Holomorphically Projective Mappings of (Pseudo-) Kähler Manifolds
Preserve the Class of Differentiability

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Abstract. In this paper we study fundamental equations of holomorphically projective mappings of (pseudo-) Kähler manifolds with respect to the smoothness class of metrics $C^r$, $r \geq 1$. We show that holomorphically projective mappings preserve the smoothness class of metrics.

1. Introduction

First we study the general dependence of holomorphically projective mappings of classical and pseudo-Kähler manifolds (shortly Kähler) on the smoothness class of the metric. We present well known facts, which were proved by Otsuki, Tashiro [31], Tashiro, Ishihara [44], Domashev, Mikeš [8], Mikeš [19, 20], A.V. Aminova, D. Kalinin [2–5], etc., see [6, 9, 25, 28, 35, 36, 45]. To the theory of holomorphically projective mappings and their generalization are devoted many publications, eg. [1, 7, 10, 14–18, 21–23, 26, 29, 30, 32, 33, 38–41]. In these results no details about the smoothness class of the metric were stressed. They were formulated “for sufficiently smooth” geometric objects.

The following results are connected to the paper [11] where it was proved that holomorphically projective mappings preserve the smoothness class $C^r$ of the metrics in the case $r \geq 2$. In the following paper we generalize this result to the case $r \geq 1$.

2. Main Results

Let $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ be (pseudo-) Kähler manifolds, where $M$ and $\bar{M}$ are $n$-dimensional manifolds with dimension $n \geq 4$, $g$ and $\bar{g}$ are metrics, $F$ and $\bar{F}$ are structures. All the manifolds are assumed to be connected.

Definition 2.1. A diffeomorphism $f$: $K_n \rightarrow \bar{K}_n$ is called a holomorphically projective mapping of $K_n$ onto $\bar{K}_n$ if $f$ maps any holomorphically planar curve in $K_n$ onto a holomorphically planar curve in $\bar{K}_n$.

We obtain the following theorem.
Theorem 2.2. If the (pseudo-) Kähler manifold $K_n (K_n \in C', r \geq 1)$ admits a holomorphically projective mapping onto $K_n \in C'$, then $K_n$ belongs to $C'$.

Briefly, this means that:

holomorphically projective mappings preserve the class of smoothness of the metric.

The analogous property for geodesic mappings of (pseudo-) Riemannian manifolds is proved in [12].

Here and later $K_n = (M, g, \bar{F}) \in C'$ denotes that $g \in C'$, i.e. in a coordinate neighborhood $(U, x)$ for the components of the metric $g$ holds $g_{ij}(x) \in C'$. If $K_n \in C'$, then $M \in C'^{r+1}$. This means that the atlas on the manifold $M$ has the differentiability class $C'^{r+1}$, i.e. for non disjoint charts $(U, x)$ and $(U', x')$ on $U \cap U'$ it is true that the transformation $x' = x'(x) \in C'^{r+1}$.

The differentiability class $r$ is equal to $0, 1, 2, \ldots, \infty, \omega$, where $0, \infty$ and $\omega$ denotes continuous, infinitely differentiable, and real analytic functions respectively.

Remark 2.3. It’s easy to prove that the Theorem 2.2 is valid also for $r = \infty$ and for $r = \omega$. This follows from the theory of solvability of differential equations. Of course we can apply this theorem only locally, because differentiability is a local property.

Remark 2.4. A minimal requirement for holomorphically projective mappings is $K_n, \bar{K}_n \in C^1$.

Mikeš, see [19, 21, 22, 24, 25], [28, p. 82] found equidistant Kähler metrics $g$ in canonical coordinates $x$:

$$g_{ab} = g_{a+m b+m} = \partial_{ab} f + \partial_{a+m b+m} f \quad \text{and} \quad g_{ab+m} = \partial_{a+m b+m} f - \partial_{a+m b} f,$$

where $a = 1, 2, \ldots, m, m = n/2$, $f = \exp(2x^1) \cdot G(x^2, x^3, \ldots, x^n, x^{2+m}, x^{3+m}, \ldots, x^{2m})$, $G \in C^3$, which admit holomorphically projective mappings. Evidently, if $G \in C'^{r+1}$ ($r \in N$), $G \in C^\omega$ and $C^\omega$, then $K_n \in C'$, $K_n \in C^\omega$ and $K_n \in C^\omega$, respectively. From these metrics we can easily see examples of non trivial holomorphically projective mappings $K_n \rightarrow K_n$, where $K_n, \bar{K}_n \in C'$ and $\not\in C'^{r+1}$ for $r \in N$; $K_n, \bar{K}_n \in C^\omega$ and $\not\in C^\omega$; $K_n, \bar{K}_n \in C^\omega$.

3. (Pseudo-) Kähler Manifolds

In the following definition we introduce generalizations of Kähler manifolds.

Definition 3.1. An $n$-dimensional (pseudo-) Riemannian manifold ($n \geq 4$) is called a (pseudo-) Kähler manifold $K_n = (M, g, \bar{F})$, if beside the metric tensor $g$, a tensor field $F$ of type $(1, 1)$ is given on the manifold $M$, called a structure $F$, such that the following conditions hold:

$$F^2 = - \text{Id}; \quad g(X, FX) = 0; \quad \nabla F = 0,$$

where $X$ is an arbitrary vector of $TM$, and $\nabla$ denotes the covariant derivative in $K_n$.

These spaces were first considered as $A$-spaces by P.A. Shirokov, see [34]. Independently such spaces with positive definite metric were studied by E. Kähler [13]. The tensor field $F$ is called a complex structure [45].

The following lemma specifies the properties of the differentiability of geometrical objects on (pseudo-) Kähler manifolds.

Lemma 3.2. If $K_n = (M, g, F) \in C'$, i.e. $g \in C'$, then $F \in C'$, for $r \in N$ and $r = \infty, \omega$.

Proof. Let $K_n \in C'$, i.e. the components of the metric $g_{ij}(x) \in C'$ in a coordinate chart $x$. It is a priori valid that $F^g \in C^1$. The formula $\nabla F = 0$ can be written $\partial_k F^g = \Gamma^g_{i j k} F^g_I$, where $\Gamma^g_{i j k} = \frac{1}{2} (\partial_i g_{j k} + \partial_j g_{k i} - \partial_k g_{i j})$, $\partial_k = \partial_k / \partial x^k$, and $\Gamma^g_{i j} = \Gamma^g_{i j k}$ are Christoffel symbols of the first and second kind, respectively. It holds that $\Gamma_{i j k}$ and $\Gamma^g_{i j} \in C'^{r+1}$. From this equation follows immediately $F^g_I(x) \in C'$, i.e. $F \in C'$. 


Moreover, due to the differentiability of \( g \in C^r \) according to (1), each point has a coordinate neighborhood \((U, \chi) \in C^{r+1}\) in which the structure \( F \) has the following canonical form:

\[
F^{a+m}_{b+m} = -F^a_{b+m} = \delta^a_b, \quad F^a_b = F^{a+m}_{b+m} = 0, \quad a, b = 1, \cdots, m; \quad m = \frac{n}{2}.
\]

(2)

We get, as an immediate consequence, that the dimension is even, \( n = 2m \). Such a coordinate system will be called canonical.

