of the Ryll-Nardzewski fixed point theorem, see Theorem 6.20 in [2], shows that \( WAP(L^\infty(G)) \) has a unique invariant mean \( m \). If \( f \in L^\infty(G) \), then \( \{ _T t; s \in G \} = \{ T(f); s \in G \} \) is relatively weakly compact. Hence \( T(f) \in WAP(L^\infty(G)) \). It follows that \( m \circ T \) is a invariant mean on \( L^\infty(G) \), and so \( G \) is \( \Gamma \)-amenable.  

For a locally compact group \( G \), \( L^1(G)^* \) will always denote the second conjugate algebra of \( L^1(G) \) equipped with the first Arens multiplication. Let also \( L^1_0(G) \) be the subspace of \( L^\infty(G) \) consisting of all functions \( f \in L^\infty(G) \) that vanish at infinity. It is known that \( L^1_0(G) \) is a closed ideal of \( L^\infty(G) \) invariant under conjugation and translation, containing \( C_0(G) \) as a closed subspace, see Proposition 2.7 in [15]. Furthermore \( L^1_0(G)^* \) is a closed subalgebra of \( L^\infty(G)^* \) with respect to first Arens product, see Corollary 2.10 in [15]. Information about the first Arens product can be found in [4].

It is known that if \( G \) is a noncompact locally compact group, then \( L^\infty(G)^* \) cannot have any non-zero weakly compact left multipliers \( T \) with \( \langle T(n), 1 \rangle \neq 0 \), for some \( n \in L^\infty(G)^* \), see Theorem 4.1 in [10]. On the other hand \( G \) is amenable if and only if there is a non-zero weakly compact right multiplier on \( L^\infty(G)^* \), see Theorem 2.1 in [11].

\textbf{Theorem 4.2.} (i) A locally compact group \( G \) is compact if and only if there is a non-zero weakly compact linear operator \( T \) from \( L^\infty_0(G) \) to \( L^\infty(G) \) such that \( T(f \cdot h) = f \cdot T(h) \) for every \( f \in L^1(G) \) and \( h \in L^\infty_0(G) \).

(ii) A locally compact group \( G \) is amenable if and only if there is a non-zero weakly compact linear operator \( T \) from \( L^\infty(G) \) to itself such that \( T(f \cdot h) = f \cdot T(h) \) for every \( f \in L^1(G) \) and \( h \in L^\infty(G) \).

\textit{Proof.} (i) If \( G \) is compact, Proposition 4.6 in [17] implies that existence of a mean \( m \) on \( L^\infty(G) = L^\infty_0(G) \) such that \( \langle m, f \cdot h \rangle = \langle m, h \rangle \) for all \( h \in L^\infty(G) \) and \( f \in P^1(G) \). Define \( T : L^\infty_0(G) \to L^\infty_0(G) \) by \( T(h) = \langle m, h \rangle 1 \). It is
routine to verify that \( T \) is a weakly compact linear operator. Further \( T(f \cdot h) = f \cdot T(h) \) for all \( h \in L^\infty(G) \) and \( f \in L^1(G) \).

