An Inequality for Similarity Condition Numbers of Unbounded Operators with Schatten - von Neumann Hermitian Components

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Abstract. Let $H$ be a linear unbounded operator in a separable Hilbert space. It is assumed the resolvent of $H$ is a compact operator and $H - H^*$ is a Schatten - von Neumann operator. Various integro-differential operators satisfy these conditions. Under certain assumptions it is shown that $H$ is similar to a normal operator and a sharp bound for the condition number is suggested.

We also discuss applications of that bound to spectrum perturbations and operator functions.

1. Introduction and Statement of the Main Result

Let $\mathcal{H}$ be a separable Hilbert space with a scalar product $(.,.)$, the norm $\|\cdot\| = \sqrt{(.,.)}$ and unit operator $I$. Two operators $A$ and $\tilde{A}$ acting in $\mathcal{H}$ are said to be similar if there exists a boundedly invertible bounded operator $T$ such that $\tilde{A} = T^{-1}AT$. The constant $\kappa_T = \|T^{-1}\|\|T\|$ is called the condition number. The condition number is important in applications. We refer the reader to [5], where condition number estimates are suggested for combined potential boundary integral operators in acoustic scattering and [23], where condition numbers are estimated for second-order elliptic operators. Conditions that provide the similarity of various operators to normal and selfadjoint ones were considered by many mathematicians, cf. [1, 4, 7], [14, 15], [17]-[21], and references given therein. In many cases, the condition number must be numerically calculated, e.g. [2, 20]. The interesting generalizations of condition numbers of bounded linear operators in Banach spaces were explored in the paper [13].

In the present paper we consider a class of unbounded operators in a Hilbert space with Schatten - von Neumann Hermitian components. Numerous integro-differential operators belong to that class. We suggest a sharp bound for the condition numbers of the considered operators. It generalizes and improves the bound for the condition numbers of operators with Hilbert-Schmidt Hermitian components from [11]. We also discuss applications of the obtained bound to spectrum perturbations and norm estimates for operator functions.

Introduce the notations. For a linear operator $A$ in $\mathcal{H}$, $\text{Dom}(A)$ is the domain, $A'$ is the adjoint of $A$; $\sigma(A)$ denotes the spectrum of $A$ and $A^{-1}$ is the inverse to $A$; $R_\lambda(A) = (A - I\lambda)^{-1}$ ($\lambda \notin \sigma(A)$) is the resolvent; $A_k := (A - A')/2i$; $\lambda_k(A)$ ($k = 1, 2, ...$) are the eigenvalues of $A$ taken with their multiplicities and enumerated as $|\lambda_k(A)| \leq |\lambda_{k+1}(A)|$, and $\rho(A, \lambda) = \inf_k |\lambda - \lambda_k(A)|$. By $\text{SN}_r$ ($1 \leq r < \infty$) we denote the Schatten - von Neumann ideal of compact operators $K$ with the finite norm $N_r(K) := [\text{Trace}(KK^*)^{r/2}]^{1/r}$.
Everywhere below $H$ is an invertible operator in $\mathcal{S}$, with the following properties: $\text{Dom}(H) = \text{Dom}(H^*)$, and there are an $r \in [1, \infty)$ and an integer $p \geq 1$, such that

$$H^{-1} \in SN_r$$ and $H_I \in SN_{2p}$. \hfill (1.1)

Note that instead of the condition $H^{-1} \in SN_r$, in our reasonings below, one can require the condition $(H - aI)^{-1} \in SN_r$ for some point $a \notin \sigma(H)$. Since $H^{-1}$ is compact, $\sigma(H)$ is purely discrete. It is assumed that all the eigenvalues $\lambda_j(H)$ of $H$ are different. For a fixed integer $m$ put

$$\delta_m(H) = \inf_{j=1,2,...,j=m} |\lambda_j(H) - \lambda_m(H)|.$$

It is further supposed that

$$\zeta_q(H) := \left[ \sum_{j=1}^{\infty} \frac{1}{\delta_j^q(H)} \right]^{1/q} < \infty \quad (\frac{1}{q} + \frac{1}{2p} = 1)$$ \hfill (1.2)

for an integer $p \geq 1$. Hence it follows that

$$\delta(H) := \inf_m \delta_m(H) = \inf_{j,k;j,k=1,2,...} |\lambda_j(H) - \lambda_k(H)| > 0.$$ \hfill (1.3)

Denote also

$$u_p(H) := \sqrt{2\zeta_2(H)} \sum_{m=0}^{p-1} \sum_{k=0}^\infty \beta_p \frac{\delta_j^{2p+m+1}(H)}{N_{2p}^{2p+m}(H)} \frac{1}{\sqrt{k!}},$$

where

$$\beta_p := 2 \left( 1 + \frac{2p}{e^{2/\sqrt{2n}}}. \right).$$ \hfill (1.4)

Now we are in a position to formulate our main result.

