A Pair of Fractional Powers of Hankel-Clifford Transformations of Arbitrary Order

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Abstract. The main objective of this paper is to extend a pair of fractional powers of Hankel-Clifford transformations to arbitrary values of $\nu$. Moreover, we obtain some interesting results for these extension. To illustrate some problems are given.

1. Introduction

Prasad et al. [9] introduced a pair of fractional powers of $\alpha$ ($0 < \alpha < \pi$) of Hankel-Clifford transformations of order $\nu \geq 0$ depending on an arbitrary real parameter $\mu$, which is a generalization of a pair of Hankel-Clifford transformations [1, 6, 7]. In this work the fractional powers of first Hankel-Clifford transformation is defined as:

$$(h_{1,\nu,\mu}^\alpha f)(y) = f_{1,\nu,\mu}^{\alpha}(y) = \int_0^\infty K_1^\alpha(x, y)f(x)dx,$$  

where,

$$K_1^\alpha(x, y) = \begin{cases} \gamma_{\nu,\mu}^\alpha C_{\nu,\mu}(xy \csc^2 \alpha) e^{(x+y)\cot \alpha} y^\mu, & \alpha \neq n\pi, \\ C_{\nu,\mu}(xy)^\mu, & \alpha = \frac{\pi}{2}, \\ \delta(x - y), & \alpha = n\pi, \end{cases}$$

where $n \in \mathbb{Z}$, $\gamma_{\nu,\mu}^\alpha = \frac{-\sin(\nu - \frac{\pi}{2})}{\sin(\alpha - \frac{\pi}{2})}$, $C_{\nu,\mu}(x) = x^{-\mu/2}J_\nu(2 \sqrt{x})$ and $J_\nu$ is the Bessel function of first kind of order $\nu$.

Analogously, the fractional powers of $\alpha$ ($0 < \alpha < \pi$) of the second Hankel-Clifford transformation is defined by:

$$(h_{2,\nu,\mu}^\alpha g)(y) = g_{2,\nu,\mu}^{\alpha}(y) = \int_0^\infty K_2^\alpha(x, y)g(x)dx = y^{-\mu}(h_{1,\nu,\mu}^\alpha (x^\mu g))(y),$$

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where,

\[
K_2^\alpha(x, y) = \begin{cases}
\gamma_{\alpha,\mu}^2 C_{\mu,\nu}(xy \csc^2 \alpha) e^{i(xy + y)} \cot \alpha \chi^\mu, & \alpha \neq n\pi, \\
C_{\mu,\nu}(xy) \chi^\mu, & \alpha = \frac{\pi}{2}, \\
\delta(x - y), & \alpha = n\pi,
\end{cases}
\]  

(4)

where \( n \) and \( \frac{\pi}{2} \) as above.

The fractional powers of first and second Hankel-Clifford transformation are reduced to a pair of Hankel-Clifford transformation [1, 7, 10] by choosing \( \nu = \mu = \pi/2 \). The first and the second Hankel-Clifford (or fractional Hankel-Clifford) transformations are adjoint of each other.

For \( \mu = 0 \) and \( \alpha = \pi/2 \), the transformations defined in (1) and (3) coincide and is denoted by \( h_{\nu} \) and for \( \varphi \in L^2(I) \), it is defined by

\[
(h_{\nu} \varphi)(y) = \hat{\varphi}_1(y) = \int_0^\infty f_1(2\sqrt{xy}) \varphi(x) dx, \quad 0 < y < \infty,
\]

which is adjoint of itself. Hence \( h_{\nu} \) is known as Hankel-Clifford transformation.

The inverse of (1) and (3) respectively are defined as follows:

\[
f(x) = \left( (h_{\nu}^\mu)^{-1} f_{\nu}^\mu \right)(x) = \int_0^\infty K_1^\mu(y, x) f_{\nu}^\mu(y) dy
\]

and

\[
g(x) = \left( (h_{\nu}^\mu)^{-1} g_{\nu}^\mu \right)(x) = \int_0^\infty K_2^\mu(y, x) g_{\nu}^\mu(y) dy,
\]

where \( K_1^\mu(y, x) \) and \( K_2^\mu(y, x) \) are same as \( K_1^\mu(y, x) \) and \( K_2^\mu(y, x) \) respectively. Throughout this paper we denote complex conjugate by \( \gamma \).

We shall need the following operational formulas [7],

\[
D_x^r C_{\mu}(x) = (-1)^r C_{\mu+r}(x),
\]

\[
D_x^r [x^{\nu-\tau} C_{\mu+r}(x)] = x^\nu C_{\mu}(x), \quad \forall \, r \in \mathbb{N}_0,
\]

where \( C_{\mu}(x) = x^{-\mu/2} f_{\mu}(2\sqrt{x}) \).

We have the following differential and integral operators [9]:

\[
R_{1,\nu,\alpha}(x) = e^{ix \cot \alpha} x^{\nu\alpha} D_x x^{\nu} e^{-ix \cot \alpha}, \quad D_x = \frac{d}{dx},
\]

(9)

\[
S_{1,\nu,\alpha}(x) = e^{ix \cot \alpha} x^{\nu\mu} D_x x^{\nu} e^{-ix \cot \alpha},
\]

(10)

\[
\Delta_{1,\nu,\alpha}(x) = S_{1,\nu,\alpha}(x) R_{1,\nu,\alpha}(x) = e^{ix \cot \alpha} x^{\nu\mu} D_x x^{\nu+1} D_x x^{\nu} e^{-ix \cot \alpha}
\]

\[
= xD_x^2 + [(1 - \mu) - 2i x \cot \alpha] D_x - \left[(1 + \mu)i \cot \alpha / x + x \cot^2 \alpha + \frac{\nu^2 - \mu^2}{4x}\right],
\]

(11)

\[
R_{1,\nu,\alpha}^{-1} \varphi(x) = e^{ix \cot \alpha} x^{\nu\mu} \int_0^x x_1^{-(\mu\nu+1)} e^{-ix_1 \cot \alpha} \varphi(x_1) dx_1
\]

(12)

\[
R_{2,\nu,\alpha}^{-1} \varphi(x) = -e^{ix \cot \alpha} x^{\nu\mu} D_x x^{\nu} e^{-ix \cot \alpha},
\]

(13)

\[
S_{2,\nu,\alpha}(x) = -e^{ix \cot \alpha} x^{\nu\mu} D_x x^{\nu} e^{-ix \cot \alpha},
\]

(14)

\[
\Delta_{2,\nu,\alpha}(x) = R_{2,\nu,\alpha}^{-1} S_{2,\nu,\alpha}^{-1} x^{\nu\mu} D_x x^{\nu+1} D_x x^{\nu} e^{-ix \cot \alpha}
\]

\[
= xD_x^2 + [(1 + \mu) - 2i x \cot \alpha] D_x - \left[(1 - \mu)i \cot \alpha / x + x \cot^2 \alpha + \frac{\nu^2 - \mu^2}{4x}\right],
\]

(15)

\[
S_{2,\nu,\alpha}^{-1} \varphi(x) = e^{ix \cot \alpha} x^{\nu\mu} \int_0^\infty x_1^{-(\mu\nu+1)} e^{-ix_1 \cot \alpha} \varphi(x_1) dx_1
\]

