

Stancu-Type Generalizations of the Chan-Chyan-Srivastava Operators

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Abstract. We give the Stancu-type generalization of the operators which is given by Erkus-Duman and Duman in this study. We derive approximation theorems via A-statistical Korovkin-type result. We also give rate of convergence of the operators via the modulus of smoothness, the modulus of continuity, and Lipschitz class functional.

1. Introduction

Erkus et al. [8] gave the following operators

\[ L_n^{(\beta)}(f;x) = \left( \prod_{i=1}^{r}(1 - xu_i^{(j)})^{\beta} \right) \sum_{m=0}^{\infty} \sum_{k_1 + k_2 + \ldots + k_r = m} f \left( \frac{k_i}{m + k_r - 1} \right) \prod_{j=1}^{r} \left( u_n^{(j)}(1) \right) \frac{k_i!}{k_j!} x^m, \]

where \( f \in C[0,1], \) \( \beta := (u^{(1)}, \ldots, u^{(r)}) \) with \( u_n^{(j)} := \left( u_n^{(j)} \right)_{n \in \mathbb{N}} \) and \( 0 < u_n^{(j)} < 1 \) and \( n \in \mathbb{N}, r \in \mathbb{N}, x \in [0,1], \)

\( (\lambda)_k = \lambda (\lambda + 1) \ldots (\lambda + k - 1), (\lambda)_0 = 1. \)

The operators \( L_n^{(\beta)} \) are obtained from the definition of Chan-Chyan-Srivastava (CCS) polynomials (see [2, 16])

\[ g_{m}^{(\alpha_1, \ldots, \alpha_r)}(x_1, \ldots, x_r) = \sum_{k_1 + \ldots + k_r = m} \left\{ \prod_{i=1}^{r} (\alpha_i)_k \frac{x_i^k}{k_i!} \right\}, \]

which have been generated by

\[ \prod_{i=1}^{r}(1 - tx_i)^{-\alpha_i} = \sum_{m=0}^{\infty} g_{m}^{(\alpha_1, \ldots, \alpha_r)}(x_1, \ldots, x_r) t^m, \]

where \(|t| < \min(|x_1|^{-1}, \ldots, |x_r|^{-1})\) and \( \alpha_1, \ldots, \alpha_r \) are complex parameters.
Erkus-Duman and Duman [7] considered a new generalization of the Kantorovich-type of the operators \(L_n^{(r)}\) given as

\[
K_n^{(r)}(f;x) = \left\{ \prod_{i=1}^{r} (1 - xu_n^{(i)})^p \right\} \sum_{m=0}^{\infty} \sum_{k_1+k_2+\ldots+k_r=m} (n + k_r - 1) \left\{ \prod_{j=1}^{r} \frac{(u_n^{(j)})^{k_j} (n)_{k_j}}{k_j!} \int_{\frac{t}{n+1}} f(t) \, dt \right\} x_m, \quad (4)
\]

where all parameters are stated above. They also defined the \(s\)-th order generalization of the operators \(K_n^{(r)}\) which was first introduced by Kirov and Popova [14]. They proved the approximation theorems for the operators \(K_n^{(r)}\) via \(A\)-statistical convergence. Further they obtained rates of convergence in care of modulus of continuity, Lipschitz class functionals and Peetre’s K-functional. Finally, they gave the valuable remarks and conclusions.

Many mathematicians observed and developed the approximation theory not only by describing new positive linear operators on a variety of functions spaces but also by generalizing the operators on behalf of the classical operators. So many powerful results have been obtained (see, [4, 6, 13, 17–19]). Our main interest of the present paper are to give Stancu-type generalization of the operators \(K_n^{(r)}\) as

\[
K_n^{(r)}(f;x) = \left\{ \prod_{i=1}^{r} (1 - xu_n^{(i)})^p \right\} \sum_{m=0}^{\infty} \sum_{k_1+k_2+\ldots+k_r=m} (n + k_r + \beta - 1) \times \left\{ \prod_{j=1}^{r} \frac{(u_n^{(j)})^{k_j} (n)_{k_j}}{k_j!} \int_{\frac{t}{n+1}} f(t) \, dt \right\} x_m, \quad (5)
\]

where \(f \in C[0,1], \gamma(r) := (u^{(1)}, \ldots, u^{(r)})\) with \(u^{(l)} := \{u_n^{(l)}\}_{n \in \mathbb{N}}\) and \(0 < u_n^{(i)} < 1\) and \(n \in \mathbb{N}, r \in \mathbb{N}, x \in [0,1]\), and \(0 \leq \alpha \leq \beta\).

We start \(r = 2\) in (5). Then, by taking \(u^{(1)} = u := \{u_n\}_{n \in \mathbb{N}}, u^{(2)} = v := \{v_n\}_{n \in \mathbb{N}},\) and \(\gamma(2) = (u,v),\) \(0 < u_n, v_n < 1\) for each \(n \in \mathbb{N},\) we see

\[
K_n^{\gamma(2)}(f;x) = (1 - xu_n)^{\alpha} (1 - xv_n)^{\alpha} \sum_{m=0}^{\infty} \sum_{k=0}^{m} (n + k + \beta - 1) \times \left\{ \frac{(u_n)^{m-k}(v_n)^k (m-k)_{k}}{(m-k)!} \int_{\frac{t}{n+1}} f(n + k + \beta - 1) \, dt \right\} x_m, \quad (6)
\]

where \(\alpha + 2 < n\) and \(0 \leq \alpha \leq \beta\).

2. Auxiliary Results

Lemma 2.1. For the operator and \(\|\cdot\|\) is supremum norm on \(C[0,1],\) we have

\[
\left\| K_n^{\gamma(2)}(e_0;x) - e_0 \right\| = 0 \quad \text{for each} \quad n \in \mathbb{N}. \quad (7)
\]
Proof. By the definition of the operators (6) and generating functions (3) we find that

$$K_n^{(2)}(e_0; x) = (1 - xu_n)^\alpha(1 - xv_n)^\beta \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{n + k + \beta - 1}{m - k} \int_{1/n}^{1} \frac{1}{\sqrt{t(1-t)}} e^{-nt} dt \chi^m$$

$$= (1 - xu_n)^\alpha(1 - xv_n)^\beta \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(u_*)^{m-k}v_*^{m-k}(n+1)}{(m-k)!k!} \chi^m$$

$$= 1.$$

Lemma 2.2. For the operator and \(\|\cdot\|_{[0,1]}\) is supremum norm, one can write for each \(n \in \mathbb{N}\)

$$\|K_n^{(2)}(e_1) - e_1\| \leq 1 - v_n + \frac{2\alpha + 1}{2n}.$$  \((8)\)

Proof. Let \(x \in [0,1]\). Using \((n)_k = (n+k-1)(n)_{k-1}\) and \(\frac{1}{n+k-1} \leq \frac{1}{n}\), then we can write

$$K_n^{(2)}(e_1; x) = (1 - xu_n)^\alpha(1 - xv_n)^\beta \sum_{m=0}^{\infty} \sum_{k=0}^{m} (n + k + \beta - 1) \left\{ \frac{(u_*)^{m-k}v_*^{m-k}(n+1)}{(m-k)!k!} \right\} \int_{1/n}^{1} \frac{1}{\sqrt{t(1-t)}} e^{-nt} dt \chi^m$$

$$= \frac{1}{2}(1 - xu_n)^\alpha(1 - xv_n)^\beta \sum_{m=0}^{\infty} \sum_{k=0}^{m} 2k + 2\alpha + 1 \kappa_n \chi^m$$

$$= xv_n(1 - xu_n)^\alpha(1 - xv_n)^\beta \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(u_*)^{m-k}v_*^{m-k}(n+1)}{(m-k)!k!} \chi^m + \frac{2\alpha + 1}{2} (1 - xu_n)^\alpha$$

$$\times (1 - xv_n)^\beta \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{k + n - 1} \left\{ \frac{(u_*)^{m-k}v_*^{m-k}(n+1)}{(m-k)!k!} \right\} \chi^m$$

$$\leq xv_n + \frac{2\alpha + 1}{2n}.$$  

Using above inequality we can get

$$K_n^{(2)}(e_1; x) - e_1(x) \leq x(v_n - 1) + \frac{2\alpha + 1}{2n}. \quad (9)$$