**Theorem 1.1.** Let conditions (1.1) and (1.2) be fulfilled. Then there are an invertible operator $T$ and a normal operator $D$ acting in $\mathcal{S}$, such that

$$THx = DTx \quad (x \in \text{Dom}(H)).$$ 

Moreover,

$$\kappa_T := ||T^{-1}||||T|| \leq e^{2p(H)}.$$ \hfill (1.6)

The proof of this theorem is divided into a series of lemmas which are presented in the next three sections. The theorem is sharp: if $H$ is selfadjoint, then $u_p(H) = 0$ and we obtain $\kappa_T = 1$.

As it is shown below, one can replace (1.6) by the inequality

$$\kappa_T \leq e^{2p(H)}.$$ \hfill (1.7)

where

$$\hat{u}_p(H) := \sqrt{2\zeta_2(H)} \sum_{m=0}^{p-1} \sum_{k=0}^\infty \beta_p^{2p+m+1}(H) \frac{1}{\delta_j^{2p+m}(A)} \frac{1}{\sqrt{k!}}.$$ 

In addition, below we show that in our considerations instead of $\beta_p$ defined by (1.4) in the case

$$p = 2^m - 1, \quad m = 2, 3, ..., \quad \text{one can take } \hat{\beta}_p = 2(1 + \text{ctg}(\frac{\pi}{4p})) \text{ and } \hat{\beta}_1 = \sqrt{2}.$$ \hfill (1.8)

instead of $\beta_1$.

To illustrate Theorem 1.1, consider the operator $H = S + K$, where $K \in SN_{2p}$ and $S$ is a positive definite selfadjoint operator with a discrete spectrum, whose eigenvalues are different and

$$\lambda_{j+1}(S) - \lambda_j(S) \geq b_0 f^a \quad (b_0 = \text{const} > 0; \alpha > 1/q = (2p - 1)/(2p); j = 1, 2, ...).$$ \hfill (1.9)
Since $S$ is selfadjoint we have
\[ \sup_k \inf_j |\lambda_k(H) - \lambda_j(S)| \leq ||K||, \]
cf. [16]. Thus, if
\[ 2||K|| < \inf_j(\lambda_{j+1}(S) - \lambda_j(S)), \]
then $\delta(H) \geq \inf_j(\lambda_{j+1}(S) - \lambda_j(S) - 2||K||)$ and (1.2) holds with
\[ \zeta_q(H) \leq \zeta_q(S, K), \]
where $\zeta_q(S, K) := \left\{ \sum_{j=1}^{\infty} (\lambda_{j+1}(S) - \lambda_j(S) - 2||K||)^{-1/q} \right\}^{1/q} < \infty$.

**Example 1.2.** Consider in $L^2(0, 1)$ the spectral problem
\[ u^{(4)}(x) + (Ku)(x) = \lambda u(x) \quad (\lambda \in \mathbb{C}, 0 < x < 1); \ u(0) = u(1) = u''(0) = u''(1) = 0, \]
where $K \in SN_{2p, p} \geq 1$ for an arbitrary $p \geq 1$. So $H$ is defined by $H = d^4/dx^4 + K$ with
\[ \text{Dom}(H) = \{ v \in L^2(0, 1) : \ v^{(4)} \in L^2(0, 1), \ v(0) = v(1) = v''(0) = v''(1) = 0 \}. \]
Take $S = d^4/dx^4$ with $\text{Dom}(S) = \text{Dom}(H)$. Then $\lambda_j(S) = \pi^4 j^4$ ($j = 1, 2, ...$) and $\lambda_{j+1}(S) - \lambda_j(S) \geq 4\pi^4 j^3$. If $||K|| < 2\pi^4$, then $\delta(H) \geq 4\pi^4 - 2||K||$ and
\[ \zeta_q(H) \leq \sum_{j=1}^{\infty} (4\pi^4 j^3 - 2||K||)^{-q} < \infty. \]

Now one can directly apply Theorem 1.1.