(16)
From (12) and (16) respectively, we have

\[
R_{1,\nu,\mu}^{-1} \ldots R_{1,\nu+m-1,\mu}^{-1} \varphi(x) = e^{ix \cot \alpha} \frac{ix}{\sin x} \int_{\infty}^{\infty} \ldots \int_{\infty}^{\infty} x_m^{-\gamma_{\nu,\mu}} e^{-ix_m \cot \alpha} \varphi(x_m) dx_m \ldots dx_1,
\]

(17)

\[
S_{2,\nu,\mu}^{-1} \ldots S_{2,\nu+m-1,\mu}^{-1} \varphi(x) = (-1)^m e^{ix \cot \alpha} \frac{x}{\sin x} \int_{\infty}^{\infty} \ldots \int_{\infty}^{\infty} x_m^{-\gamma_{\nu,\mu}} e^{-ix_m \cot \alpha} \varphi(x_m) dx_m \ldots dx_1.
\]

(18)

We observe that \( \Delta_1^{\nu,\mu} \) and \( \Delta_2^{r,\nu,\mu} \) are adjoint of \( \Delta_2^{r,\nu,\mu} \) and \( \Delta_1^{\nu,\mu} \) respectively.

1.1. The spaces \( \mathcal{H}_{1,\nu,\mu}^a(l) \) and \( \mathcal{H}_{2,\nu,\mu}^a(l) \) and their dual

A complex valued \( C^\infty \)-function \( \varphi \) defined on \( I = (0, \infty) \) is in \( \mathcal{H}_{1,\nu,\mu}^a(l) \) if

\[
\gamma_{1,\nu,\mu}^{1,2}(\varphi) = \sup_{x \in I} \left| x^\nu D_{x}^\nu e^{ix \cot \alpha} \varphi(x) \right| = \sup_{x \in I} \left| x^\nu D_{x}^\nu e^{ix \cot \alpha} \varphi(x) \right| < \infty,
\]

(19)

for each pair of non-negative integers \( q \) and \( k \). The topology over \( \mathcal{H}_{1,\nu,\mu}^a(l) \) is generated by the family \( \{ \gamma_{1,\nu,\mu}^{1,2} \}_{q,k \in \mathbb{N}} \) of semi-norms.

On the other hand, \( \mathcal{H}_{2,\nu,\mu}^a(l) \) consists of all complex valued \( C^\infty \)-functions \( \psi \) defined on \( I \) which satisfies

\[
\gamma_{2,\nu,\mu}^{1,2}(\psi) = \sup_{x \in I} \left| x^\nu D_{x}^\nu e^{ix \cot \alpha} \psi(x) \right| = \sup_{x \in I} \left| x^\nu D_{x}^\nu e^{ix \cot \alpha} \psi(x) \right| < \infty,
\]

(20)

for each pair of non-negative integers \( q \) and \( k \). The topology over \( \mathcal{H}_{2,\nu,\mu}^a(l) \) is generated by the family \( \{ \gamma_{2,\nu,\mu}^{1,2} \}_{q,k \in \mathbb{N}} \) of semi-norms.

Also, \( (\mathcal{H}_{1,\nu,\mu}^a)'(l) \) and \( (\mathcal{H}_{2,\nu,\mu}^a)'(l) \) represent the dual of \( \mathcal{H}_{1,\nu,\mu}^a(l) \) and \( \mathcal{H}_{2,\nu,\mu}^a(l) \) respectively and their members are generalized functions of slow growth. Hence, \( (\mathcal{H}_{1,\nu,\mu}^a)'(l) \) and \( (\mathcal{H}_{2,\nu,\mu}^a)'(l) \) are too complete.

Main goal of this paper is to define a pair of fractional powers of Hankel-Clifford transformations for all real values of the order \( \nu \) and real parameter \( \mu \) and \( \alpha (0 < \alpha < \pi) \) according to the method developed in [4, 5, 8, 11] for Hankel transformations.

2. Fractional Powers of First Hankel-Clifford Transformation of Arbitrary Order

Let \( \nu, \mu \) be any real numbers and \( \alpha (0 < \alpha < \pi) \) and \( m \) be a positive integer such that \( \nu + \mu + m \geq 0 \). We define the extended fractional powers of first Hankel-Clifford transformation \( h_{1,\nu,\mu,m}^a \) of any \( \varphi \in \mathcal{H}_{1,\nu,\mu}^a(l) \) by

\[
\Phi(y) = (h_{1,\nu,\mu,m}^a \varphi)(y) = (-1)^m e^{-i(\alpha - \frac{\pi}{2})m}(y \csc^2 \alpha)^{-m/2} \left[ h_{1,\nu,\mu,m}^a(R_{1,\nu,\mu,m}^a \varphi) \right](y).
\]

(21)

The inverse transformation \( (h_{1,\nu,\mu,m}^a)^{-1} \) of any \( \Phi \in \mathcal{H}_{1,\nu,\mu}^a(l) \) is defined by

\[
\varphi(x) = ((h_{1,\nu,\mu,m}^a)^{-1} \Phi(y))(x) = (-1)^m e^{-i(\alpha - \frac{\pi}{2})m} R_{1,\nu,\mu,m}^{-1} \left( (h_{1,\nu,\mu,m}^a)^{-1} ((y \csc^2 \alpha)^{-m/2} \Phi) \right)(x).
\]

(22)

**Theorem 2.1.** The extended fractional powers of first Hankel-Clifford transformation \( h_{1,\nu,\mu,m}^a \) as defined by (21), is an isomorphism from \( \mathcal{H}_{1,\nu,\mu}^a(l) \) onto whatever be the real number \( \nu \). Moreover, \( h_{1,\nu,\mu,m}^a \) coincides with \( h_{1,\nu,\mu}^a \) if \( \nu + \mu \geq 0 \).

**Proof.** The theorem follows from the fact that \( \varphi \rightarrow R_{1,\nu,\mu,m}^a \varphi \) is an isomorphism from \( \mathcal{H}_{1,\nu,\mu}^a(l) \) onto \( \mathcal{H}_{1,\nu,\mu,m}^a(l) \), \( \varphi \rightarrow h_{1,\nu,\mu,m}^a \varphi \) is an isomorphism on \( \mathcal{H}_{1,\nu,\mu,m}^a(l) \) and \( \varphi \rightarrow (y \csc^2 \alpha)^{-m/2} \varphi \) is an isomorphism from \( \mathcal{H}_{1,\nu,\mu,m}^a(l) \) onto \( \mathcal{H}_{1,\nu,\mu,m}^a(l) \). (See Ref. [9], Proposition 3.6(a) and first part of the Theorem 4.2.)