On the one hand, we have

$$K_n^{(2)}(e_1; x) \geq (1 - xu_n)^\alpha(1 - xv_n)^\beta \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{k}{k + n - 1} \left\{ \frac{(u_*)^{m-k}v_*^{m-k}(n+1)}{(m-k)!k!} \right\} \chi^m$$

$$\geq xv_n. \quad (10)$$

Then we can write

$$K_n^{(2)}(e_1; x) - e_1(x) \geq x(v_n - 1).$$

Thus by (9) and (10) and taking supremum over \(x \in [0,1]\), we obtain

$$\|K_n^{(2)}(e_1) - e_1\| \leq 1 - v_n + \frac{2\alpha + 1}{2n}. \quad \Box$$
Lemma 2.3. For the operator and \( \| \cdot \|_{C[0,1]} \) is supremum norm, one can write for each \( n \in \mathbb{N} \)

\[
\left\| K_n \gamma^{(2)}(e_2 - e_2) \right\| \leq 2(1 - v_n) + \frac{2(\alpha + 1)v_n}{n} + \frac{3\alpha^2 + 3\alpha + 1}{3n^2}.
\]

(11)

Proof. Let \( x \in [0, 1] \). From (6), \( (n)_k = (n + k - 1)(n - 1) \), and \( \frac{1}{n+k-1} \leq \frac{1}{n} \), we have

\[
K_n^{\gamma^{(2)}}(e_2; x) = (1 - xu_n)^{\alpha}(1 - xv_n)\sum_{m=0}^{\infty} \sum_{k=0}^{m} \left( x + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{k}{k + n - 1} \right) \left( \frac{(n)_k}{(m)_k} \right)^{\alpha} x^m
\]

\[
= (1 - xu_n)^{\alpha}(1 - xv_n)\sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{k}{k + n - 1} \left( \frac{(n)_k}{(m)_k} \right)^{\alpha} x^m
\]

\[
+ (2\alpha + 1)(1 - xu_n)^{\alpha}(1 - xv_n)\sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{k}{k + n - 1} \left( \frac{(n)_k}{(m)_k} \right)^{\alpha} x^m
\]

\[
+ \frac{(3\alpha^2 + 3\alpha + 1)}{3} (1 - xu_n)^{\alpha}(1 - xv_n)\sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{k}{k + n - 1} \left( \frac{(n)_k}{(m)_k} \right)^{\alpha} x^m
\]

\[
\leq xv_n(1 - xu_n)^{\alpha}(1 - xv_n)\sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{k}{k + n - 1} \left( \frac{(n)_k}{(m)_k} \right)^{\alpha} x^m
\]

\[
+ (2\alpha + 1)v_n(1 - xu_n)^{\alpha}(1 - xv_n)\sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{k + n - 1} \left( \frac{(n)_k}{(m)_k} \right)^{\alpha} x^m
\]

\[
+ \frac{(3\alpha^2 + 3\alpha + 1)}{3n^2}
\]

\[
\leq xv_n(1 - xu_n)^{\alpha}(1 - xv_n)\sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{k - 1 + 1}{k + n - 1} \left( \frac{(n)_k}{(m)_k} \right)^{\alpha} x^m
\]

\[
+ \frac{(2\alpha + 1)v_n}{n} + \frac{(3\alpha^2 + 3\alpha + 1)}{3n^2}
\]

\[
= x^2v_n^2(1 - xu_n)^{\alpha}(1 - xv_n)\sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{k + n - 2}{k + n - 1} \left( \frac{(n)_k}{(m)_k} \right)^{\alpha} x^m
\]

\[
+ xv_n(1 - xu_n)^{\alpha}(1 - xv_n)\sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{k + n - 1} \left( \frac{(n)_k}{(m)_k} \right)^{\alpha} x^m
\]

\[
+ \frac{(2\alpha + 1)v_n}{n} + \frac{(3\alpha^2 + 3\alpha + 1)}{3n^2}
\]

\[
\leq x^2v_n^2 + \frac{2(\alpha + 1)v_n}{n} + \frac{(3\alpha^2 + 3\alpha + 1)}{3n^2}.
\]