2. Auxiliary Results

Let $B_0$ be a bounded linear operator in $\mathcal{S}$ having a finite chain of invariant projections $P_k$ ($k = 1, ..., n; \ n < \infty$):
\[ 0 \subset P_1 \mathcal{S} \subset P_2 \mathcal{S} \subset ... \subset P_n \mathcal{S} = \mathcal{S} \tag{2.1} \]
and
\[ P_k B_0 P_k = B_0 P_k \quad (k = 1, ..., n). \tag{2.2} \]
That is, $B_0$ maps $P_k \mathcal{S}$ into $P_k \mathcal{S}$ for each $k$. Put
\[ \Delta P_k = P_k - P_{k-1} \quad (P_0 = 0) \quad \text{and} \quad A_k = \Delta P_k B_0 \Delta P_k. \]

It is assumed that the spectra $\sigma(A_k)$ of $A_k$ in $\Delta P_k \mathcal{S}$ satisfy the condition
\[ \sigma(A_k) \cap \sigma(A_j) = \emptyset \quad (j \neq k; \ j, k = 1, ..., n). \tag{2.3} \]

**Lemma 2.1.** One has
\[ \sigma(B_0) = \bigcup_{k=1}^{n} \sigma(A_k). \]
For the proof see [11].

Under conditions (2.1), (2.2) put
\[ Q_k = I - P_k, \ B_k = Q_k B_0 Q_k \quad \text{and} \quad C_k = \Delta P_k B_0 Q_k. \]

Since $B_j$ is a block triangular operator matrix, according to the previous lemma we have
\[ \sigma(B_j) = \bigcup_{k=j+1}^{n} \sigma(A_k) \quad (j = 0, ..., n). \]

Under this condition, according to the Rosenblum theorem from [22], the equation
\[ A_j X_j - X_j B_j = -C_j \quad (j = 1, ..., n - 1) \tag{2.4} \]
has a unique solution (see also [6, Section I.3] and [3]). We need also the following result proved in [11].
Lemma 2.2. Let condition (2.3) hold and $X_j$ be a solution to (2.4). Then

$$(I - X_{n-1})(I - X_{n-2}) \cdots (I - X_1) B_0 (I + X_1)(I + X_2) \cdots (I + X_{n-1}) =$$

$$A_1 + A_2 + \ldots + A_n.$$  \hfill (2.5)

Take

$$T_n = (I + X_1)(I + X_2) \cdots (I + X_{n-1}).$$  \hfill (2.6)

It is simple to see that the inverse to $I + X_j$ is the operator $I - X_j$. Thus,

$$T_n^{-1} = (I - X_{n-1})(I - X_{n-2}) \cdots (I - X_1)$$

and (2.5) can be written as

$$T_n^{-1}B_0T_n = \text{diag} (A_k)_{k=1}^n.$$  \hfill (2.8)

By the inequalities between the arithmetic and geometric means we get

$$||T_n|| \leq \prod_{k=1}^{n-1} (1 + ||X_k||) \leq \left(1 + \frac{1}{n-1} \sum_{k=1}^{n-1} ||X_k||\right)^{n-1}$$

and

$$||T_n^{-1}|| \leq \left(1 + \frac{1}{n-1} \sum_{k=1}^{n-1} ||X_k||\right)^{n-1}.$$  \hfill (2.10)

Furthermore, we need the following result

Theorem 2.3. Let $M$ be a linear operator in $\mathcal{S}_p$, such that $\text{Dom} (M) = \text{Dom} (M^*)$ and $M_i = (M - M^*)/2i \in \mathcal{S}_p$ for some integer $p \geq 1$. Then

$$||R_\lambda(M)|| \leq \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \frac{(|\beta_p N_{2p}(M_i)|)^{m+p}}{p^m+1(M,\lambda) \sqrt{k!}} \quad (\lambda \notin \sigma(M)).$$  \hfill (2.11)

Moreover, one has

$$||R_\lambda(M)|| \leq \sqrt{p} \sum_{m=0}^{p-1} \frac{(|\beta_p N_{2p}(M_i)|)^{m}}{p^m+1(M,\lambda)} \exp \left[\frac{(\beta_p N_{2p}(M_i))^{2p}}{2p^{2p}(M,\lambda)}\right] \quad (\lambda \notin \sigma(M)).$$  \hfill (2.12)

For the proof in the case $p > 1$ see [8, Theorem 7.9.1]. The case $p = 1$ is proved in [8, Theorem 7.7.1]. Besides, $\beta_p$ can be replaced by $\hat{\beta}_p$ according to (1.8).