Now, we prove the last part of theorem. By definition (21) for \( m = 1 \), we have

\[
(h_{1,\nu,\mu}^a \varphi)(y) = (-1)e^{-i(\alpha - \frac{\pi}{2}) \frac{1}{2}}(y \csc^2 \alpha)^{-1/2}(h_{1,\nu,\mu,1}^a \varphi)(y).
\]
Using the relations [9, Proposition 3.4],
\[ h_{1,v,\pm}^a \left( R_{1,v,\pm,\alpha}^a \phi \right) (y) = -e^{(a-\frac{1}{2}) (y \csc^2 \alpha)^{1/2}} \left( h_{1,v,\pm}^a \phi \right) (y), \]
we have
\[ (h_{1,v,\pm}^a \phi)(y) = (h_{1,v,\pm}^a \phi)(y). \]
Similarly for \( m = 2, \)
\[ (h_{1,v,\pm,2}^a \phi)(y) = (-1)^2 e^{-2(a-\frac{1}{2}) (y \csc^2 \alpha)^{1/2}} \left( h_{1,v,\pm+2}^a R_{1,v,\pm+1,\mu,\alpha}^a R_{1,v,\pm,\alpha}^a \phi \right)(y) = (h_{1,v,\pm}^a \phi)(y). \]
Proceeding in this way, we have
\[ (h_{1,v,\pm,\mu}^a \phi)(y) = (h_{1,v,\pm}^a \phi)(y). \]
This completes the proof of theorem. \( \square \)

**Theorem 2.2.** The extended inverse transformation \( (h_{1,v,\pm}^a)^{-1} \), as defined by (22), is an isomorphism from \( H_{1,v,\pm}^a(I) \) onto itself whatever be the real number \( v \). Moreover, \( (h_{1,v,\pm}^a)^{-1} \) coincides with \( (h_{1,v,\pm}^a)^{-1} \) if \( v + \mu \geq 0. \)

**Proof.** The theorem follows from the fact that \( \phi \to (y \csc^2 \alpha)^{m/2} \phi \) is an isomorphism from \( H_{1,v,\pm}^a(I) \) onto \( H_{1,v+m,\pm}^a(I) \) and \( R_{1,v,\pm,\alpha}^a \cdot R_{1,v+m-1,\mu,\alpha}^a (y \csc^2 \alpha)^{m/2} \phi \to \phi \) is an isomorphism from \( H_{1,v,\pm}^a(I) \) onto \( H_{1,v,\pm}^a(I) \). Hence, \( \phi \to (h_{1,v,\pm+m,\mu}^a)^{-1} \phi \) is an isomorphism on \( H_{1,v,\pm}^a(I) \) and (See Ref. [9], Proposition 3.6(a) and second part of the Theorem 4.2).

Now, we prove the last part of theorem. From definition (22) and (17), we have
\[
\left( (h_{1,v,\pm,\mu}^a)^{-1} \phi \right) (y) = (-1)^m e^{(a-\frac{1}{2})^m} e^{-iy \cot \alpha} \frac{y^\mu}{\sin y} \int_0^y \cdots \int_0^{y_{m-1}} y_m^{-\frac{i(x-y)}{m}} e^{iy_n \cot \alpha} (h_{1,v,\pm+m,\mu}^a)^{-1} \left( (x \csc^2 \alpha)^{m/2} \phi \right)(y_m) dy_m \cdots dy_1
\]
\[ = (-1)^m e^{-iy \cot \alpha} e^{-iy \cot \alpha} \frac{y^\mu}{\sin y} \int_0^y \cdots \int_0^{y_{m-1}} e^{-ix \cot \alpha} (x \csc^2 \alpha)^{m/2} C_{v-m}(y_m \csc^2 \alpha) \phi(x) dx dy_m \cdots dy_1. \]
Interchanging the order of integration between \( y \) and \( x_m \) and using (7), we have
\[
\left( (h_{1,v,\pm,\mu}^a)^{-1} \phi \right) (y) = (-1)^m e^{y \cot \alpha} e^{-iy \cot \alpha} \frac{y^\mu}{\sin y} \int_0^y \cdots \int_0^{y_{m-1}} e^{-ix \cot \alpha} (x \csc^2 \alpha)^{m/2} C_{v-m}(y_m \csc^2 \alpha) \phi(x) dx dy_m \cdots dy_1.
\]
\[ \times (-1)^m e^{-iy \cot \alpha} e^{-iy \cot \alpha} \frac{y^\mu}{\sin y} \int_0^y \cdots \int_0^{y_{m-1}} e^{-ix \cot \alpha} (x \csc^2 \alpha)^{m/2} C_{v-m}(y_m \csc^2 \alpha) \phi(x) dx dy_m \cdots dy_1. \] (23)
Proceeding in this way, we have
\[
[(h_{1,v,\pm,\mu}^a)^{-1} \phi](y) = [(h_{1,v,\pm}^a)^{-1} \phi)](y).
\]
This completes the proof of theorem. \( \square \)

**Lemma 2.3.** For any positive integers \( m \) and \( n \) both greater than \(-(v + \mu)\), we have \( h_{1,v,\pm,m}^a = h_{1,v,\pm,n}^a \) on \( H_{1,v,\pm}^a(I) \).
Proof. Note that the definition $h^α_{1,v,\mu,m}$ is independent of choice of $m$ so long as $\nu + \mu + m \geq 0$. Indeed if $m > n \geq -(\nu + \mu)$, then $h^α_{1,v+n,\mu,m-n} = h^α_{1,v,\mu}$ by Theorem 2.1.

Hence,

$$h^α_{1,v,\mu,m}(\varphi)(y) = \begin{cases} (-1)^m e^{-i(\alpha-\frac{\nu}{2})m}(y \csc^2 \alpha)^{-m/2} \left[ h^{α}_{1,v+\mu,m} \left( R^*_1, v+m-1, \mu, \alpha \right) \cdots R^*_1, \nu, \mu, \alpha \right] (y) & \text{if } \varphi \in \mathcal{H}^{α}_{1,v,\mu}(I) \\ \phi & \text{if } \varphi \not\in \mathcal{H}^{α}_{1,v,\mu}(I) \end{cases}$$

This completes the proof. \(\Box\)

Now, we obtained some interesting operational formulae for the transformation $h^α_{1,v,\mu,m}$ as:

**Proposition 2.4.** Let $\nu$ and $\mu$ be the real numbers and $m$ be a positive integer such that $\nu + \mu + m \geq 0$. Then for $\varphi \in \mathcal{H}^{α}_{1,v,\mu}(I)$, we have

$$\Delta_1, v, \mu, (h^α_{1,v,\mu,m}(\varphi))(y) = h^α_{1,v,\mu,m} \left( -\left( \chi \csc^2 \alpha \right) \varphi \right)(y),$$

$$h^α_{1,v+1,\mu,m} \left( R^*_1, \nu, \mu, \alpha \varphi \right)(y) = -e^{i(\alpha-\frac{\nu}{2})m}(y \csc^2 \alpha)^{1/2} \left( h^α_{1,v,\mu,m} \varphi \right)(y),$$

$$h^α_{1,v,\mu,m} \left( \Delta^*_1, v, \mu, \alpha \varphi \right)(y) = -(y \csc^2 \alpha) \left( h^α_{1,v,\mu,m} \varphi \right)(y).$$

If $\varphi \in \mathcal{H}^α_{1,v+1,\mu}(I)$, then

$$h^α_{1,v,\mu,m} \left( S^*_1, v, \mu, \alpha \varphi \right)(y) = e^{-i(\alpha-\frac{\nu}{2})m}(y \csc^2 \alpha)^{1/2} \left( h^α_{1,v+1,\mu,m} \varphi \right)(y).$$