(12)
Thus we get
\[ K_n^{*\gamma(2)}(e_2;x) - e_2(x) \leq x^2 (v_n^2 - 1) + \frac{2(\alpha + 1)xv_n}{n} + \frac{(3\alpha^2 + 3\alpha + 1)}{3n^2}. \]

In the mean time, it is clear that
\[ 0 \leq K_n^{*\gamma(2)}((t - x)^2;x) = K_n^{*\gamma(2)}(e_2;x) - 2xK_n^{*\gamma(2)}(e_1;x) + x^2. \]

By (10), we get
\[ K_n^{*\gamma(2)}(e_2;x) - e_2(x) \geq 2xK_n^{*\gamma(2)}(e_1;x) - 2x^2 \geq 2x(\alpha x_n) - 2x^2 \geq 2x^2(x_n - 1). \]

From (12) and (13), we have
\[ \left| K_n^{\gamma(2)}(e_2;x) - e_2(x) \right| \leq 2x^2 (1 - v_n^2) + \frac{2(\alpha + 1)xv_n}{n} + \frac{(3\alpha^2 + 3\alpha + 1)}{3n^2}. \]

3. A–Statistical Convergence

A–Statistical convergence was first given by Freedman and Sember [10]. If \( A := (a_{jm}) \) is a non-negative regular summability matrix, \( x := (x_n) \) is a sequence which converge to a number \( L \) by means of \( A–statistically \) provided that,
\[ \lim_{n \to \infty} \sum_{n \mid x_n - L \geq \epsilon} a_{jn} = 0, \]
where every \( \epsilon > 0 \). This is shown by \( st_A - \lim x = L \). Replacing the matrix \( A \) by the identity matrix, \( A–statistical \) convergence coincide with classical convergence.

If a sequence is \( A–statistically \) convergent, then this sequence is ordinary convergent to the same value. Besides converse is not always right. Kolk [15] showed that \( A–statistical \) convergence is stronger than Classical convergence for \( A = (a_{jm}) \), any non-negative regular matrix, providing \( \lim_{n \to \infty} \max_{j} a_{jm} = 0. \)

Furthermore, for \( A = C_1, C_1 \) is the Cesàro matrix of order one, \( C_1–statistical \) convergence is equal to statistical convergence (see [9, 11]).

**Theorem 3.1.** For \( A = (a_{jm}) \) which is a non-negative regular summability matrix, one can write
\[ st_A - \lim_{n} \left\| K_n^{\gamma(2)}(f) - f \right\| = 0, \text{ all } f \in C[0,1], \] (14)
where \( \| \cdot \|_{C[0,1]} \) is supremum norm iff
\[ st_A - \lim_{n} v_n = 1. \] (15)

**Proof.** For all \( f \in C[0,1] \), assume that (14) holds. Then we get
\[ st_A - \lim_{n} \left\| K_n^{\gamma(2)}(e_1) - e_1 \right\| = 0. \]
From (9), it follows that
\[ 0 \leq x(1 - v_n) \leq e_1 - K_n^{(2)}(e_1; x) + \frac{2\alpha + 1}{2n}, \]
which gives
\[ 0 \leq 1 - v_n \leq \|e_1 - K_n^{(2)}(e_1)\| + \frac{2\alpha + 1}{2n}. \]  
(16)

Using the hypothesis and (16), we easily get (15). Let \( v = (v_n)_{n \in \mathbb{N}} \) be a sequence satisfying \( st_A - \lim_n v_n = 1 \). From lemma 2.1, it is clear that
\[ \lim_n \|K_n^{(2)}(e_0) - e_0\| = 0. \]

However, for \( A = (a_{jn}) \), because every convergent sequence is \( A \)-statically convergent to the same value, we have
\[ st_A - \lim_n \|K_n^{(2)}(e_0) - e_0\| = 0. \]  
(17)