3. The Finite Dimensional Case

In this section we apply Lemma 2.3 to an $n \times n$-matrix $A$ whose eigenvalues are different and are enumerated in the increasing way of their absolute values. We define

$$\delta(A) := \min_{j,k=1,\ldots,n:k\neq j} |\lambda_j(A) - \lambda_k(A)| > 0.$$  \hfill (3.1)

Hence, there is an invertible matrix $T_n \in \mathbb{C}^{n \times n}$ and a normal matrix $D_n \in \mathbb{C}^{n \times n}$, such that

$$T_n^{-1}A T_n = D_n.$$  \hfill (3.2)

Furthermore, for a fixed $m \leq n$ put

$$\delta_m(A) = \inf_{m=1,2,\ldots,n:k\neq j} |\lambda_j(A) - \lambda_m(A)|.$$
Let \( \{ e_k \} \) be the Schur basis (the orthogonal normal basis of the triangular representation) of matrix \( A \):

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  0 & a_{22} & a_{23} & \cdots & a_{2n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & a_{nn}
\end{pmatrix}
\]

with \( a_{jj} = \lambda_j(A) \). Take \( P_j = \sum_{k=1}^j (., e_k)e_k \). \( B_0 = A, \Delta P_k = (., e_k)e_k \),

\[
Q_j = \sum_{k=j+1}^n (., e_k)e_k, A_k = \Delta P_k A \Delta P_k = \lambda_k(A) \Delta P_k,
\]

\[
B_j = Q_j A Q_j = \begin{pmatrix}
  a_{j+1,j+1} & a_{j+1,j+2} & \cdots & a_{j+1,n} \\
  0 & a_{j+2,j+2} & \cdots & a_{j+2,n} \\
  \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & a_{nn}
\end{pmatrix},
\]

\[
C_j = \Delta P_j A Q_j = \begin{pmatrix}
  a_{j+1,j+1} & a_{j+1,j+2} & \cdots & a_{j+1,n} \\
  0 & a_{j+2,j+2} & \cdots & a_{j+2,n} \\
  \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & a_{nn}
\end{pmatrix}
\]

and

\[
D_n = \text{diag}(\lambda_k(A)).
\]

In addition,

\[
A = \begin{pmatrix}
  A_1(A) & C_1 \\
  0 & B_1
\end{pmatrix}, B_1 = \begin{pmatrix}
  A_2(A) & C_2 \\
  0 & B_2
\end{pmatrix}, \ldots, B_j = \begin{pmatrix}
  A_{j+1}(A) & C_{j+1} \\
  0 & B_{j+1}
\end{pmatrix}
\]

\((j < n)\). So \( B_j \) is an upper-triangular \((n-j) \times (n-j)\)-matrix. Equation (2.4) takes the form

\[
\lambda_j(A) X_j - X_j B_j = -C_j.
\]

Since \( X_j = X_j Q_j \), we can write \( X_j (A)(A) Q_j - B_j) = C_j \). Therefore

\[
X_j = C_j (\lambda_j(A) Q_j - B_j)^{-1}.
\]

The inverse operator is understood in the sense of subspace \( Q_j \mathbb{C}^n \). Hence,

\[
\| X_j \| \leq \| C_j \| \| (\lambda_j(A) Q_j - B_j)^{-1} \|.
\]

Besides, due to (2.11)

\[
\| (\lambda_j(A) Q_j - B_j)^{-1} \| \leq \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \left( \beta_{p}^m N_2(B_j)^{kp+m} / \delta_j^{kp+m}(A) \sqrt{k!} \right),
\]

where \( B_{ji} \) is the imaginary Hermitian component of \( B_j \).