**Proof.** First we prove (24). Since $\varphi \in \mathcal{H}^α_{1,v,\mu}(I)$, then $\varphi \in \mathcal{H}^α_{1,v+1,\mu}(I) \subset \mathcal{H}^α_{1,v,\mu}(I)$. Moreover,

$$R^*_1, v+m-1, \mu, \alpha \varphi = e^{-i\cot \alpha} x^{\frac{\nu + \mu}{\alpha}} D^m x^{-\frac{\nu + \mu}{\alpha}} e^{ix \cot \alpha}$$

Now, from definition (21), relation (28) and using (7), we have

$$\Delta_1, v, \mu, \left( h^α_{1,v,\mu,m} \varphi \right)(y) = S_1, v, \mu, R^*_1, \nu, \mu, \left( h^α_{1,v,\mu,m} \varphi \right)(y)$$

$$= \begin{cases} (-1)^m e^{i\cot \alpha} y^{\frac{\nu}{\alpha}} D^m y^{\nu+1} D_y y^{\nu/\alpha} e^{-i(\alpha-\frac{\nu}{2})m}(y \csc^2 \alpha)^{-m/2} & \text{if } \varphi \in \mathcal{H}^{α}_{1,v,\mu}(I) \\ \phi & \text{if } \varphi \not\in \mathcal{H}^{α}_{1,v,\mu}(I) \end{cases}$$

$$\times \chi^{\mu+m}(x \csc^2 \alpha) e^{ix \cot \alpha} x^{\frac{\nu + \mu}{\alpha}} D^m x^{-\frac{\nu + \mu}{\alpha}} e^{ix \cot \alpha} \varphi(x) dx$$

$$= (-1)^m (csc^2 \alpha)^{-m/2} e^{ix \cot \alpha} y^{\frac{\nu}{\alpha}} D_y y^{\nu+1} D_y y^{\nu/\alpha} e^{-i(\alpha-\frac{\nu}{2})m}(y \csc^2 \alpha)^{-m/2}$$

$$\times D^m x^{\frac{\nu + \mu}{\alpha}} e^{ix \cot \alpha} \varphi(x) dx$$

$$= (-1)^m (csc^2 \alpha)^{-m/2} e^{ix \cot \alpha} y^{\frac{\nu}{\alpha}} D_y y^{\nu+1} e^{ix \cot \alpha} \int_0^{\infty} D_y C_{v+m}(x \csc^2 \alpha)x^{x^{\nu/\alpha}}$$

$$\times D^m x^{-\frac{\nu + \mu}{\alpha}} e^{ix \cot \alpha} \varphi(x) dx$$

$$= (-1)^m (csc^2 \alpha)^{-m/2} e^{ix \cot \alpha} y^{\frac{\nu}{\alpha}} D_y y^{\nu+1} e^{ix \cot \alpha} \int_0^{\infty} C_{v+m}(x \csc^2 \alpha)x^{x^{\nu/\alpha}}$$

$$\times D^m x^{-\frac{\nu + \mu}{\alpha}} e^{ix \cot \alpha} \varphi(x) dx$$

$$= (-1)^m (csc^2 \alpha)^{-m/2} e^{ix \cot \alpha} y^{\frac{\nu}{\alpha}} D_y y^{\nu+1} e^{ix \cot \alpha} \int_0^{\infty} C_{v+m}(x \csc^2 \alpha)x^{x^{\nu/\alpha}}$$

$$\times D^m x^{-\frac{\nu + \mu}{\alpha}} e^{ix \cot \alpha} \varphi(x) dx.$$
Now, differentiating under the sign of integration and then using (8), the right-hand side of above equation can be written as

\[
(-1)^{m+1}(\csc^2 \alpha)^{-(v+\mu+m)/2} e^{ix \cot \alpha} y^{\nu} \gamma_{\nu, \mu} \int_0^\infty D_y [y^{-m}(xy \csc^2 \alpha)^{v+m+1} C_{v+m+1}(xy \csc^2 \alpha)] D_x^m x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x) dx
\]

\[
= (-1)^{m+1}(\csc^2 \alpha)^{-(v+\mu+m)/2} e^{ix \cot \alpha} y^{\nu} \gamma_{\nu, \mu} \int_0^\infty (-1)^m y^{-m-1}(xy \csc^2 \alpha)^{v+m+1} C_{v+m+1}(xy \csc^2 \alpha)
\]

\[
\times D_x^m x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x) dx + (-1)^{m+1}(\csc^2 \alpha)^{-(v+\mu+m)/2} e^{ix \cot \alpha} y^{\nu} \gamma_{\nu, \mu} \int_0^\infty y^{-m}(x \csc^2 \alpha)
\]

\[
\times (x \csc^2 \alpha)^{v+m} C_{v+m}(x \csc^2 \alpha) D_x^m x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x) dx.
\]

The first of the two integrals, in the above relation, is integrated by parts yield

\[
\Delta_{1,\nu,\mu,a} \left( h_{1,\nu,\mu,a}^a \right)(y)
\]

\[
= (-1)^m m(\csc^2 \alpha)(y \csc^2 \alpha)^{-m/2} e^{ix \cot \alpha} y^{\nu} \gamma_{\nu, \mu} \int_0^\infty C_{v+m,\mu}(xy \csc^2 \alpha) x^{-(v+\mu) + 1} D_x^m x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x) dx
\]

\[
+ (-1)^{m+1} (\csc^2 \alpha)(y \csc^2 \alpha)^{-m/2} e^{ix \cot \alpha} y^{\nu} \gamma_{\nu, \mu} \int_0^\infty C_{v+m,\mu}(xy \csc^2 \alpha) x^{-(v+\mu) + 1} D_x^m x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x) dx. \tag{29}
\]

From [9], we have

\[
D_x^m x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x) = m D_x^{m-1} x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x) + x D_x^m x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x). \tag{30}
\]

We now consider the right-hand side of (24), to which we invoke (21) and using (30), we have

\[
k_{1,\nu,\mu,a}^a \left( -x \csc^2 \alpha \varphi \right)(y) = (-1)^{m+1} e^{-(\alpha - \frac{2}{\mu})m}(y \csc^2 \alpha)^{-m/2} \gamma_{\nu, \mu} y^{\nu} \int_0^\infty C_{v+m,\mu}(xy \csc^2 \alpha)
\]

\[
\times e^{ix \cot \alpha} x^{-(v+\mu) + 1} D_x^m x^{-(v+\mu)} e^{ix \cot \alpha} (x \csc^2 \alpha) \varphi(x) dx
\]

\[
= (-1)^{m+1} (\csc^2 \alpha)(y \csc^2 \alpha)^{-m/2} e^{ix \cot \alpha} y^{\nu} \gamma_{\nu, \mu} \int_0^\infty C_{v+m,\mu}(xy \csc^2 \alpha) e^{ix \cot \alpha}
\]

\[
\times x^{-(v+\mu) + 1} \left[ m D_x^{m-1} x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x) + x D_x^m x^{-(v+\mu)} e^{ix \cot \alpha} \varphi(x) \right] dx,
\]

which is equivalent to (29). This proves (24).