Due to the conditions (1) and (2), the components of the metric tensor and Christoffel symbols of the second kind in a canonical coordinate system satisfy

\[
g_{a+m,b+m} = g_{ab}, \quad g_{ab+m} = -g_{a+mb}, \quad \Gamma^a_{b+c} = \Gamma^{a+m}_{b+c+m} = -\Gamma^a_{b+c+m'}, \quad \Gamma^{a+m}_{b+c+m} = \Gamma^a_{b+c} = -\Gamma^{a+m}_{b+c}.
\]

(3)

Obviously, the coordinate transformation \( x^h = x^{h'}(x) \) preserves a canonical coordinate system if and only if the Jacobi matrix \( J = (\partial x^h/\partial x^j) \) satisfies

\[
\frac{\partial x^a}{\partial x^b} = \frac{\partial x^a}{\partial x^b} \quad \text{and} \quad \frac{\partial x^{a+m}}{\partial x^b} = -\frac{\partial x^a}{\partial x^{b+m}}.
\]

(4)

Let us set \( z^i = x^i + ix^{a+m} \), \( z^a = x^a + ix^{a+m} \) (where \( i \) is the imaginary unit). Then (4) can be interpreted as Cauchy-Riemann conditions for the complex functions \( z^a = z^a(z^1, \cdots, z^m) \), and we will call this transformation analytic.

4. Holomorphically Projective Mappings \( K_n \to K_n \) of Class \( C^1 \)

Assume the (pseudo-) Kähler manifolds \( K_n = (M, g, F) \) and \( K_n = (\tilde{M}, \tilde{g}, \tilde{F}) \) with metrics \( g \) and \( \tilde{g} \), structures \( F \) and \( \tilde{F} \), Levi-Civita connections \( V \) and \( \tilde{V} \), respectively. Here \( K_n, \tilde{K}_n \in C^1 \), i.e. \( g, \tilde{g} \in C^1 \) which means that their components \( g_{ij}, \tilde{g}_{ij} \in C^1 \).

Likewise, as in [31], see [6], [35, p. 205], [36], [25], [28, p. 240], we introduce the following notations.

**Definition 4.1.** A curve \( \ell \) in \( K_n \) which is given by the equation \( \ell = \ell(t), \lambda = dt/dt (\neq 0), t \in I \), where \( t \) is a parameter is called holomorphically planar, if under the parallel translation along the curve, the tangent vector \( \lambda \) belongs to the two-dimensional distribution \( D = \text{Span} \{\lambda, F\} \) generated by \( \lambda \) and its conjugate \( F\lambda \), that is, it satisfies

\[
\nabla_t \lambda = a(t)\lambda + b(t)F\lambda,
\]

where \( a(t) \) and \( b(t) \) are some functions of the parameter \( t \).

Particularly, in the case \( b(t) = 0 \), a holomorphically planar curve is a geodesic.

We recall the Definition 2.1: A diffeomorphism \( f: K_n \to \tilde{K}_n \) is called a holomorphically projective mapping of \( K_n \) onto \( \tilde{K}_n \) if \( f \) maps any holomorphically planar curve in \( K_n \) onto a holomorphically planar curve in \( \tilde{K}_n \).

Assume a holomorphically projective mapping \( f: K_n \to \tilde{K}_n \). Since \( f \) is a diffeomorphism, we can suppose local coordinate charts on \( M \) or \( \tilde{M} \), respectively, such that locally \( f: K_n \to \tilde{K}_n \) maps points onto points with the same coordinates, and \( M = \tilde{M} \).

A manifold \( K_n \) admits a holomorphically projective mapping onto \( K_n \) if and only if the following equations [28, 36]:

\[
\nabla_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X - \psi(F)FY - \psi(FY)FX
\]

(5)

hold for any tangent fields \( X, Y \) and where \( \psi \) is a differential form. In local form:

\[
\Gamma^a_{ij} = \Gamma^{a}_{ij} + \psi_1 \delta^a_{ij} + \psi_2 \delta^a_{ij} - \psi_1 \delta^a_{ij} - \psi_1 \delta^a_{ij},
\]

where \( \Gamma^a_{ij} \) and \( \Gamma^{a}_{ij} \) are the Christoffel symbols of \( K_n \) and \( \tilde{K}_n \), \( \psi_1, \psi_2, F_1^i \) are components of \( \psi, F \) and \( \delta^a_{ij} \) is the Kronecker delta, \( \psi_1 = \psi(x^a \partial_x^i, \delta^a_{ij} = F^a_i \). Here and in the following we will use the conjugation operation of indices in the way

\[
A_{i...j} = A_{i...j}, F^{k}_{i...} = A_{i...k} \Gamma^{k}_{i},
\]

\[
A_{i...j} = A_{i...j}, F^{k}_{i...} = A_{i...k} \Gamma^{k}_{i},
\]
If $ψ ≡ 0$, then $f$ is affine or trivially holomorphically projective. Beside these facts it was proved [28, 36] that $F = ± F$; for this reason we can suppose that $F = F$.

