A Generalization of Power and Alternating Power Sums to Any Appell Polynomials

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Abstract. The classical power sum and alternating power sum identities can be stated as
\[ \sum_{i=0}^{m} s_{n}(i) = \frac{1}{n+1} (B_{n+1} (m+1) - B_{n+1}), \]
\[ \sum_{i=0}^{m} (-1)^{i} s_{n}(i) = \frac{1}{2} ((-1)^{m} E_{n} (m+1) + E_{n}), \]
where \( s_{n}(x) = x^{n} \) is the simplest possible Appell polynomial for the Sheffer pair \((1, t)\). The impetus for this research starts from the question that what if we replace \( s_{n}(x) = x^{n} \) by any Appell polynomial. In this paper, we give a generalization of power and alternating power sums to any Appell polynomials.

1. Introduction

As is well known, the Bernoulli polynomials are defined by the generating function
\[ \frac{t}{e^{t} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n} (x) \frac{t^{n}}{n!}, \quad \text{(see [3–11])}. \] (1.1)

When \( x = 0 \), \( B_{n} = B_{n} (0) \) are called the Bernoulli numbers.

The ordinary Euler polynomials are also given by the generating function
\[ \frac{2}{e^{t} + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n} (x) \frac{t^{n}}{n!}, \quad \text{(see [1, 17, 20])}. \] (1.2)

When \( x = 0 \), \( E_{n} = E_{n} (0) \) are called the Euler numbers.

There have been numerous investigations on Bernoulli and Euler polynomials and their related ones. Some interesting generalizations of these polynomials were obtained in the recent papers [13, 14, 18]. For
their connections to many interesting integrals and series associated with zeta functions, one is referred to the detailed and systematic treatment in [19]; for explicit computations of the related numbers, one may want to see [15].

Let \( \mathcal{F} \) be the set of all formal power series in the variable \( t \):

\[
\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}; a_k \in \mathbb{C} \right\}. \tag{1.3}
\]

For \( \mathbb{P} = \mathbb{C}[x] \), let \( \mathbb{P}^* \) be the vector space of all linear functionals on \( \mathbb{P} \). \( \langle L | p(x) \rangle \) denotes the action of the linear functional \( L \) on \( p(x) \) which satisfies \( \langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle \), and \( \langle cL | p(x) \rangle = c \langle L | p(x) \rangle \), where \( c \) is a complex constant. The linear functional \( \langle f(t) | \cdot \rangle \) on \( \mathbb{P} \) is defined by

\[
\langle f(t) | x^n \rangle = a_n, \quad (n \geq 0), \tag{1.4}
\]

where \( f(t) \in \mathcal{F} \), (see [16]).

From (1.3) and (1.4), we have

\[
\langle t^k | x^n \rangle = n! \delta_{nk}, \quad (n, k \geq 0), \tag{1.5}
\]

where \( \delta_{nk} \) is the Kronecker’s symbol.

Let us set \( f_L(t) = \sum_{k=0}^{\infty} \frac{(L | t^k)}{k!} \). Then, by (1.4), we get \( \langle f_L(t) | x^n \rangle = \langle L | x^n \rangle \). So, the map \( L \mapsto f_L(t) \) is a vector space isomorphism from \( \mathbb{P}^* \) onto \( \mathcal{F} \). Henceforth, \( \mathcal{F} \) denotes both the algebra of formal power series in \( t \) and the vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional. We call \( \mathcal{F} \) the umbral algebra and the umbral calculus is the study of umbral algebra.

The order \( o(f(t)) \) of a power series \( f(t) \neq 0 \) is the smallest integer \( k \) for which the coefficient of \( t^k \) does not vanish (see [16]). If the order of \( f(t) \neq 0 \) is zero, then \( f(t) \) is called an invertible series; if the order of \( f(t) \) is one, then \( f(t) \) is called a delta series. For \( f(t), g(t) \in \mathcal{F} \) with \( o(f(t)) = 1 \) and \( o(g(t)) = 0 \), there exists a unique sequence \( s_n(x) \) (\( \deg s_n(x) = n \)) such that \( \langle g(t)f(t) | x^n \rangle = n! \delta_{nk} \) for \( k \geq 0 \). Such a sequence \( s_n(x) \) is called the Sheffer sequence for \( g(t), f(t) \) which is denoted by \( s_n(x) \sim (g(t), f(t)) \). Also, \( s_n(x) \sim (g(t), f(t)) \) if and only if

\[
\frac{1}{g(\overline{f}(t))}e^{y\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad (y \in \mathbb{C}), \quad (\text{see [16]}), \tag{1.6}
\]

where \( \overline{f}(t) \) is the compositional inverse of \( f(t) \) with \( \overline{f}(f(t)) = f(\overline{f}(t)) = t \).