But \( N_2(B_{ji}) = N_2(Q_j A_j Q_j) \leq N_2(A_j) (j \geq 1) \). So

\[
\| (\lambda_j(A) Q_j - B_j)^{-1} \| \leq \frac{\tau_j(A)}{\delta_j(A)}
\]

where

\[
\tau_j(A) = \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \left( \beta_{p}^m N_2(A_j)^{kp+m} / \delta_j^{kp+m}(A) \sqrt{k!} \right).
\]

Consequently,

\[
\| X_j \| \leq \tau_j(A) \frac{\| C_j \|}{\delta_j(A)}.
\]
Take $T_n = \hat{T}_n$ as in (2.6) with $X_k$ defined by (3.5). Besides (2.9) and (2.10) imply
\[
\|T_n\| \leq \left(1 + \frac{1}{n-1} \sum_{j=1}^{n-1} \|X_j\|\right)^{n-1} \leq \left(1 + \frac{\tau(A)}{(n-1) \sum_{j=1}^{n-1} \|C_j\| \delta_j(A)}\right)^{n-1}
\]  \hspace{1cm} (3.6)
and
\[
\|T_n^{-1}\| \leq \left(1 + \frac{\tau(A)}{(n-1) \sum_{j=1}^{n-1} \|C_j\| \delta_j(A)}\right)^{n-1} \cdot \hspace{1cm} (3.7)
\]
But by the H"older inequality,
\[
\sum_{j=1}^{n-1} \frac{\|C_j\|}{\delta_j(A)} \leq \left(\sum_{j=1}^{n-1} \|C_j\|^2 p\right)^{1/2p} \zeta_q(A) \left(1/(2p) + 1/q = 1\right), \hspace{1cm} (3.8)
\]
where
\[
\zeta_q(A) := \left(\sum_{k=1}^{n-1} \frac{1}{\delta_k^q(A)}\right)^{1/q}.
\]
In addition,
\[
\|C_j\|^2 \leq \sum_{k=j+1}^{n} |a_{jk}|^2, j < n; \ C_n = 0,
\]
and
\[
4\|A_1 e_j\|^2 = \|(A - A') e_j\|^2 = \|a_{jj} - a_{\tilde{j}j}\|^2 + 2 \sum_{k=j+1}^{n} |a_{jk}|^2 \geq 2\|C_j\|^2; j < n.
\]
Thus, $\|C_j\| \leq \sqrt{2}\|A_1 e_j\|, j \leq n$ and therefore
\[
\sum_{j=1}^{n-1} \|C_j\|^2 p \leq 2^p \sum_{j=1}^{n-1} \|A_1 e_j\|^2 p.
\]
But from Lemmas II.4.1 and II.3.4 [12], it follows that
\[
\sum_{j=1}^{n-1} \|A_1 e_j\|^2 p \leq N_{2p}^p(A_1).
\]
Therefore relations (3.6)-(3.8) with the notation
\[
\psi_{n,p}(A) = \left(1 + \frac{\tau(A) \sqrt{2}N_{2p}^p(A_1)\zeta_q(A)}{n-1}\right)^{n-1}
\]
imply $\|T_n\| \leq \psi_{n,p}(A)$ and $\|T_n^{-1}\| \leq \psi_{n,p}(A)$.

We thus have proved the following.

**Lemma 3.1.** Let condition (3.1) be fulfilled. Then there is an invertible operator $T_n$, such that (3.2) holds with $\kappa_{T_n} := \|T_n^{-1}\|/\|T_n\| \leq \psi_{n,p}^2(A)$. 


According to (2.12) one can replace \( \tau(A) \) by

\[
\tau(A) := \sqrt[n]{\sum_{m=0}^{n-1} \frac{(\beta_p N_{2p}(A))^{m}}{\delta^m(A)}} \exp \left[ \frac{(\beta_p N_{2p}(A))^{2p}}{2 \delta^{2p}(A)} \right]
\]

and therefore

\[
\kappa_{\tau} \leq \hat{\psi}^2_{r.p}(A), \quad (3.9)
\]

where

\[
\hat{\psi}_{r.p}(A) = \left( 1 + \frac{\hat{\tau}(A) \sqrt{2} N_{2p}(A) \zeta_p(A)}{n-1} \right)^{n-1}.
\]

The previous lemma and (3.9) improve the bound from [9, 10] for the condition numbers of matrices with large \( n \).

4. Proof of Theorem 1.1

Recall the Keldysh theorem, cf. [12, Theorem V. 8.1].

**Theorem 4.1.** Let \( A = S(I + K) \), where \( S = S^* \in SN \), for some \( r \in [1, \infty) \) and \( K \) is compact. In addition, let from \( Af = 0 \) \( (f \in \mathcal{S}) \) it follows that \( f = 0 \). Then \( A \) has a complete system of root vectors.

We need the following result.

**Lemma 4.2.** Under the hypothesis of Theorem 1.1, operator \( H^{-1} \) has a complete system of root vectors.

**Proof.** We can write \( H = H_R + i H_I \) with the notation \( H_R = (H + H^*)/2 \). For any real \( c \) with \( -c \notin \sigma(H) \cup \sigma(H_R) \) we have

\[
(H + cl)^{-1} = (I + i(H_R + cl)^{-1}H_I)^{-1}(H + cl)^{-1}.
\]

But \( (I + i(H_R + cl)^{-1}H_I)^{-1} - I = K_0 \), where \( K_0 = -i(H_R + cl)^{-1}H_I(I + i(H_R + cl)^{-1}H_I)^{-1} \) is compact. So

\[
(H + cl)^{-1} = (H_R + cl)^{-1}(I + K_0) \quad (4.1)
\]

Due to (1.1) \( (H + cl)^{-1} = H^{-1}(I + cH^{-1})^{-1} \in SN_r \). Hence

\[
(H_R + cl)^{-1} = (I + i(H_R + cl)^{-1}H_I)(H + cl)^{-1} \in SN_r
\]

and therefore by (4.1) and the Keldysh theorem operator \( (H + cl)^{-1} \) has a complete system of roots vectors. Since \( (H + cl)^{-1} \) and \( H^{-1} \) commute, \( H^{-1} \) has a complete system of roots vectors, as claimed. \( \square \)

From the previous lemma it follows that there is an orthonormal (Schur) basis \( \{ \hat{e}_k \}_{k=1}^\infty \), in which \( H^{-1} \) is represented by a triangular matrix (see [12, Lemma I.4.1]). Denote \( \hat{P}_k = \sum_{j=1}^k \langle \cdot, \hat{e}_j \rangle \hat{e}_j \). Then

\[
H^{-1}\hat{P}_k = \hat{P}_k H^{-1}\hat{P}_k \quad (k = 1, 2, ...).
\]

Besides,

\[
\Delta\hat{P}_k H^{-1} \Delta\hat{P}_k = \lambda_k^{-1}(H)\Delta\hat{P}_k \quad (\Delta\hat{P}_k = \hat{P}_k - \hat{P}_{k-1}, \; k = 1, 2, ...; \hat{P}_0 = 0). \quad (4.2)
\]

Put

\[
D = \sum_{k=1}^\infty \lambda_k \Delta\hat{P}_k \quad (\Delta\hat{P}_k = \hat{P}_k - \hat{P}_{k-1}, \; k = 1, 2, ...) \text{ and } V = H - D.
\]

We have

\[
H\hat{P}_k f = \hat{P}_k H\hat{P}_k f \quad (k = 1, 2, ...; \; f \in Dom(H)). \quad (4.3)
\]
Indeed, $H^{-1}\hat{P}_k$ is an invertible $k \times k$ matrix, and therefore, $H^{-1}\hat{P}_k \mathcal{S}$ is dense in $\hat{P}_k \mathcal{S}$. Since $\Delta \hat{P}_k = 0$ for $j > k$, we have $0 = \Delta \hat{P}_k H H^{-1} \hat{P}_k = \Delta \hat{P}_k H \hat{P}_k H^{-1} \hat{P}_k$. Hence $\Delta \hat{P}_k H f = 0$ for any $f \in \hat{P}_k H$. This implies (4.3).

Furthermore, put $H_n = H P_n$. Due to (4.3) we have

$$||H_n f - H f|| \to 0 \quad (f \in \text{Dom}(H)) \quad \text{as} \quad n \to \infty.$$  \hfill (4.4)

From Lemma 3.1 and (4.4) with $A = H_n$ it follows that in $\hat{P}_n \mathcal{S}$ there is an invertible operator $T_n$ such that $T_n H_n = \hat{P}_n D T_n$ and

$$||T_n|| \leq \psi_{n,p}(H_n) := (1 + \frac{\tau(H_n) \sqrt{2N_{2p}(H_n)}}{n})^{p-1}$$

where

$$\tau(H_n) = \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \frac{(\beta_p N_{2p}(H_n))^k}{\delta^{p+m}(H_n) \sqrt{k}}.$$  

It is clear, that

$$\tau(H_n) \sqrt{2N_{2p}(H_n)} \zeta_q(H_n) \leq \tau(H) \sqrt{2N_{2p}(H)} \zeta_q(H) = u_p(H)$$

and therefore

$$||T_n|| \leq (1 + \frac{u_p(H)}{n})^{p-1} \leq e^{\psi_{n,H}}.$$  

Similarly, $||T_n^{-1}|| \leq e^{\psi_{n,H}}$.