It is known that

$$ψ = \nabla_i \Psi, \quad \Psi = \frac{1}{2(n + 2)} \ln \left| \det g \right|.$$ 

Equations (5) are equivalent to the following equations

$$V_2 g(\mathbf{X}, \mathbf{Y}) = 2ψ \mathbf{g}(\mathbf{X}, \mathbf{Y}) + ψ(\mathbf{X}) \mathbf{g}(\mathbf{Y}, \mathbf{Z}) + ψ(\mathbf{Y}) \mathbf{g}(\mathbf{X}, \mathbf{Z}) + ψ(F \mathbf{X}) \mathbf{g}(\mathbf{F} \mathbf{Y}, \mathbf{Z}) + ψ(F \mathbf{Y}) \mathbf{g}(\mathbf{F} \mathbf{X}, \mathbf{Z}).$$

In local form:

$$V_k g_{ij} = 2ψ_k g_{ij} + ψ_i g_{jk} + ψ_j g_{ik} + ψ_j g_{ik},$$

where $g_{ij}$ are components of the metric $g$ on $\bar{K}$.

The above formulas are well known for $\bar{K}$, locally.

The above formulas are the criterion for holomorphically projective mappings $\bar{K}_n \to \bar{K}_n$, globally as well as locally.

5. Holomorphically Projective Mapping for $K_n \in C^2 \to \bar{K}_n \in C^1$

I. Hinterleitner [11] proved the theorem:

**Theorem 5.1.** If a (pseudo-) Kähler manifold $K_n \in C^r$, $r \geq 2$, admits a holomorphically projective mapping onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^2$.

It is easy to see that Theorem 2.2 follows from Theorem 5.1 and the following theorem.

**Theorem 5.2.** If $K_n \in C^2$ admits a holomorphically projective mapping onto $\bar{K}_n \in C^1$, then $\bar{K}_n \in C^2$.

**Proof.** We will suppose that the (pseudo-) Kähler manifold $K_n = (\bar{M}, g, F) \in C^2$ admits a holomorphically projective mapping $f$ onto the (pseudo-) Kähler manifold $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F}) \in C^1$. Furthermore, we can assume that $\bar{M} = M$ and $\bar{F} = F$. The corresponding points $x \in M$ and $\bar{x} = f(x) \in M$ have common coordinates $(x^1, x^2, \ldots, x^n)$, shortly $x$, in the coordinate chart $(U, x)$, $U \subset M$.

We study the coordinate neighborhood $(U, x)$ of any point $p$ at $M$. Moreover, we suppose that the coordinate system $x$ is canonical (2). On $(U, x)$ formulae (5)–(9) hold, and formula (7) may be written in the following form

$$\partial_k a_{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k + \bar{\lambda}^i F^j_k + \bar{\lambda}^j F^i_k - f^i_j,$$

(10)
where $d^i = a_{ik}g^{kj}j, l^i = \lambda_\mu g^{\mu i}, \lambda^i = \lambda^\mu_{\nu}g^{\mu i}$, and $f^{ij}_k = a \Gamma^i_{jk} + a \Gamma^i_{kj}$.

The components $g_{ij}(x) \in C^2$ and $g^{ij}(x) \in C^1$ on $U \subset M$ and from that facts follows that the functions $g^{ij}(x) \in C^2$, $g^{ij}(x) \in C^1$, $\Psi(x) \in C^1$, $\psi_i(x) \in C^0$, $d^i(x) \in C^1$, $\lambda^i(x) \in C^0$, and $\Gamma^i_{jk}(x) \in C^1$. It is easy to see, that $f^{ij}_k \in C^1$.