Also, for a given \( \varepsilon > 0 \), define the following sets:
\[ D := \left\{ n : \|K_n^{(2)}(e_1) - e_1\| \geq \varepsilon \right\} \]
\[ D_1 := \left\{ n : 1 - v_n \geq \frac{\varepsilon}{2} \right\} \]
\[ D_2 := \left\{ n : \frac{2\alpha + 1}{n} \geq \varepsilon \right\}. \]

From lemma 2.2, it is simple to show that \( D \subseteq D_1 \cup D_2 \). We get
\[ \sum_{n \in D} a_{jn} \leq \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn} \]
for each \( j \in \mathbb{N} \). Then letting \( j \to \infty \) in (16) and using theorem condition, we finalize that
\[ \lim_j \sum_{n \in D} a_{jn} = 0, \]
which guarantees that
\[ st_A - \lim_n \|K_n^{(2)}(e_1) - e_1\| = 0. \]  
(18)

In the meanwhile, using the following sets
\[ D := \left\{ n : \|K_n^{(2)}(e_2) - e_2\| \geq \varepsilon \right\}, \]
\[ D_1 := \left\{ n : 1 - v_n^2 \geq \frac{\varepsilon}{6} \right\}, \]
\[ D_2 := \left\{ n : \frac{(\alpha + 1)v_n}{n} \geq \frac{\varepsilon}{6} \right\}, \]
\[ D_3 := \left\{ n : \frac{3\alpha^2 + 3\alpha + 1}{n^2} \geq \varepsilon \right\}, \]
and applying lemma 2.3, we see that $D \subseteq D_1 \cup D_2 \cup D_3$. For each $j \in \mathbb{N}$, we have

$$\sum_{n \in D} a_{jn} \leq \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn} + \sum_{n \in D_3} a_{jn}. \quad (19)$$

Since $st_A - \lim v_n = 1$, observe that $st_A - \lim (1 - v_n^2) = 0$ and $st_A - \lim \frac{v_n}{n-1} = 0$. Using the above and taking the limit as $j \to \infty$ in (19), we finalize that

$$\lim_{j \to \infty} \sum_{n \in D} a_{jn} = 0,$$

which guarantees that

$$st_A - \lim_{n} \|K_n^{\gamma(2)}(e_2) - e_2\| = 0. \quad (20)$$

Now, combining (17),(18) and (20) and considering the statistical Korovkin theorem ([5, 12]), we complete the proof. \qed

4. Rate of Convergence

We denote the usual modulus of continuity of $f \in C_{\mathbb{R}}[0, \infty)$ by

$$\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x + h) - f(x)|.$$ 

Theorem 4.1. For all $f \in C[0, 1]$ and each $n \in \mathbb{N}$, we get

$$\|K_n^{\gamma(2)}(f) - f\| \leq 2\omega(f, \delta_n). \quad (21)$$

Here $\omega(f, \delta_n)$ is the classical modulus of continuity of $f$ and

$$\delta_n = \left\{ \frac{2(1 - v_n - v_n^2)}{\alpha + 1}\frac{v_n}{n} + \frac{3\alpha^2 + 3\alpha + 1}{3n^2} + \frac{2\alpha + 1}{n} \right\}^{1/2}. \quad (22)$$

Proof. Let $x \in [0, 1]$ and $f \in C[0, 1]$. One can easily have that, for any $\delta > 0$,

$$\left| K_n^{\gamma(2)}(f; x) - f(x) \right| \leq K_n^{\gamma(2)}(\|f(y) - f(x)\|; x) \leq \omega(f, \delta) \left\{ 1 + \frac{1}{\delta} (K_n^{\gamma(2)}(\alpha_1 - x_2; x))^{1/2} \right\}. \quad (23)$$

Using the Cauchy-Schwarz inequality, one can write

$$\left| K_n^{\gamma(2)}(f; x) - f(x) \right| \leq \omega(f, \delta) \left\{ 1 + \frac{1}{\delta} (K_n^{\gamma(2)}(\alpha_1, \alpha_2; x))^{1/2} \right\}. \quad (24)$$

Also we see that

$$K_n^{\gamma(2)}(\alpha_1 - x_2; x) \leq \left| K_n^{\gamma(2)}(\alpha_2; x) - \alpha_2(x) \right| + 2\alpha \left| K_n^{\gamma(2)}(\alpha_1(x); x) - \alpha_1(x) \right|.$$

Then using lemma 2.2 and lemma 2.3, we deduce

$$\|K_n^{\gamma(2)}(\alpha_1 - x_2; x)\| \leq 2(1 - v_n^2) + \frac{2(\alpha + 1)v_n}{n} + \frac{3\alpha^2 + 3\alpha + 1}{3n^2} + 2(1 - v_n) + \frac{2\alpha + 1}{n - 1}. \quad (24)$$

Now, using inequality (24) and taking $\delta = \delta_n$ given by (22), we conclude from (23) that

$$\|K_n^{\gamma(2)}(f) - f(x)\| \leq 2\omega(f, \delta_n)$$

which is the desired result. \qed
Theorem 4.3. For all $f \in C[0,1]$ belongs to $Lip_M(\alpha) (M > 0$ and $0 < \alpha \leq 1)$, if

$$|f(x) - f(y)| \leq M|x - y|^{\alpha} , \ x, y \in [0,1].$$

(25)

We shall show the rate of convergence of the operators $K_n^{(2)}$ via functions belong to Lipschitz class with the next theorem.

**Theorem 4.2.** For all $f \in Lip_M(\alpha)$ and for each $n \in \mathbb{N}$, we get

$$\left\|K_n^{(2)}(f) - f\right\| \leq M\delta_n^{2},$$

(26)

where $Lip_M(\alpha)$ $(M > 0$ and $0 < \alpha \leq 1)$ is Lipschitz space and $\delta_n$ is the same as in (22).

**Proof.** From (25), we have

$$\left|K_n^{(2)}(f; x) - f(x)\right| \leq Mk_n^{(2)}\left((y - x)^{\alpha} ; x\right).$$

From the H"{o}lder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2 - \alpha}$, one can easily see that

$$\left|K_n^{(2)}(f; x) - f(x)\right| \leq M\left(K_n^{(2)}((y - x)^{2} ; x)\right)^{\alpha/2}. $$

(27)

Following way in the proof of Theorem 4.1, one make a deduction from (22) and (27) that

$$\left\|K_n^{(2)}(f) - f(x)\right\| \leq M\delta_n^{2}$$

which is the desired results. \qed

Let $C_0[0,\infty)$ be the space of all real valued continuous bounded function $f$ on $[0,\infty)$. Here the norm is given by $\|f\| = \sup_{x \in [0,\infty]} |f(x)|$. Further we consider the Peetre’s $K$-functional (see [1]) as

$$K_2(f, \delta) = \inf_{g \in C_0[0,\infty]} \left\{\|f - g\| + \delta\|g''\|\right\},$$

where $\delta > 0$ and $C_0^2[0,\infty) := \{g \in C_0[0,\infty), g', g'' \in C_0[0,\infty]\}$.

From [3], there exist an absolute constant $C > 0$ satisfying the property

$$K_2(f, \delta) \leq C\omega_2(f, \sqrt{\delta}),$$

where $\omega_2(f, \delta)$, the second-order modulus of smoothness, of $f \in C_0[0,\infty)$ given by

$$\omega_2(f, \delta) = \sup_{0< h \leq \delta} \sup_{x \in [0,\infty)} \left|f(x + 2h) - 2f(x + h) + f(x)\right|.$$ 

**Theorem 4.3.** For all $f \in C[0,1]$ and for each $n \in \mathbb{N}$, we get

$$\left\|K_n^{(2)}(f) - f\right\| \leq C\omega_2\left(f, \sqrt{\epsilon_n}\right) + \omega_2\left(1, v_n + \frac{2\alpha + 1}{2n}\right),$$

(28)

where

$$\epsilon_n = (5 - 3v_n - 2v_n^2) + \frac{2(\alpha + 1)v_n}{n} + \frac{3\alpha^2 + 3\alpha + 1}{3n^2} + \frac{3(2\alpha + 1)}{2n}$$

and $C = 4L, L > 0$. 

Proof. We define the new operators $\bar{K}_n^{-\gamma(r)}$ as
\[
\bar{K}_n^{-\gamma(r)} (f; x) = K_n^{-\gamma(r)} (f; x) - f \left( \frac{2\alpha + 1}{2n} \right) + f(x).
\] (29)
So from (9) and (10), we have
\[
\bar{K}_n^{-\gamma(r)} (t - x; x) = 0
\] (30)
For $g \in C_0^1 [0, \infty)$, $t \in [0, \infty)$, using Taylor’s expansion, it is hold that
\[
g(t) = g(x) + g'(x) (t - x) + \int_x^t (t - u) g''(u) du.
\]
In the view of (30), we obtain
\[
\bar{K}_n^{-\gamma(r)} (g; x) - g(x) = \bar{K}_n^{-\gamma(r)} \left( \int_x^t (t - u) g''(u) du; u \right).
\]
From (29), we have
\[
\left| \bar{K}_n^{-\gamma(r)} (g; x) - g(x) \right| \leq \left| \bar{K}_n^{-\gamma(r)} \left( \int_x^t (t - u) g''(u) du; x \right) \right| + \left| \int_x^t \left( x \frac{2\alpha + 1}{2n} - u \right) g''(u) du \right|
\]
\[
\leq \left| \bar{K}_n^{-\gamma(r)} ((t - x)^2; x) \right| + \left| \left( x (1 - v_n) + \frac{2\alpha + 1}{2n} \right) \right| \left| g'' \right|
\]
\[
\leq \left| \bar{K}_n^{-\gamma(r)} ((t - x)^2; x) \right| + \left| \left( x (1 - v_n) + \frac{2\alpha + 1}{2n} \right) \right| \left| g'' \right|
\]
Using (24), we get
\[
\left| \bar{K}_n^{-\gamma(r)} (g; x) - g(x) \right| \leq \left[ 2(1 - v_n) + \frac{2(\alpha + 1)v_n}{n} + 3(1 - v_n) + \frac{3\alpha^2 + 3\alpha + 1}{3n^2} + \frac{3(2\alpha + 1)}{2n} \right] \left| g'' \right|
\]
From (29) and the above inequality, we have
\[
\left| K_n^{-\gamma(r)} (f; x) - f(x) \right| \leq \left| K_n^{-\gamma(r)} (f - g; x) - (f - g)(x) \right| + \left| K_n^{-\gamma(r)} (g; x) - g(x) \right| + \left| f \left( x \frac{2\alpha + 1}{2n} \right) - f(x) \right|
\]
\[
\leq 4 \left| f - g \right| + \left| f \left( x \frac{2\alpha + 1}{2n} \right) - f(x) \right| \left[ 2(1 - v_n) + \frac{2(\alpha + 1)v_n}{n} + 3(1 - v_n) \right] + \left| \left( x (1 - v_n) + \frac{2\alpha + 1}{2n} \right) \right| \left| g'' \right|
\]
\[
\leq 4 \left| f - g \right| + \left| f \left( x \frac{2\alpha + 1}{2n} \right) - f(x) \right| \left[ 2(1 - v_n) + \frac{2(\alpha + 1)v_n}{n} + 3(1 - v_n) \right] + \left| \left( x (1 - v_n) + \frac{2\alpha + 1}{2n} \right) \right| \left| g'' \right|
\]
Taking infinitum both side of this inequality over all $g \in C_0^1 [0, \infty)$, we have
\[
\left| K_n^{-\gamma(r)} (f; x) - f(x) \right| \leq 4K_2 (f; \epsilon_n) + \omega \left( f, 1 - v_n + \frac{2\alpha + 1}{2n} \right) + \omega \left( f, 1 - v_n + \frac{2\alpha + 1}{2n} \right)
\]
\[
\leq C \omega_2 \left( f, \sqrt{v_n} \right) + \omega \left( f, 1 - v_n + \frac{2\alpha + 1}{2n} \right),
\]
where
\[
\epsilon_n = (5 - 3v_n - 2v_n^2) + \frac{2(\alpha + 1)v_n}{n} + \frac{3\alpha^2 + 3\alpha + 1}{3n^2} + \frac{3(2\alpha + 1)}{2n}
\]
and $C = 4L, L > 0$. Hence the theorem is proved. \qed
References