So there is a weakly convergent subsequence $T_n$, whose limit we denote by $T$. It is simple to check that $T_n = P_n T$. Since projections $P_n$ converge strongly, subsequence $\{T_n\}$ converges strongly. Thus $T_n H_n f \to TH f$ strongly and, therefore $\hat{P}_n D T_n f = T_n H_n f \to TH f$ strongly. Letting $n_i \to \infty$ hence we arrive at the required result. \hfill \Box

**Inequality (1.7)** follows from (3.9) according to the above arguments.

5. Operators with Hilbert - Schmidt Components

In this section in the case $p = 1$ we slightly improve Theorem 1.1. Besides, the misprint in the main result from [11] is corrected.

Denote

$$g(H) := \sqrt{2} [N_2^2(H)] - \sum_{k=1}^{\infty} \{ \text{Im} \lambda_k(H)^2 \}^{1/2} \leq \sqrt{2} N_2(H),$$

and

$$\tau_2(H) := \sum_{k=0}^{\infty} \frac{g^{k+1}(H)}{\sqrt{k!} \delta^{k}(H)}.$$  

**Theorem 5.1.** Let conditions (1.1) and (1.2) be fulfilled with $p = 1$. Then there are an invertible operator $T$ and a normal operator $D$ acting in $\mathcal{S}$, such that (1.5) holds. Moreover,

$$k_T \leq 2 \zeta_2(H) \tau_2(H).$$  \hfill (5.1)

**Proof.** Let $A$ be an $n \times n$-matrix whose eigenvalues are different. Define $\delta(A)$, $\delta_n(A)$ and $\zeta_2(A)$ as in Section 3. We have

$$g(A) := \sqrt{2} [N_2^2(A)] - \sum_{k=1}^{n} \{ \text{Im} \lambda_k(A)^2 \}^{1/2}.$$  

Put

$$\tau_2(A) := \sum_{k=0}^{n-2} \frac{g^{k+1}(A)}{\sqrt{k!} \delta^{k}(A)} \quad \text{and} \quad \gamma_n(A) := \left(1 + \frac{\zeta_2(A) \tau_2(A)}{n-1}\right)^{2(n-1)}.$$
Due to Lemma 3.1 from [11], there are an invertible matrix $M_n \in \mathbb{C}^{m \times m}$ and a normal matrix $D_n \in \mathbb{C}^{m \times m}$, such that $M_n^{-1}AM_n = D_n$, and
\begin{equation}
||M_n^{-1}|| = \gamma_n(A).
\end{equation}

Now take $H_n$ and $\hat{P}_n$ as in the proof of Theorem 1.1 from which it follows that in $\hat{P}_n\hat{S}$ there is an invertible operator $T_n$ such that $T_nH_n = \hat{P}_nDT_n$. Besides, according to (5.2)
\begin{equation}
||T_n^{-1}|| = \left( 1 + \frac{\tilde{\varepsilon}_2(H_n)\tau_2(H_n)}{\rho_n - 1} \right)^{2(n-1)}
\end{equation}

with
\begin{equation}
\tau_2(H_n) = \sum_{k=0}^{\infty} \frac{g^{k+1}(H_n)}{\sqrt{k!}g_k(H_n)}.
\end{equation}

It is simple to see that $\varepsilon_2(H_n) \leq \varepsilon_2(H)$, $\tau_2(H_n) \leq \tau_2(H)$ and thus
\begin{equation}
||T_n^{-1}|| \leq e^{2\varepsilon_2(H)\tau_2(H)}.
\end{equation}

Hence taking into account (4.4) and that a subsequence of $\{T_n\}$ strongly converges (see the proof of Theorem 1.1), we arrive at the required result. $\Box$

6. Applications of Theorem 1.1

Rewrite (1.5) as $Hx = T^{-1}DTx$. Let $\Delta P_k$ be the eigenprojections of the normal operator $D$ and $E_k = T^{-1}\Delta P_kT$. Then
\begin{equation}
Hx = \sum_{k=1}^{\infty} \lambda_k(H)E_kx \quad (x \in \text{Dom}(H)).
\end{equation}

Let $f(z)$ be a scalar function defined and bounded on the spectrum of $H$. Put
\begin{equation}
f(H) = \sum_{k=1}^{\infty} f(\lambda_k(H))E_k
\end{equation}

and
\begin{equation}
\gamma_p(H) = e^{2\varepsilon_p(H)}.
\end{equation}

Theorem 1.1 immediately implies.