In the canonical coordinate system $x$ we can calculate the following derivatives for fixed different indices $a, b = 1, \ldots, m, m = n/2$:

\[
\begin{align*}
\partial x a^{ab} &= \lambda^a - f^a_b, \\
\partial x b^{ab} &= -\lambda^{a+m} - f^{ab}_b, \\
\partial x b^{ab+m} &= \lambda^{a+m} - f^{ab}_b, \\
\partial x b^{ab+} &= -\lambda^{a+m} - f^{ab+}_b.
\end{align*}
\]

Eliminating $\lambda^a$ and $\lambda^{a+m}$ we obtain the equations

\[
\begin{align*}
\partial x a^{ab} - \partial x b^{ab+m} &= -f^a_b + f^{ab+}_b, \\
\partial x b^{ab+} + \partial x b^{ab+} &= -f^{ab+}_b - f^{ab+}_b.
\end{align*}
\]

We denote $w = \alpha_i^b + i \cdot \alpha_i^{a+b+m}$, $z = x^i + i \cdot x^{a+b+m}$, where $i$ is the imaginary unit. Then (12) can be rewritten

\[
\partial x w = F = (-f^a_b + f^{ab+}_b) + i \cdot (-f^{ab+}_b - f^{ab+}_b),
\]

and because $F \in C^1$, then exists $\partial^2_x w$.

So there are the second partial derivatives of the functions $\alpha_i^b$ and $\alpha_i^{a+b+m}$ of the variables $x^i$ and $x^{a+b+m}$; and, clearly, also of $x^i$ and $x^{a+b+m}$. After this from formula (11) follows that $\lambda^a \in C^1$; and equations (10) imply that $a^i, \alpha_i \in C^2$. Finally, formula (9) shows that $g_{ij} \in C^3$. \(\square\)

6. Holomorphically Projective Mapping $K_n \to K_n$ of Class $C^2$  

Let $K_n$ and $K_\nu \subset C^2$ be (pseudo-) Kähler manifolds, then for holomorphically projective mappings $K_n \to K_n$ the Riemann and the Ricci tensors transform in the following way:

\[
\begin{align*}
(a) & \quad R^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_i \psi_{jk} + \delta^h_j \psi_{ki} - 2 \delta^h_k \psi_{ji}, \\
(b) & \quad R_{ij} = R_{ij} - (n + 2) \psi_{ij},
\end{align*}
\]

where $\psi_{ij} = \psi_{ij} - \psi_{ji}$). Here the Ricci tensor is defined by $R_{ik} = \alpha_i^k$. In many papers it is defined with the opposite sign $[19, 25, 35, 46]$, etc.

The tensor of the holomorphically projective curvature, which is defined in the following form

\[
P^h_{ijk} = R^h_{ijk} + \frac{1}{n + 2} \left( \delta^h_k R_{ij} - \delta^h_i R_{jk} + \delta^h_j R_{ki} - 2 \delta^h_k R_{ji} \right),
\]

is invariant with respect to holomorphically projective mappings, i.e. $P^h_{ijk} = P^h_{ijk}$.

The above mentioned formulae can be found in the papers $[6, 28, 35]$.

The integrability conditions of equations (7) have the following form

\[
a_{i\mu} R^a_{\mu i} + a_{\mu j} R^{a}_{\mu i} = g_{\mu k} \nabla_i \lambda_j + g_{\mu k} \nabla_i \lambda_j - g_{\mu k} \nabla_i \lambda_j + g_{\mu k} \nabla_i \lambda_j + g_{\mu k} \nabla_i \lambda_j - g_{\mu k} \nabla_i \lambda_j - g_{\mu k} \nabla_i \lambda_j - g_{\mu k} \nabla_i \lambda_j.
\]

After contraction with $g^{\mu}$ we get:

\[
a_{\mu k} R^\mu_k + a_{\mu k} R^\mu_k = -\nabla_k \lambda_i - (n - 1) \nabla_k \lambda_i,
\]

where $R^\mu_k = g^{\mu} R^{a}_{\mu i}, R^\mu_k = g^{\mu} R_{\mu k}$ and $\mu = \nabla_k \lambda_i g^{\nu k}$. 