To prove (25), we employ Lemma 2.3 to obtain

\[
\Delta_{1,\nu,\mu,a} \left( R_{1,\nu,\mu,a}^a \right)(y)
\]

\[
= (-1)^m e^{-(\alpha - \frac{2}{\mu})m}(y \csc^2 \alpha)^{-m/2} \gamma_{\nu, \mu} \left( R_{1,\nu,\mu+1,\mu}^a ... R_{1,\nu,\mu,a}^a \right) \varphi(y)
\]

\[
= -e^{(\alpha - 1/2)m}(y \csc^2 \alpha)^{1/2}(-1)^{m+1} e^{-(\alpha - \frac{2}{\mu})(m+1)}(y \csc^2 \alpha)^{-(m+1)/2} \gamma_{\nu, \mu} \left( R_{1,\nu,\mu+1,\mu}^a ... R_{1,\nu,\mu+1,\mu}^a \right) \varphi(y)
\]

\[
= -e^{(\alpha - 1/2)(y \csc^2 \alpha)^{1/2}} \left( \gamma_{1,\nu,\mu+1,\mu}^a \varphi \right)(y).
\]

This proves (25).
Next, we prove (27). Let $\varphi \in \mathcal{H}_{1,v+1,\mu}^a(l)$, then we have

\[
\begin{align*}
&\int_{[0,1]} \phi_1 \, dx = (-1)^m e^{-i(\alpha - \frac{\mu}{2})m} \left( R_{1,v+1,\mu,\eta}^a \right) (y) \\
&\times \int_{0}^{\infty} C_{\nu+1,\mu} (xy \csc^2 \alpha) e^{\varphi(x)dx} \\
&\times \int_{0}^{\infty} C_{\nu+1,\mu} (xy \csc^2 \alpha) e^{\varphi(x)dx}.
\end{align*}
\]

In view of (30), (31) can be obtained in the form

\[
\begin{align*}
&\int_{[0,1]} \phi_1 \, dx = (-1)^m e^{-i(\alpha - \frac{\mu}{2})m} \left( R_{1,v+1,\mu,\eta}^a \right) (y) \\
&\times \int_{0}^{\infty} C_{\nu+1,\mu} (xy \csc^2 \alpha) e^{\varphi(x)dx} \\
&\times \int_{0}^{\infty} C_{\nu+1,\mu} (xy \csc^2 \alpha) e^{\varphi(x)dx}.
\end{align*}
\]

Further, continuing the proceedings to prove the relation (27), We prove that $e^{-i(\alpha - \frac{\mu}{2})m} \left( R_{1,v+1,\mu,\eta}^a \right) (y)$ is equivalent to (32). Since $\varphi \in \mathcal{H}_{1,v+1,\mu}^a(l)$. Then, we have

\[
\begin{align*}
e^{-i(\alpha - \frac{\mu}{2})m} \left( R_{1,v+1,\mu,\eta}^a \right) (y) \\
&\times \int_{0}^{\infty} C_{\nu+1,\mu} (xy \csc^2 \alpha) e^{\varphi(x)dx} \\
&\times \int_{0}^{\infty} C_{\nu+1,\mu} (xy \csc^2 \alpha) e^{\varphi(x)dx}.
\end{align*}
\]

Now, using the formula

\[
D_x \left[ C_{\nu+1,\mu} (xy \csc^2 \alpha) \right] = -(y \csc^2 \alpha) C_{\nu+1,\mu+1} (xy \csc^2 \alpha),
\]
and integrating by parts, we have

$$e^{-\beta x^2}(y \csc^2 \alpha)^{1/2}(h^\alpha_{1,1,\nu,\mu} \varphi)(y)$$

$$= (-1)^{\nu+1}(y \csc^2 \alpha)^{(\nu-\mu)/2} \gamma_{\nu,\mu} y^{\nu-\mu} \int_0^\infty D_y \{C_{\nu+m}(xy \csc^2 \alpha)\} e^{iy \cot a} x^{\nu+m+1} D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x) dx$$

$$= (-1)^{\nu} (y \csc^2 \alpha)^{m-\mu/2} \gamma_{\nu,\mu} y^{\nu-\mu} \int_0^\infty C_{\nu+m}(xy \csc^2 \alpha) e^{iy \cot a} D_x [x^{\nu+m+1} D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x)] dx$$

$$= (-1)^{\nu} (y \csc^2 \alpha)^{m-\mu/2} \gamma_{\nu,\mu} y^{\nu-\mu} \int_0^\infty C_{\nu+m}(xy \csc^2 \alpha) e^{iy \cot a} [x^{\nu+m+1} D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x) dx$$

$$+ (\nu + m + 1) x^{\nu+m} D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x)] dx.$$ (33)

By using the formula $C_{\nu+m}(x) = x^{(\nu+m)/2} C_\nu(x)$, relation (33) can be made equivalent to (32). This proves (27). Finally, combining (25) and (27), we obtain (26). □

**Problem 2.5.** With $\nu$, $\mu$ and $m$ as Proposition 2.4 and for all $\varphi \in H_{1,1,\nu,\mu}(I)$, prove that

$$R_{1,2,\nu,\mu}(h^\alpha_{1,1,\nu,\mu} \varphi)(y) = e^{i(x-\beta \bar{z})} h^\alpha_{1,1,\nu,\mu} (-x \csc^2 \alpha)^{1/2} \varphi(y),$$

$$S_{1,2,\nu,\mu}(h^\alpha_{1,1,\nu,\mu} \varphi)(y) = e^{i(x-\beta \bar{z})} h^\alpha_{1,1,\nu,\mu} (x \csc^2 \alpha)^{1/2} \varphi(y).$$

**Proof.** Applying $R_{1,2,\nu,\mu}$ to both sides of (21) and then using (28) with formula (7), we have

$$R_{1,2,\nu,\mu}(h^\alpha_{1,1,\nu,\mu} \varphi)(y)$$

$$= e^{i\beta \bar{z}} y^{\nu+m} D_y y^{\nu-\mu} e^{-i \beta \bar{z}} e^{i \beta \bar{z}} (x \csc^2 \alpha)^{-\mu/2} (h^\alpha_{1,1,\nu,\mu} \varphi)(y)$$

$$\times y^{\nu} \int_0^\infty C_{\nu+m}(xy \csc^2 \alpha) e^{iy \cot a} [x^{\nu+m+1} D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x)] dx$$

$$= (-1)^{\nu} e^{i \beta \bar{z}} y^{\nu+m} D_y y^{\nu-\mu} \gamma_{\nu,\mu} \int_0^\infty C_{\nu+m}(xy \csc^2 \alpha) x^{\nu+m+1} D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x) dx$$

$$= (-1)^{\nu+1} e^{i \beta \bar{z}} y^{\nu+m} \gamma_{\nu,\mu} \int_0^\infty C_{\nu+m}(xy \csc^2 \alpha) x^{\nu+m+1} D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x) dx$$

This proves (34).

To prove (35), we have form (21) and (28)

$$S_{1,2,\nu,\mu}(h^\alpha_{1,1,\nu,\mu} \varphi)(y)$$

$$= e^{i \beta \bar{z}} y^{\nu} \int_0^\infty C_{\nu+m+1,\mu}(xy \csc^2 \alpha) e^{iy \cot a} [x^{\nu+m+1} D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x)] dx$$

$$= (-1)^{\nu} e^{i \beta \bar{z}} y^{\nu+m} \gamma_{\nu,\mu} \int_0^\infty C_{\nu+m+1,\mu}(xy \csc^2 \alpha) x^{\nu+m+1} D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x) dx$$

$$\times (xy \csc^2 \alpha)^{m+1} C_{\nu+m+1}(xy \csc^2 \alpha) D_x x^{-(\nu+m+1)} e^{ix \cot a} \varphi(x) dx$$

This proves (35).
This proves (35).

Using Theorem 2.1, we have Theorem 2.6.

Let \( \mu \) and \( \alpha \) (0 < \alpha < \pi) are real parameters. Then for any positive integer \( m \) such that \( \nu + \mu + m \geq 0 \),

\[
R_{1,v+1,\mu,m} \left( h_{1,v+1,\mu,m}^{\nu} \right)(y) = (-1)^m e^{-i(\alpha - 2\pi)m} h_{1,v+1,\mu,m}^{\nu} \left( (x \csc^2 \alpha)^m \right)(y).
\]

Moreover, \( h_{1,v+1,\mu,m}^{\nu/2} = [h_{1,v+1,\mu,m}^{\nu/2}]^{-1} \).

**Proof.** Applying \( R_{1,v+1,\mu,m} \) to both side of (34), we have

\[
R_{1,v+1,\mu,m} \left( h_{1,v+1,\mu,m}^{\nu} \right)(y) = e^{-i(\alpha - 2\pi)m} R_{1,v+1,\mu,m} h_{1,v+1,\mu,m}^{\nu} \left( (x \csc^2 \alpha)^{1/2} \right)(y).
\]

Repeating this process, we have

\[
R_{1,v+m-1,\mu,\alpha-1} \left( h_{1,v+m,\mu,\alpha}^{\nu} \right)(y) = e^{-i(\alpha - 2\pi)m} R_{1,v+m-1,\mu,\alpha} h_{1,v+m,\mu,\alpha}^{\nu} \left( (x \csc^2 \alpha)^{m/2} \right)(y).
\]

Using Theorem 2.1, we have

\[
R_{1,v+m-1,\mu,\alpha} \left( h_{1,v+m,\mu,\alpha}^{\nu} \right)(y) = e^{-i(\alpha - 2\pi)m} h_{1,v+m,\mu,\alpha}^{\nu} \left( (x \csc^2 \alpha)^{m/2} \right)(y).
\]

This proves (36).

If \( \alpha = \pi/2 \), we have

\[
R_{1,v+m-1,\mu,\pi/2,\pi/2} \left( h_{1,v+m,\mu,\pi/2}^{\nu/2} \right)(y) = (-1)^m \left( h_{1,v+m,\mu,\pi/2}^{\nu/2} \right)(x^{m/2} \phi)(y).
\]

Using the fact that \( h_{1,v+1,\mu,m}^{\nu/2} = [h_{1,v+1,\mu,m}^{\nu/2}]^{-1} \) and from (22), we get

\[
\left( h_{1,v+1,\mu,m}^{\nu/2} \right)^{-1} \phi(y) = (-1)^m R_{1,v+m-1,\mu,\pi/2,\pi/2} \left( h_{1,v+m-1,\mu,\pi/2}^{\nu/2} \right)(x^{m/2} \phi)(y).
\]
Hence, we conclude that
\[ h_{1,\nu,\mu}^{n/2} = [h_{1,\nu,\mu,m}^{n/2}]^{-1}. \]
This completes the proof of theorem. □

3. Fractional Powers of Second Hankel-Clifford Transformation of Arbitrary Order

Let \( \nu, \mu \) be any real numbers and \( \alpha \) (\( 0 < \alpha < \pi \)) and \( n \) be a positive integer such that \( \nu + \mu + n \geq 0 \). We define the extended fractional powers of second Hankel-Clifford transformation \( h_{2,\nu,\mu}^{a} \) of any \( \Psi \in \mathcal{H}_{2,\nu,\mu}(I) \) by
\[
\Psi(y) = \left( h_{2,\nu,\mu}^{a} \psi \right)(y) = (-1)^{n}e^{i(\alpha - \frac{\pi}{2})n} (y \csc^{2} \alpha)^{-n/2} \left[ h_{2,\nu+n,\mu}^{a} \left( S_{2,\nu+n+1,\mu,\nu}^{a} \cdots S_{2,\nu,\mu}^{a} \psi \right) \right](y). \tag{37}
\]
The inverse transformation \( (h_{2,\nu,\mu}^{a})^{-1} \) of any \( \Psi \in \mathcal{H}_{2,\nu,\mu}(I) \) is defined by
\[
\psi(x) = \left( (h_{2,\nu,\mu}^{a})^{-1} \Psi \right)(x) = (-1)^{n}e^{-i(\alpha - \frac{\pi}{2})n} (y \csc^{2} \alpha)^{n/2} \left[ h_{2,\nu+n,\mu}^{a} \left( h_{2,\nu,\mu}^{a} \psi \right) \right](x). \tag{38}
\]
**Theorem 3.1.** The fractional powers of second Hankel-Clifford transformation \( h_{2,\nu,\mu}^{a} \), as defined by (37), is an isomorphism from \( \mathcal{H}_{2,\nu,\mu}(I) \) onto itself whatever be the real number \( \nu \). Moreover, \( h_{2,\nu,\mu}^{a} \) coincides with \( h_{2,\nu,\mu}^{a} \) if \( \nu + \mu \geq 0 \).

**Proof.** The proof of theorem is similar to that of Theorem 2.1. □

Now, we obtained some interesting operational formulae for the transformation \( h_{2,\nu,\mu}^{a} \) as:

**Proposition 3.2.** Let \( \nu \) and \( \mu \) be the real numbers and \( n \) be a positive integer such that \( \nu + \mu + n \geq 0 \). Then for \( \psi \in \mathcal{H}_{2,\nu,\mu}(I) \), we have
\[
S_{2,\nu,\mu} \left( h_{2,\nu,\mu}^{a} \psi \right)(y) = e^{-i(\alpha - \frac{\pi}{2})} h_{2,\nu+1,\mu}^{a} \left( -(x \csc^{2} \alpha)^{1/2} \psi \right)(y), \tag{39}
\]
\[
h_{2,\nu,\mu}^{a} \left( S_{2,\nu,\mu}^{a} \psi \right)(y) = -e^{i(\alpha - \frac{\pi}{2})} (y \csc^{2} \alpha)^{1/2} \left( h_{2,\nu,\mu}^{a} \psi \right)(y), \tag{40}
\]
\[
h_{2,\nu,\mu}^{a} \left( \Delta_{2,\nu,\mu}^{a} \psi \right)(y) = -(y \csc^{2} \alpha) \left( h_{2,\nu,\mu}^{a} \psi \right)(y), \tag{41}
\]
\[
\Delta_{2,\nu,\mu} \left( h_{2,\nu,\mu}^{a} \psi \right)(y) = h_{2,\nu,\mu}^{a} \left( -(x \csc^{2} \alpha)^{1/2} \psi \right)(y). \tag{42}
\]
If \( \psi \in \mathcal{H}_{2,\nu+1,\mu}(I) \) then
\[
h_{2,\nu,\mu}^{a} \left( K_{2,\nu,\mu}^{a} \psi \right)(y) = e^{i(\alpha - \frac{\pi}{2})} (y \csc^{2} \alpha)^{1/2} \left( h_{2,\nu+1,\mu}^{a} \psi \right)(y), \tag{43}
\]
\[
K_{2,\nu,\mu} \left( h_{2,\nu,\mu}^{a} \psi \right)(y) = e^{i(\alpha - \frac{\pi}{2})} h_{2,\nu,\mu}^{a} \left( (x \csc^{2} \alpha)^{1/2} \psi \right)(y). \tag{44}
\]
**Remark 3.3.** Similar results can be proved as Lemma 2.3 and Theorem 2.6 for \( h_{2,\nu,\mu}^{a} \) and \( \psi \in \mathcal{H}_{2,\nu,\mu}(I) \).


In this section, we have investigated a pair of generalized fractional powers of Hankel-Clifford transformation of arbitrary order on the dual spaces of \( \mathcal{H}_{1,\nu,\mu}(I) \) and \( \mathcal{H}_{2,\nu,\mu}(I) \).

As before \( n \) is any positive integer such that \( n \geq -(\mu + \nu) \). The generalized fractional powers of first Hankel-Clifford transformation of arbitrary order \( (h_{1,\nu,\mu}^{a})' \) is defined on \( (\mathcal{H}_{2,\nu,\mu}(I))' \), as the adjoint of \( h_{2,\nu,\mu}^{a} \) on \( \mathcal{H}_{2,\nu,\mu}(I) \), by
\[
\left( (h_{1,\nu,\mu}^{a})' f, \psi \right) = \left( f, h_{2,\nu,\mu}^{a} \psi \right), \tag{45}
\]
for \( f \in (\mathcal{H}_{2,\nu,\mu}(I))' \), \( \psi \in \mathcal{H}_{2,\nu,\mu}(I) \).

Hence from (45) and Theorem 3.1, we have the following theorem:
Theorem 4.1. The generalized fractional powers of first Hankel-Clifford transformation of arbitrary order $v$, defined in (45), is an isomorphism from $(\mathcal{H}_{2,\nu,\mu}^{a})'(I)$ into itself.

This leads to the following transformation formulae:

Proposition 4.2. For any real number $v$ and $f \in (\mathcal{H}_{2,\nu,\mu}^{a})'(I)$, we have

\[
(h_{1,\nu,\mu}^{a})'(\Lambda_{1,\nu,\mu}f)(y) = -(y \csc^{2} \alpha)\left((h_{1,\nu,\mu}^{a})'f\right)(y), \quad \text{(46)}
\]

\[
\Lambda_{1,\nu,\mu}(h_{1,\nu,\mu}^{a})'(y) = (h_{1,\nu,\mu}^{a})'\left(-x \csc^{2} \alpha f\right)(y). \quad \text{(47)}
\]

Proof. Let $\Psi \in \mathcal{H}_{1,\nu,\mu}^{a}(I)$. Then form (45) and (42), we have

\[
\langle (h_{1,\nu,\mu}^{a})'(\Lambda_{1,\nu,\mu}f), \Psi \rangle = \langle \Lambda_{1,\nu,\mu}f, h_{2,\nu,\mu}^{a}(\Psi) \rangle = \langle f, \Delta_{2,\nu,\mu,\alpha}^{a}h_{2,\nu,\mu}^{a}(\Psi) \rangle = \langle f, h_{2,\nu,\mu}^{a}\left(-y \csc^{2} \alpha \Psi\right) \rangle = \langle -(y \csc^{2} \alpha)\left((h_{1,\nu,\mu}^{a})'f\right), \Psi \rangle.
\]

In the sense of equality in distributions, we conclude the proof of (46). By the similar arguments, we can prove (47). \(\square\)

Analogously, the generalized fractional powers of second Hankel-Clifford transformation of arbitrary order $(h_{2,\nu,\mu}^{a})'$ is defined on $(\mathcal{H}_{1,\nu,\mu}^{a})'(I)$, as the adjoint of $h_{1,\nu,\mu}^{a}$ on $\mathcal{H}_{1,\nu,\mu}^{a}(I)$, by

\[
\langle (h_{2,\nu,\mu}^{a})'f, \Phi \rangle = \langle f, h_{1,\nu,\mu}^{a}(\Phi) \rangle \quad \text{(48)}
\]

for $f \in (\mathcal{H}_{1,\nu,\mu}^{a})'(I)$, $\Phi \in \mathcal{H}_{1,\nu,\mu}^{a}(I)$.

Remark 4.3. Similar results can also be proved as Theorem 4.1 and Proposition 4.2 for $(h_{2,\nu,\mu}^{a})'$ and $\Phi \in \mathcal{H}_{1,\nu,\mu}^{a}(I)$.

5. Applications

In this section, applications of a pair of fractional powers of Hankel-Clifford transformations of arbitrary order are given.

Problem 5.1. If the generalized function $\delta(x-a)$, $a > 0$ is defined on $\mathcal{H}_{1,\nu,\mu}^{a}(I)$, then

(i) \(\left(h_{1,\nu,\mu}^{a}\right)\delta(x-a) = y^{\alpha_{1},\nu,\mu}y^{(\nu(a))\cot \alpha}C_{\nu,\mu}(ay \csc^{2} \alpha)\),

(ii) \(\left(h_{1,\nu,\mu}^{a}\right)^{-1}\delta(x-a) = y^{\alpha_{1},\nu,\mu}y^{-(\nu(a))\cot \alpha}C_{\nu,\mu}(ay \csc^{2} \alpha)\).

Proof. By definition (21) and (28), we have

\[
\left(h_{1,\nu,\mu}^{a}\right)\delta(x-a) = (-1)^{m}e^{-\frac{y^{(\nu(a))\cot \alpha}}{\nu(a)}}y^{\alpha_{1},\nu,\mu} \left(h_{1,\nu,\mu}^{a}(R_{1,\nu,\mu}^{a}R_{1,\nu,\mu}^{a}\cdots R_{1,\nu,\mu}^{a}\delta(x-a))\right) y^{\alpha_{1},\nu,\mu} y^{\nu(a)}x^{\nu(a)}\cot \alpha \delta(x-a)dx.
\]

Integrating by parts repeatedly $m$ times and using the formula (8), we obtain

\[
\left(h_{1,\nu,\mu}^{a}\right)\delta(x-a) = y^{
u(a)+\frac{\nu(a)}{2}}x^{\nu(a)+\frac{\nu(a)}{2}}C_{\nu,\mu}(xy \csc^{2} \alpha)\delta(x-a)dx.
\]

\[
= y\nu(a)y^{\nu(a)}x^{\nu(a)}\cot \alpha \delta(x-a)dx.
\]
Hence, by the properties of \(\delta(x-a)\), we have the required result (i).