To prove the converse, let \( T \) be a non-zero weakly compact operator on \( L^\infty(G) \) which commutes with left convolution operators. Let \( \pi : L^\infty(G)^* \to LUC(G)^* \) be the canonical projection. Recall that \( LUC(G)^* \) is a Banach algebra by an Arens-type product and that \( L^1(G) \subseteq LUC(G)^* \). Information about the Arens product and about \( LUC(G)^* \) can be found in [15]. For every \( n \in L^\infty(G)^* \) and \( g \in L^\infty(G) \), we denote by \( ng \) the function in \( L^\infty(G) \) defined by \( (ng)(x) = (n, \hat{\phi} \cdot g) \) for all \( \phi \in L^1(G) \), where \( \hat{\phi}(x) = \hat{\phi}(x^{-1}) \Delta(x^{-1}) \) for all \( x \in G \). The space \( L^\infty(G) \) is left introverted in \( L^\infty(G) \); that is, for each \( n \in L^\infty(G)^* \) and \( g \in L^\infty(G) \), \( ng \in L^\infty(G) \). This lets us endow \( L^\infty(G)^* \) with the first Arens product defined by \( (mn, ng) = (m, ng) \) for all \( m, n \in L^\infty(G)^* \) and \( g \in L^\infty(G) \). Then \( L^\infty(G)^* \) with this product is a Banach algebra. This Banach algebra was introduced and studied by Lau and Pym, [15]. By Theorem 2.8 in [15], \( n(L^\infty(G)^*) = M(G) \). This shows that \( L^1(G)L^\infty(G)^* = L^1(G)n(L^\infty(G)^*) \subseteq L^1(G) \). We show that for each \( f \in L^1(G) \), \( T^*(f) \in L^1(G) \). Let \( \{F_\alpha\} \) be a net in \( L^\infty(G)^* \) and \( F_\alpha \to F \) in the weak * topology of \( L^\infty(G)^* \). If \( f \in L^1(G) \), then \( f = f_1 \ast f_2 \) for some \( f_1 \) and \( f_2 \) in \( L^1(G) \), by Cohen’s factorization theorem. It is known that \( \langle f_2, f_1 \ast h \rangle = \langle f_2, h \rangle f_1 \) for all \( h \in L^\infty(G) \), see [15]. Consequently if \( h \in L^\infty(G) \),

\[
\langle T^*(f)F_\alpha, h \rangle = \langle T^*(f), F_\alpha(h) \rangle = \langle f_1 \ast f_2, T(F_\alpha(h)) \rangle = \langle f_2, T(F_\alpha(h)) \rangle = \langle f_2, T(F_\alpha(h)) \rangle = \langle f_2, F_\alpha(h) \rangle = \langle f_2, F_\alpha(h) \rangle = \langle f_2, T(F_\alpha(h)) \rangle = \langle f_2, F_\alpha(h) \rangle = \langle f_2, F_\alpha(h) \rangle = \langle f_2, F_\alpha(h) \rangle.
\]

Hence \( T^*(f)F_\alpha \to T^*(f)F \), showing that \( T^*(f) \) is in the topological center of \( L^\infty(G)^* \). By Theorem 2.11 in [15], \( T^*(f) \in L^1(G) \). Clearly \( T^* \) is a left multiplier on \( L^\infty(G) \). On the other hand, \( T \) is weakly compact. It follows that \( T^*: L^\infty(G)^* \to L^\infty(G)^* \) is weakly compact, see Theorem 17.2 in [1]. So \( T^* \) restricted to \( L^1(G) \) is weakly compact. Since for a noncompact group \( G \), there are no weakly compact multipliers from \( L^1(G) \) to \( L^1(G) \), we conclude that \( G \) is compact (see Theorem in [9] and Theorem 1 in [20]).

(ii) Since \( \text{Conv}(L^\infty(G), L^\infty(G)) \subseteq \text{Hom}(L^\infty(G), L^\infty(G)) \), by [15]. An argument similar to the one in the proof of Theorem 4.1, shows that \( G \) is amenable if and only if there is a non-zero weakly compact linear operator \( T \) from \( L^\infty(G) \) to itself such that \( T(f \cdot h) = f \cdot T(h) \) for every \( f \in L^1(G) \) and \( h \in L^\infty(G) \). \( \square \)

**Theorem 4.3.** Let \( G \) be a locally compact group, and let \( \Gamma = G \times [e] \). Then the following properties are equivalent:

(i) \( G \) is amenable;

(ii) There exist a continuous linear mapping \( P \) of \( \mathcal{B}(L^2(G)) \) onto \( \mathcal{M}_T(L^2(G)) \) such that the following hold:

1. \( \|P\| = 1 \), \( P(1) = 1 \).
2. \( P(L_sT\lambda_s \cdot v) = L_sP(T)\lambda_s \cdot v = P(T) \) for every \( T \in \mathcal{B}(L^2(G)) \) and \( s \in G \), here \( L_s \) is the left translation operator in \( \mathcal{B}(L^2(G)) \) defined by \( L_s(\phi) = s \cdot \phi \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( G \) be amenable, or equivalently \( \Gamma \)-amenable (see [16]). By Theorem 4.19 in [17], there exists a mean \( \mu \) on \( L^\infty(G) \) such that \( \langle m, h \rangle = \langle \mu, h \rangle \) for every \( h \in L^\infty(G) \) and \( m \in G \). Now if \( \phi, \psi \in L^2(G) \) and \( h^T_{\phi, \psi} : G \to \mathbb{C} \) is given by the formula \( h^T_{\phi, \psi}(x) = (L_x \cdot TL_x \phi)(\psi) \), then \( \|h^T_{\phi, \psi}\| \leq \|T\| \|\phi\| \|\psi\| \). This shows that \( h^T_{\phi, \psi} \in L^\infty(G) \). Let \( \phi \in L^2(G) \). Obviously the linear map \( \psi \mapsto \langle m, h^T_{\phi, \psi} \rangle \) from \( L^2(G) \) into \( \mathbb{C} \) is continuous. Thus by the Riesz Representation Theorem, there exists a unique \( P(T)\phi \in L^2(G) \) such that \( P(T)\phi \psi = \langle m, h^T_{\phi, \psi} \rangle \). For all \( \phi, \psi \in L^2(G) \), \( s \in G \) and every \( x \in G \),

\[
\begin{align*}
    h^T_{L_s\phi, L_s\psi}(x) &= (L_x \cdot TL_x \lambda_s \phi)(\psi) = (L_{x^{-1}}L_x \cdot TL_x \lambda_s \phi)(\psi) \\
    &= (L_{x^{-1}}L_x \cdot TL_x \phi)(\psi) = h^T_{\phi, \psi}(x).
\end{align*}
\]
Thus \( \langle m, h^T_{0, \phi, \psi} \rangle = \langle m, h^T_{0, \phi, \psi} \rangle \), that is,

\[
(L_{-1} P(T)L_m \phi | \psi) = (P(T)L_m \phi | L_m \psi) = (P(T)\phi | \psi).
\]

Since this holds for all \( \phi, \psi \in L^2(G) \), we conclude that \( L_{-1} P(T)L_m = P(T) \), that is, \( P(T) \in \mathcal{M}_r(L^2(G)) \). The mapping \( T \mapsto P(T) \) from \( \mathcal{B}(L^2(G)) \) onto \( \mathcal{M}_r(L^2(G)) \) is clearly linear and \( \|P\| = 1 \). It is not hard to see \( P(T)L_{-1} = L_m P(T)L_{-1} = P(T) \) for every \( T \in \mathcal{B}(L^2(G)) \) and \( s \in G \).

\[ (ii) \Rightarrow (i). \] Let us assume that there exists a linear mapping \( P \) of \( \mathcal{B}(L^2(G)) \) onto \( \mathcal{M}_r(L^2(G)) \) satisfying the conditions of Theorem. For \( h \in L^\infty(G) \) define \( m_h \in \mathcal{B}(L^2(G)) \) by \( m_h(\phi) = h\phi \). Consider a fixed positive \( \phi_0 \in L^2(G) \) with \( \|\phi_0\|_2 = 1 \). If \( h \in L^\infty(G) \), let \( (m, h) = (P(m_h)\phi_0 | \phi_0) \). Clearly \( m \) is a mean on \( L^\infty(G) \). For all \( h \in L^\infty(G), s \in G, \) and \( \phi \in L^2(G), \) we have \( \phi_0 = L_m h \phi \). It follows that \( m_h = L_m m_h L_{-1} \). By assumption, \( P(m_h) = P(L_m m_h L_{-1}) = P(m_h) \) and so \( \langle m, h \rangle = \langle P(m_h)\phi_0 | \phi_0 \rangle = \langle P(m_h)\phi_0 | \phi_0 \rangle = \langle m, h \rangle \). Therefore \( m \) is a left invariant mean on \( L^\infty(G) \), and so \( G \) is \( \Gamma \)-amenable.

**Corollary 4.4.** Let \( G \) be an amenable locally compact group, and let \( \Gamma = G \times \{e\} \). Then \( \mathcal{M}_r(L^2(G)) \) is invariantly complemented in \( \mathcal{B}(L^2(G)) \), that is, \( \mathcal{M}_r(L^2(G)) \) is the range of a continuous projection on \( \mathcal{B}(L^2(G)) \) commuting with translations.

**Proof.** The statement follows from Theorem 4.3 and its proof.

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**References**