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We contract this formula with $F^j_k$ and from the properties of the Riemann and the Ricci tensors of $K_n$ we obtain

$$\nabla_k \lambda_i = \nabla_k \lambda_i,$$

and ([8, 25, 28, 35])

$$n \nabla_k \lambda_i = \mu g_{ik} - a_b R^b_k - a_k R^b_i \ .$$

Because $\lambda_i$ is a gradient-like covector, from equation (17) follows $a_b R^b_k = a_b R^b_i$.

From (16) follows that the vector field $\lambda_i (\equiv \lambda_a P^a)$ is a Killing vector field, i.e.

$$\nabla_j \lambda_i + \nabla_i \lambda_j = 0.$$

But the other side of the equations (16) can be written in the form $\nabla_a \lambda^b P_a^b = \nabla_a \lambda^b P_a^b_i$. In the canonical coordinate system $x$ they are given by

$$\partial_b \lambda^a - \partial_{b+m} \lambda^{a+m} = 0 \quad \text{and} \quad \partial_{b+m} \lambda^{a+m} + \partial_b \lambda^{a+m} = 0, \quad a, b = 1, \ldots, m, \ m = n/2.$$

These are Cauchy-Riemann equations, which implies that the functions $\lambda^b(x)$ are real analytic. After this differentiation of the Killing equations we obtain $\nabla_j (\nabla_i \lambda^b) = \lambda^b R^b_{j\mu}$, and by contraction with $F^j_h$, we finally obtain

$$\nabla_{ij} = -2 \lambda^b R^b_{ij}.$$

These equations were found earlier under the assumption $K_n \subset C^3$ and $K_n \subset C^3$, [20], see [35, p. 212], [28, pp. 247–248].

From that we proof the following theorem

**Theorem 6.1.** A Kähler manifold $K_n \subset C^2$ admits holomorphically projective mappings onto $K_n \subset C^1$ if and only if the system of differential equations

$$\begin{align*}
\nabla_k a_{ij} &= \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_k g_{ij}, \\
\n\nabla_k \lambda_i &= \mu g_{ik} - a_b R^b_k - a_k R^b_i, \\
\n\nabla_{ij} &= -2 \lambda^b R^b_{ij} ,
\end{align*}$$

has a solution $a_{ij}$, $\lambda_i$ and $\mu$ satisfying the following conditions

$$a_{ij} = a_{ij} = a_{ij}, \quad \det(a_{ij}) \neq 0. \quad (19)$$

**Remark 6.2.** Moreover if $K_n \subset C^3$, it follows that $K_n \subset C^3$, the function $\lambda_i \subset C^3$ and $\mu \subset C^{-1}$.

**Remark 6.3.** If $K_n \subset C^\infty$, then $K_n \subset C^\infty$, and if $K_n \subset C^\omega$, then $K_n \subset C^\omega$.

Theorem 6.1 was proved in the case $K_n, \bar{K}_n \subset C^3$, see [20].

The family of differential equations (18) is linear with coefficients of intrinsic character in $K_n$ and independent of the choice of coordinates. If the metric tensor $g$ and the structure tensor $F$ of the Kähler manifold $K_n$ are real then for the initial data

$$a_{ij}(x_0) = a_{ij}, \quad \lambda_i(x_0) = \lambda_i, \quad \mu(x_0) = \mu,$$

the system (18) has at most one solution. Accounting that the initial data must satisfy (19), it follows that the general solution of (18) depends on $r_{hpm}$ significant parameters, where $r_{hpm} \leq (n/2 + 1)^2$. 