In particular, if \( s_n(x) \sim (g(t), f(t)) \), then \( s_n(x) \) is called the Appell sequence for \( g(t) \) if \( s_n(x) \sim (1, f(t)) \), then \( s_n(x) \) is called the associated sequence for \( f(t) \) (see [16]).

For \( f(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \), we have

\[
f(t) = \sum_{k=0}^{\infty} \langle f(t) | x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k | p(x) \rangle \frac{x^k}{k!}. \tag{1.7}
\]

By (1.7), we get

\[
t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad e^y p(x) = p(x + y). \tag{1.8}
\]

From (1.8), we note that

\[
\langle e^y | p(x) \rangle = p(y). \tag{1.9}
\]
The classical power sum and alternating power sum identities can be respectively stated as

\[
\sum_{i=0}^{m} s_n(i) = \frac{1}{n+1} (B_{n+1} (m+1) - B_{n+1}), \tag{1.10}
\]

\[
\sum_{i=0}^{m} (-1)^i s_n(i) = \frac{1}{2} \left( (-1)^m E_n (m+1) + E_n \right), \quad (n, m \in \mathbb{Z}_{\geq 0}), \tag{1.11}
\]

where \(s_n(x) \sim (1, t)\) is the simplest possible Appell polynomial. Here we would like to mention that, as a refinement of (1.10), some problem related with power sums along arithmetic progressions had been treated in [2].

The impetus for this research starts from the question that what if we replace \(s_n(x) = x^n\) by any Appell polynomial.

In this paper, we will introduce Barnes’ multiple Bernoulli and Appell mixed-type polynomials, and Barnes’ multiple Euler and Appell mixed-type polynomials. Then we will establish one main identity for each of them connecting a sum for the Appell polynomial and that for the mixed-type polynomial. As a consequence, the direct generalization of the identities in (1.10) and (1.11) will follow from these as special simple cases.

2. A Generalization of Power and Alternating Power Sums to any Appell Polynomials

Assume that \(s_n(x) \sim (g(t), t), r_n(x) \sim (h(t), t)\) are Appell sequences. Then we denote the Appell sequence for \(g(t) h(t)\) by \(sr_n(x)\) which are called \(s_n(x)\) and \(r_n(x)\) mixed-type polynomials. That is,

\[
sr_n(x) \sim (g(t) h(t), t). \tag{2.1}
\]

From (1.6) and (2.1), we have

\[
\frac{1}{g(t) h(t)} e^x = \sum_{n=0}^{\infty} sr_n(x) \frac{t^n}{n!}. \tag{2.2}
\]

Thus, by (2.2), we get

\[
sr_n(x) = \frac{1}{g(t) h(t)} x^n = \frac{1}{g(t)} r_n(x) = \frac{1}{h(t)} s_n(x), \tag{2.3}
\]

and

\[
l sr_n(x) = n sr_{n-1}(x), \quad (n \geq 1). \tag{2.4}
\]

The following facts will be used frequently: for any polynomial \(p(x)\), we have

\[
e^x \int_x^{x+y} p(u) \, du = \int_x^{x+y} e^u p(u) \, du, \tag{2.5}
\]

\[
\int_0^{y} \frac{e^u - 1}{u} p(u) \, du = \int_0^{y} \frac{e^u - 1}{u} p(u) \, du, \tag{2.6}
\]

\[
\frac{e^{(m+1)t} - 1}{e^t - 1} p(x) = \sum_{i=0}^{m} p(x + ai), \tag{2.7}
\]

\[
\frac{e^{(m+1)t} - 1}{e^t - 1} p(x) = \sum_{i=0}^{m} p(ai), \tag{2.8}
\]
Proof. \[ \frac{(-1)^m e^{(m+1)nt} + 1}{e^{nt} + 1} p(x) = \sum_{i=0}^{m} (-1)^i p(x + ai), \] (2.9)

and

\[ \left( \frac{(-1)^m e^{(m+1)nt} + 1}{e^{nt} + 1} \right) p(x) = \sum_{i=0}^{m} (-1)^i p(ai). \] (2.10)

Lemma 1. