Teichmüller Space of a Countable Set of Points on the Riemann Sphere

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Abstract. We introduce the Teichmüller space \( T(E) \) of an ordered countable set \( E \) of infinite number of distinct points on the Riemann sphere. We discuss the relation between the Teichmüller distance on \( T(E) \) and a natural one on the configuration space for \( E \). Also we give a system of global holomorphic coordinates for \( T(E) \) when \( E \) is determined from a finitely generated semigroup consisting of Möbius transformations with the totally disconnected forward limit set.

1. Introduction

Let \( E = \{z_k\}_{k=1}^{\infty} \) be an ordered countable set of an infinite number of distinct points on \( \hat{\mathbb{C}} \). We define a natural kind of the deformation space of \( E \) as follows.

Definition 1.1. Let \( QC(E) \) be the set of all ordered countable sets \( E' = \{z'_k\}_{k=1}^{\infty} \) of an infinite number of distinct points on \( \hat{\mathbb{C}} \) such that there are quasiconformal self-homeomorphisms \( f \) of \( \hat{\mathbb{C}} \) which are order-preserving from \( E \) onto \( E' \) in a sense that \( f(z_k) = z'_k \) for every \( k \).

We say that two points \( E_1 \) and \( E_2 \) of \( QC(E) \) are equivalent if there is a conformal self-homeomorphism, or equivalently a Möbius transformation, \( \phi \) of \( \hat{\mathbb{C}} \) which is order-preserving from \( E_1 \) onto \( E_2 \).

The Teichmüller space \( T(E) \) of \( E \) consists of all equivalence classes \( [E] \) of \( E \in QC(E) \).

The Teichmüller metric on \( T(E) \) is defined as usual.

Definition 1.2. The Teichmüller distance between \( [E_1] \) and \( [E_2] \) in \( T(E) \) is defined by setting

\[
d_T([E_1],[E_2]) = \inf_{g} \log K_g,
\]

where \( g \) moves all quasiconformal self-homeomorphisms \( g \) of \( \hat{\mathbb{C}} \) order-preserving from \( E_1 \) onto \( E_2 \).

It is clear that the Teichmüller distance \( d_T \) is actually a distance, and hence \( T(E) \) equipped with \( d_T \) is a metric space.

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Remark 1.3. We can define the Teichmüller space $T(E)$ of an ordered countable set of infinite number of points on a general Riemann surface $R$, and can equip $T(E)$ with a natural complex Banach manifold structure under some conditions. We will discuss it in a forthcoming paper. In this paper, we give some interesting examples in the next section instead.

Without loss of generality, we may assume in the sequel that an ordered countable set $E = \{z_k\}_{k=1}^\infty$ is normalized, i.e.

$$z_1 = \infty, \quad z_2 = 0, \quad z_3 = 1.$$  

Then, every point $[E']$ of $T(E)$ contains a single normalized ordered countable set, say $E'$, and hence in the sequel, we identify $T(E)$ with the set $NQC(E)$ of all normalized ones in $QC(E)$, and write $[E']$ simply as $E'$.

Definition 1.4. The configuration space of normalized ordered countable sets of infinite number of distinct points on $C - \{0, 1\}$ is the subset

$$\Sigma = \{\{w_k\}_{k=1}^\infty \in \Omega = (C - \{0, 1\})^\infty \mid w_m \neq w_n \text{ if } m \neq n\}$$

of the product space $C^\infty$.

The topology of $C^\infty$ is usually induced from component-wise convergence. In this paper, we assume that the subset $\Omega$, and hence also $\Sigma$, is equipped with the hyperbolic $\ell^\infty$ distance defined by

$$d\left([w_k], [w'_k]\right) = \sup_k d_h(w_k, w'_k)$$

for every $[w_k], [w'_k] \in \Omega$, where $d_h$ is the hyperbolic distance on $C - \{0, 1\}$. In general, $\Sigma$ is not necessarily open in $\Omega$.

Definition 1.5. Let $E$ be an ordered countable set as above. Then, there is a natural injection

$$\iota : T(E) = NQC(E) \to \Sigma,$$

by sending $E' = \{z'_k\}_{k=1}^\infty \in NQC(E)$ to $[z'_{k+3}]_{k=1}^\infty \in \Sigma$.

We have equipped $T(E)$ with the Teichmüller distance $d_T$, while the image $\iota(T(E))$ with the hyperbolic $\ell^\infty$ one. Hence, we need to clarify the relation between these two distances.

Here, we consider the following condition.

Definition 1.6. We say that a normalized ordered countable set $E = \{z_k\}$ is uniformly discrete if

$$\inf_{m,n \geq 4, m \neq n} d_h(z_m, z_n) > 0.$$  

Example 1.7. The density of the hyperbolic metric $d_h$ on $C - \{0, 1\}$ near $\infty$ is comparable with

$$\frac{1}{|z| \log |z|'}$$

and hence $z_k = a^{2^k}$ with $a > 1, k \geq 4$ give a uniformly discrete normalized ordered countable set.

Let $UD$ be the subset of $T(E)$ consisting of those which are uniformly discrete.

Lemma 1.8. The subset $UD$ is open in $T(E)$. 
Proof. Suppose that \( E' = \{ z'_m \} \in T(E) \) is uniformly discrete, and set

\[
a = \inf_{m,n \geq k, m \neq n} d_E(z'_m, z'_n) > 0.
\]

Then a classical theorem due to Teichmüller (cf. [1]) states that there is an \( \epsilon > 0 \) such that every \( K\text{-qc} \) self-homeomorphism \( f \) of \( C \) fixing 0, 1 with \( K < 1 + \epsilon \) satisfies

\[
d_E(z'_m, f(z'_m)) \leq a/3
\]

for every \( m \geq 4 \). Actually, \( \epsilon = e^{t/3} - 1 \) is available. Thus we conclude the assertion. \( \square \)

Theorem 1.9. The injection \( \iota : T(E) \to \Sigma \) is continuous and non-expanding. Moreover, \( \iota \) is a locally bi-Lipschitz homeomorphism of \( UD \) onto its image, i.e., for every \( E' \in UD \), there is a neighborhood \( V \) of \( E' \) and an \( M > 0 \) such that

\[
\frac{1}{M} d_\ell(E_1, E_2) \leq d(\iota(E_1), \iota(E_2)) \leq M d_\ell(E_1, E_2),
\]

for every \( E_1, E_2 \in V \).

Proof. Again by the classical theorem due to Teichmüller stated in the above proof, we see that

\[
d(\iota(E_1), \iota(E_2)) \leq d_\ell(E_1, E_2)
\]

for every \( E_1, E_2 \in T(E) \), which means that \( \iota \) is continuous and non-expanding.

Next, fix \( \{ w_k \}_{k=1}^\infty \) in \( \iota(UD) \). Then the assumption implies that

\[
a = \inf_{m,n} d_E(w_m, w_n) > 0.
\]

In particular, the \((a/3)\text{-neighborhoods} \) \( U_k \) of \( w_k \) with respect to the hyperbolic metric \( d_E \) are mutually disjoint in \( C - [0, 1] \). Here replacing \( a/3 \) by a smaller positive constant if necessary, we may assume that \( U_k \) is either a topological disk or an annulus for every \( k \). In the latter case, we can find a positive \( a' < a/3 \), depending only on \( a \), such that either the \( a'\text{-neighborhood of} \ w_k \) is a topological disk or the hyperbolic distance between \( w_k \) and the boundary of \( U_k \) is greater than \( a/4 \) for every \( k \).

In any case, there exists a positive \( \eta \) sufficiently smaller than \( a' \), satisfying the following condition: For every \( \{ w'_k \} \) in

\[
V = \{ \{ w'_k \} \in \Sigma | d \left( \{ w'_k \}, \{ w_k \} \right) < \eta \},
\]

we can construct explicitly a normalized quasiconformal self-homeomorphism \( f \) of \( C \), order-preserving from \( \{ w_k \} \) onto \( \{ w'_k \} \), such that \( f \) is the identical map on \( C - \bigcup_k U_k \) and the maximal dilatation of \( f \) is bounded by

\[
\exp \left( M d(\{ w'_k \}, \{ w_k \}) \right),
\]

where \( M \) is a positive constant depending only on \( a, a' \), and \( \eta \).

Thus we conclude that \( V \subset \iota(UD) \) and \( \iota \) is a locally bi-Lipschitz homeomorphism of \( UD \) to its image. \( \square \)

Remark 1.10. The continuous and non-expanding injection \( \iota \) seems not always to be locally bi-Lipschitz on \( T(E) \). Also see Remark 2.11 below.

On the other hand, the standard complex analytic structure of \( T(E) \), if exists, is given by using Beltrami coefficients, and the corresponding normalized quasiconformal homeomorphisms gives holomorphic families (cf. [4]). In particular, components \( w_k \) of \( \Sigma \) always induce some complex Banach manifold structure on \( T(E) \) if \( \iota(T(E)) \) is open in \( \Omega \).
2. Countable Sets Ordered by Möbius Action

Let $S$ be a semigroup generated by Möbius transformations of the Riemann sphere $\hat{C}$. We give a canonical order to $S$ by using the word length of the reduced word expressions and in addition by giving elements with same length the lexicographic order with respect to the ordered generators. So, we consider $S$ as an ordered set $\{\sigma_1, \ldots, \sigma_n\}$. Here we always assume that the identity $e$ is added to $S$ and $\sigma_1 = e$.

For the sake of simplicity, we restrict ourselves to the case that $S$ is finitely generated in the sequel.

Definition 2.1. Let $S = \{g_1, \ldots, g_n\}$ be a semigroup generated by a finite number of ordered non-identical Möbius transformations $g_1, \ldots, g_n$. Assume that $\infty$ is a fixed point of $g_1$ and that $g_k$ have mutually distinct fixed points in $C$.

We say that an ordered countable set $E$ of an infinite number of distinct points on $\hat{C}$ including $\infty$ is $S$-invariant if

$$g(E) \subset E \text{ for every } g \in S$$

and

$$\bigcup_{k=1}^{n} g_k(E) = E.$$

We say that the order of $E$ is induced from the $S$-action if there is an order-preserving injection

$$\tau : E \to S.$$

Remark 2.2. The second condition of $S$-invariance is equivalent to

$$E = \bigcup_{k=1}^{n} g_k(E).$$

If $S$ is a group, then the first condition implies that $g(E) = E$ for every $g \in S$, and hence the second condition is unnecessary.

For a countable set $E$ with an order induced from the $S$-action, we consider the Teichmüller space $T(E; S)$ of $E$ with $S$-action as follows.

Definition 2.3. Let $E, S = \{g_1, \ldots, g_n\}, \tau : E \to S$ be as above and QC($E; S$) the set of all triples $(E', S', \tau')$ of ordered countable sets $E' = \{\epsilon_{1k}\}_{k=1}^{\infty}$ of infinite number of distinct points on $\hat{C}$, semigroups $S' = \{g'_1, \ldots, g'_n\}$ isomorphic to $S$ by the isomorphisms $\sigma_{S'} : S \to S'$ which send $g_k$ to $g'_k$ for every $k$, and order-preserving injections $\tau' : E' \to S'$ such that $\tau'(E') = \sigma_{S'} \circ \tau(E)$ and there are quasiconformal self-homeomorphisms $f$ of $\hat{C}$ which equal $\tau^{-1} \circ \sigma_{S'} \circ \tau$ on $E$.

We say that two points $(E_1; S_1, \tau_1)$ and $(E_2; S_2, \tau_2)$ of QC($E; S$) are equivalent if there is a conformal self-homeomorphism $\phi$ of $\hat{C}$ such that $\phi(E_1) = E_2$ and

$$\tau_2 \circ \phi = (\sigma_{S_2} \circ \sigma_{S_1}^{-1}) \circ \tau_1$$
on $E_1$.

The Teichmüller space $T(E; S)$ of $E$ with $S$-action consists of all equivalence classes $[E; S, \tau]$ of $(E'; S', \tau') \in$ QC($E; S$), which we write simply by $[E]$ whenever $S'$ and $\tau'$ are clear.

The Teichmüller distance $d_T$ on $T(E; S)$ is defined similarly as before.

Now, we give two typical examples where the closures of all fixed points of elements in $S - \{e\}$ are totally disconnected. In each examples, we can give a system of global coordinates for the standard complex Banach manifold structure on $T(E; S)$.
Example 2.4. We consider a semigroup $S = [g_1, \ldots, g_n]$ generated of contractive similarities

$$
\begin{align*}
  g_1(z) &= \lambda_1 z, \\
  g_2(z) &= \lambda_2(z - 1) + 1, \\
  g_3(z) &= \lambda_3(z - \alpha_3) + \alpha_3, \\
  \vdots \\
  g_n(z) &= \lambda_n(z - \alpha_n) + \alpha_n.
\end{align*}
$$

Here, $n \geq 3, 0 < |\lambda_k| < 1$, and $0, 1, \alpha_3, \ldots, \alpha_n$ are distinct.

Assume that the attractor of the IFS (iterative function system) given by $S$ is dust-like. In other words, the forward limit set $\Lambda(S)$ is totally disconnected. For the backgrounds on IFSs, see [2] and the references of it.

The standard order of $S$ starts with $e, g_1, g_2, \ldots, g_n$, and so on. Let $E = \{z_k\}_{k=1}^\infty$ be the set consisting of $z_1 = \infty$ and all other fixed points of elements in $S$. Here, we have already normalized $S$ so that the fixed points $z_2$ and $z_3$ of $g_1$ and $g_2$ are $0$ and $1$, respectively. Also we set $z_{k+1} = \alpha_k$ for every $k = 3, \ldots, n$.

Define the injection $\tau : E \to S$ as in Definition 2.1 by sending $\infty, 0, 1, \ldots, \alpha_n$, and every other fixed point $z_k \in E \cap C$ of some $g_k$ to $e, g_1, \ldots, g_n$, and $g_k$ for every $k \geq n + 2$, respectively, where $g_{(k)}$ is assumed to have the smallest order among all $g \in S - [e]$ having $z_k$ as the fixed point, and assume that the order of $E$ is induced from the $S$-action by $\tau$. Then we can define $T(E; S)$.

By the normalization, we have a canonical continuous injection

$$
j : T(E; S) \to ND(n)
$$

by sending $[E', S'] = [g'_1, \ldots, g'_n, \tau']$ to $(g'_1, \ldots, g'_n)$. Here, $ND(n)$ is the subspace of $(CS)^n$ consisting of all $(g_1, \ldots, g_n)$ normalized as above, where $CS$ is the space of all contractive similarities.

Theorem 2.5 ([2]). The Teichmüller space $T(E; S)$ is identified with the dust-likeness locus $DL(n)$ consisting of all $S' = [g'_1, \ldots, g'_n] \in ND(n)$ with totally disconnected $\Lambda(S')$.

In particular, $T(E; S)$ is a domain in $ND(n)$, and the $(2n - 2)$ fixed points, say $z_4, \ldots, z_{2n+1}$, of

$$
g_3, \ldots, g_n, g_1 g_2, \ldots, g_1 g_n, g_2 g_1,
$$

in $E$ gives a system of global coordinates for $T(E; S)$.

Actually, the first assertion has been shown as Theorem 1.2 in [2].

Next, by definition, $z_{k+1} = \alpha_k$ for every $k = 3, \ldots, n$, and the fixed points $z_{2n+1}$ of $g_2 g_1$ and $z_{n+k}$ of $g_1 g_k$ with $k \geq 2$ are

$$
\frac{\lambda_2 - 1}{\lambda_1 \lambda_2 - 1} \quad \text{and} \quad \frac{\lambda_1 \alpha_k (\lambda_k - 1)}{\lambda_1 \lambda_k - 1},
$$

respectively. In particular,

$$
\lambda_1 = \frac{z_{2n+2}}{z_{2n+1}}, \quad \lambda_2 = \frac{z_{2n+1} - 1}{z_{n+2} - 1},
$$

and hence we can conclude that

$$
\lambda_k = \frac{z_{2n+1} - z_{n+k} - z_{n+2k+1}}{z_{n+2}(z_{n+k} - z_{k+1})}
$$

for $k = 3, \ldots, n$. It is clear that these relations gives a bi-rational homeomorphism of $T(E; S)$ in $C^{2n-2}$ with coordinates $z_4, \ldots, z_{2n+1}$ onto the domain $DL(n)$ in $C^{2n-2}$ with coordinates

$$
\lambda_1, \ldots, \lambda_n, \alpha_3, \ldots, \alpha_n.$$
**Example 2.6 (cf.[8]).** Let $G = \langle g_1, \cdots, g_n \rangle$ be a Schottky group, i.e., a finitely generated purely loxodromic free discrete group with totally disconnected limit set, generated by ordered $n \geq 2$ Möbius transformations $g_1, \cdots, g_n$. For the backgrounds on Kleinian groups, see for instance [6].

The standard order of $G$ starts with

$$e, g_1, g_1^{-1}, g_2, g_2^{-1}, \cdots, g_n, g_n^{-1},$$

and so on. We consider the set $E = \{z_k\}_{k=1}^m$ consisting of all fixed points of elements in $G - \{e\}$. We normalize $E$ so that $z_1 = \infty$ and $z_2 = 0$ are the repelling and the attracting fixed point of $g_1$, and $z_3 = 1$ is the repelling fixed point of $g_2$. Also we assume that $z_4$ is the attracting fixed point of $g_2$, and that $z_{2k-1}$ and $z_{2k}$ are the repelling and the attracting fixed points of $g_k$, respectively, for $k = 3, \cdots, n$. Recall that the sets of the fixed points of different $g \in G - \{e\}$ are mutually distinct and the attracting fixed point of $g$ is the repelling one of $g^{-1}$.

Define the injection $\tau: E \to G$ by sending the repelling fixed point $z_k$ of $g(k)$ to $g(k)$ for every $k \geq 2n + 1$, where $g(k)$ is assumed to have the smallest order among all $g \in G - \{e\}$ having $z_k$ as the repelling fixed points, and assume that the order of $E$ is induced from the $G$-action by $\tau$. Then we can define $T(E; G)$.

By the normalization, we have a canonical continuous injection

$$j: T(E; G) \to NDef(n)$$

by sending $[E'; G' = \langle g_1', \cdots, g_n' \rangle, \tau']$ to $\langle g_1', \cdots, g_n' \rangle$. Here, $NDef(n)$ is the subspace of $(PSL(2, \mathbb{C}))^n$ consisting of all $(g_1, \cdots, g_n)$ normalized as above, where $PSL(2, \mathbb{C})$ is the space of all Möbius transformations.

**Theorem 2.7.** The Teichmüller space $T(E; G)$ is identified with the normalized Schottky locus $NS(n)$ in $NDef(n)$ consisting of all normalized system of ordered generators of Schottky groups canonically isomorphic to $G$.

In particular, $T(E; G)$ is a domain in $NDef(n)$ and $3n - 3$ fixed points

$$z_{4j}, \cdots, z_{2n}, z_1, z_2, \cdots, z_{n-1}$$

in $E$ give a system of global coordinates for $T(E; G)$. Here $z_{k, j}$ is the repelling fixed points of $g_j g_k^{-1}$ for every $j$ and $k$ with $j \neq k$.

**Proof.** First, recall that $NS(n)$ is a domain in $NDef(n)$ and it is clear that $NS(n) \subset j(T(E; G))$. On the other hand, take a point $[E'; G', \tau']$ in $j(T(E; G))$. Then the definition of $T(E; G)$ implies that $G'$ is a finitely generated purely loxodromic free subgroup of $PSL(2, \mathbb{C})$ and the set of all fixed points of elements in $G' - \{e\}$ are mutually disjoint and is contained in a totally disconnected closed subset of $\overline{\mathbb{C}}$. Hence by Lemma 2.8 below, $G'$ is discrete, which means that $j([E'; G', \tau']) \in NS(n)$.

Next, the ordered fix points and the multiplier determine the Möbius transformation. Hence those of $g_1, \cdots, g_n$ give a system of global coordinates for $NS(n)$. On the other hand, the multiplier of $g_j$ is $g_j(1)$, which is the repelling fixed point $z_{1, 2}$ of $g_j g_k^{-1}$, and the multipliers of other $g_k$ are given by

$$\frac{z_k - z_{2k}}{z_{k-1} - z_{2k-1}}.$$

In particular, these relations gives a bi-rational homeomorphism of $T(E; G)$ in $\mathbb{C}^{3n-3}$ with coordinates

$$z_{4j}, \cdots, z_{2n}, z_1, z_2, \cdots, z_{n-1}$$

onto the domain $NS(n)$ in $\mathbb{C}^{3n-3}$ with coordinates consisting of the ordered fixed points and the multipliers of all $g_k$. $\square$

**Lemma 2.8.** Let $G$ be a subgroup of $PSL(2, \mathbb{C})$ which contains no elliptic elements and fixes no points in $\overline{\mathbb{C}}$. If the set of all fixed points of elements in $G - \{e\}$ is contained in a proper closed subset of $\overline{\mathbb{C}}$, then $G$ is discrete.
Proof. By the classification theorem (Proposition of [7]), the assumptions implies that $G$ is either discrete or a non-elementary subgroup of $PSL(2, \mathbb{R})$ without elliptic elements. But it is elementary to show that the latter is also discrete. \qed

Remark 2.9. We can consider an infinitely generated normalized Schottky group $G$ as in [5]. Even in this case, we can consider an ordered set $E$ consisting of all fixed points of elements in $G - \{e\}$ and the corresponding Teichmüller space $T(E; G)$, where some ordered subset of $E$ gives a system of global coordinates of $T(E; G)$.

Note that in the previous proofs of theorems, we could use the following classical variants of global coordinates.

Definition 2.10. The cross-ratio $\chi(a, b, c, d)$ of distinct 4 points $a, b, c, d$ in $\mathbb{C}$ is defined by

$$
\chi(a, b, c, d) = \frac{a - b}{a - c} \frac{d - c}{d - b}
$$

(and by taking the limit if one of them is $\infty$).

When $a, b, c, d$ are in $E$, then $\chi(a, b, c, d)$ can be considered as a function on $T(E)$, which we call the cross-ratio coordinate for $a, b, c, d \in E$.

Let $CR(E)$ be the set of all cross-ratio coordinates corresponding to 4 distinct ordered points of $E$ with the induced order, which are clearly countable. Without using the normalization, $CR(E)$ gives an continuous injection $CR : T(E) \to \Omega$.

Remark 2.11. We can define a metric on $T(E)$ by using $CR(E)$, which we called the CR-metric in [2]. Again, we know that the CR metric subordinates to the Teichmüller metric.

Finally, recall that, if $T(E)$ is finite-dimensional, then $T(E)$ is locally compact. Hence we can consider a natural kind of compactification of it.

Definition 2.12. Let $X$ be a locally compact, but non-compact, Hausdorff space, and $F$ be a set of continuous maps of $X$ to $\hat{\mathbb{C}}$.

Then, a compactification $X^*$ of $X$ such that every element in $F$ can be extended to a continuous map of $X^*$ and that the family of all extended maps separates points of $X^* - X$ is called an $F$-compactification of $X$.

Proposition 2.13. There exists an $CR(E)$-compactification $T(E)^*$ of $T(E)$, which is unique up to homeomorphisms fixing $T(E)$ point-wise.

For the reference and the proof, see [3].

References