**Corollary 6.1.** Let conditions (1.1) and (1.2) hold. Then $||f(H)|| \leq \gamma_p(H) \sup_{\lambda \in \sigma(H)} |f(\lambda_k(H))|$.

In particular, we have
\begin{equation}
||e^{-tH}|| \leq \gamma_p(H)e^{-\beta(H)t} \quad (t \geq 0),
\end{equation}

where $\beta(H) = \inf_k \Re \lambda_k(H)$ and
\begin{equation}
||R_t(H)|| \leq \frac{\gamma_p(H)}{\rho(H, \lambda)} \quad (\lambda \notin \sigma(H)).
\end{equation}

Let $A$ and $\hat{A}$ be linear operators. Then the quantity
\begin{equation}
sv(A) := \sup_{t \in \sigma(\hat{A})} \inf_{s \in \sigma(A)} |t - s|
\end{equation}
is said to be the variation of $\hat{A}$ with respect to $A$.

Now let $\hat{H}$ be a linear operator in $\hat{S}$ with $\text{Dom}(H) = \text{Dom}(\hat{H})$ and
\begin{equation}
\xi := ||\hat{H} - \hat{H}|| < \infty.
\end{equation}

From (6.1) it follows that $\lambda \notin \sigma(\hat{H})$, provided $\xi \gamma_p(H) < \rho(H, \lambda)$. So for any $\mu \in \sigma(\hat{H})$ we have $\xi \gamma_p(H) \geq \rho(H, \mu)$. This inequality implies our next result.
Corollary 6.2. Let conditions (1.1), (1.2) and (6.2) hold. Then $sv_{tH} (\tilde{H}) \leq \xi_{t\gamma_p} (H)$.

Now consider unbounded perturbations. To this end put

$$H^{-\nu} = \sum_{k=1}^{\infty} \lambda_k^{-\nu}(H) E_k \ (0 < \nu \leq 1).$$

Similarly $H^\nu$ is defined. We have

$$\|H^\nu R\lambda (H)\| \leq \frac{\gamma(H)}{\phi_\nu (H, \lambda)} \ (\lambda \notin \sigma(H)),$$

where

$$\phi_\nu (H, \lambda) = \inf_k (\lambda - \lambda_k (H)) \lambda_k^{-\nu}(H).$$

Now let $\tilde{H}$ be a linear operator in $\mathfrak{H}$ with $\text{Dom}(H) = \text{Dom}(\tilde{H})$ and

$$\xi_{\nu} := \| (H - \tilde{H}) H^{-\nu} \| < \infty.$$ (6.4)

Take into account that

$$R\lambda (H) - R\lambda (\tilde{H}) = R\lambda (H) (\tilde{H} - H) R\lambda (H) = R\lambda (\tilde{H}) (\tilde{H} - H) H^{-\nu} H R\lambda (H).$$

Thus, $\lambda \notin \sigma(\tilde{H})$, provided the conditions (6.4) and $\xi_{\nu} \gamma_p (H) < \phi_\nu (H, \lambda)$ hold. So for any $\mu \in \sigma(\tilde{H})$ we have

$$\xi_{\nu} \gamma_p (H) \geq \phi_\nu (H, \mu).$$ (6.5)

The quantity

$$\nu - \text{rsv}_{tH} (\tilde{H}) := \sup \inf \ |t - s|^{-\nu}$$

is said to be the $\nu -$ relative spectral variation of operator $\tilde{H}$ with respect to $H$. Now (6.5) implies.

Corollary 6.3. Let conditions (1.1), (1.2) and (6.4) hold. Then $\nu - \text{rsv}_{tH} (\tilde{H}) \leq \xi_{\nu} \gamma_p (H)$.

References