Next, we prove (ii). From (22) and (17), we have

\[
(h_{1,\nu,m}^a)^{-1}\delta(x-a)(y) = (-1)^m e^{(a-\frac{y}{2m})} e^{y\cot a} y \int_0^\infty \int_0^{y_1} \int_0^{y_2} \cdots \int_0^{y_{m-1}} \left[ (x \csc^2 a)^{m/2} \delta(x-a) \right](y_1) dy_m \cdots dy_2 dy_1.
\]

Now, by properties of \(\delta(x-a)\) and then using (7), we have

\[
(h_{1,\nu,m}^a)^{-1}\delta(x-a)(y) = (-1)^m \gamma_{\nu,m} e^{-i(y+y_m)\cot a} \gamma y^2 e^{(y+y_m)\cot a} C_{\nu,m}(ay \csc^2 a) y \int_0^\infty \int_0^{y_1} \int_0^{y_2} \cdots \int_0^{y_{m-1}} \left[ (x \csc^2 a)^{m/2} \delta(x-a) \right](y_1) dy_m \cdots dy_2 dy_1.
\]

Proceeding in this way, we get

\[
(h_{1,\nu,m}^a)^{-1}\delta(x-a)(y) = \gamma_{\nu,m} e^{-i(y+y_m)\cot a} y^2 e^{i(y+y_m)\cot a} C_{\nu,m}(ay \csc^2 a).
\]

This proves (ii). \(\square\)

The fractional powers of Hankel-Clifford transformations of arbitrary order can also be utilized in solving some partial differential equations. Consider the general equation [3]:

\[
a(x, y) \frac{\partial^2 \varphi}{\partial x^2} + b(x, y) \frac{\partial^2 \varphi}{\partial x \partial y} + c(x, y) \frac{\partial^2 \varphi}{\partial y^2} + d(x, y) \frac{\partial \varphi}{\partial x} + e(x, y) \frac{\partial \varphi}{\partial y} + f(x, y) \varphi = G(x, y),
\]

when

\[
a(x, y) = x, \quad b(x, y) = c(x, y) = e(x, y) = 0, \quad d(x, y) = (1 - \mu) + 2i \cot a,
\]

\[
f(x, y) = (1 - \mu)i \cot a - x \cot^2 a - \frac{y^2 - \mu^2}{4x} \quad \text{and} \quad G(x, y) = \delta(x-a),
\]

then (49) is reduced as

\[
\varphi(x) - \Delta^*_{\nu,m} \varphi(x) = \delta(x-a).
\]

Applying \(h_{1,\nu,m}^{-1}\) to both sides and using (26) and Problem 5.1(i), we get

\[
(1 + y \csc^2 a) (h_{1,\nu,m}^{-1} \varphi)(y) = \gamma_{\nu,m} \gamma \varphi(y) e^{i(y+y_m)\cot a} C_{\nu,m}(ay \csc^2 a).
\]

Therefore,

\[
\varphi(x) = \left( h_{1,\nu,m}^{-1} \right)^{-1} \gamma_{\nu,m} \gamma \varphi(y) e^{i(y+y_m)\cot a} C_{\nu,m}(ay \csc^2 a)(1 + y \csc^2 a) \quad \text{for} \quad (x, y) \in \mathbb{R}^2.
\]
Interchanging the order of integration as (23), we obtain
\[
\varphi(x) = e^{i(a-x)\cot\alpha} x^{\frac{\mu}{2}} \int_0^\infty \frac{C_{\alpha}(xy \csc^2 \alpha)(y \csc^2 \alpha)^{\mu} y \mu C_{\nu,\mu}(ay \csc^2 \alpha)}{(1 + y \csc^2 \alpha)} dy
\]
\[
= a^{-\mu/2} e^{i(a-x)\cot\alpha} x^{\mu/2} \int_0^\infty \frac{I_{\alpha}(2\sqrt{xy \csc^2 \alpha})(y \csc^2 \alpha)^{\mu} y \mu f_{\nu}(2\sqrt{ay \csc^2 \alpha})}{(1 + y \csc^2 \alpha)} dy,
\]
which on putting \( y \csc^2 \alpha = t \), we have
\[
\varphi(x) = \frac{2\rho^{(1-\alpha)\cot\alpha} x^{\mu/2}}{a^{\mu/2}(\csc^2 \alpha)^{\mu+1}} \int_0^\infty \frac{I_{\alpha}(2\sqrt{\rho t}) J_{\nu}(2\sqrt{\rho t})}{(1 + t)} dt,
\]
then from Erdelyi [2, p. 49],
\[
\varphi(x) = \frac{2\rho^{(1-\alpha)\cot\alpha} x^{\mu/2}}{a^{\mu/2}(\csc^2 \alpha)^{\mu+1}} \begin{cases} I_{\alpha}(2\sqrt{\rho x}) K_{\nu}(2\sqrt{x}), & 0 < x < 1, \\ I_{\alpha}(2\sqrt{\rho x}) K_{\nu}(2\sqrt{x}), & 1 < x < \infty, \end{cases}
\]
where \( I_{\alpha} \) and \( K_{\nu} \) are known as modified Bessel function of first and third kind respectively.

Similarly, if \( a(x, y), b(x, y), c(x, y), d(x, y), e(x, y) \) and \( f(x, y) \) are as above and \( G(x, y) = e^{-ix\cot\alpha} x^{\nu/2} C_{\nu,\mu}(ax \csc^2 \alpha) \), then we have
\[
\varphi(x) - \Delta_{\nu,\mu,\alpha}^n \varphi(x) = e^{-ix\cot\alpha} x^{\nu/2} C_{\nu,\mu}(ax \csc^2 \alpha).
\]
Now, applying \( h_{1,\nu,\mu,m}^\alpha \) to the both sides and using (26) and Problem 5.1(ii), we have
\[
(1 + y \csc^2 \alpha) \left( h_{1,\nu,\mu,m}^\alpha \varphi \right)(y) = \gamma_{\nu,\mu} e^{ia\cot\alpha} \delta(y - a).
\]
Therefore,
\[
\varphi(x) = \gamma_{\nu,\mu} e^{ia\cot\alpha} \left( h_{1,\nu,\mu,m}^\alpha \right)^{-1} \left( (y - a)(1 + y \csc^2 \alpha)^{-1} \right)(x)
\]
\[
= (-1)^m \gamma_{\nu,\mu} e^{ia\cot\alpha} \left( a(\csc^2 \alpha)^{\mu/2} \delta(y - a)(1 + y \csc^2 \alpha)^{-1} \right)(x)
\]
\[
= \gamma_{\nu,\mu} e^{ia\cot\alpha} \left( a(\csc^2 \alpha)^{\mu/2} \delta(y - a)(1 + y \csc^2 \alpha)^{-1} \right)(x) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{i\alpha x} cot \alpha
\]
\[
\times (-1)^m \left( h_{1,\nu,\mu,m}^\alpha \right)^{-1} \left( (y - a)(1 + y \csc^2 \alpha)^{-1} \right)(x) dx \ dx_1 \ dx_2 \ dx_3 \ dx_4 \ dx_5 \ dx_6.
\]
Proceeding similar as Problem 5.1(ii), we obtain
\[
\varphi(x) = (1 + a \csc^2 \alpha)^{-1} e^{-ix\cot\alpha} x^{\nu/2} C_{\nu,\mu}(ax \csc^2 \alpha).
\]
This solve our problems.

**Remark 5.2.** Analogously, applying the theory of fractional powers of second Hankel-Clifford transformation of arbitrary order \( h_{2,\nu,\mu,m}^\alpha \), we can solve some differential equation associated with Bessel type operator \( \Delta_{2,\nu,\mu,a}^n \).

**Remark 5.3.** Similar results of all theorems of Sects. 2 and 3 may be proved using the technique (23) for \( (h_{1,\nu,\mu,m}^\alpha)^{-1} \) and \( (h_{2,\nu,\mu,m}^\alpha)^{-1} \).
References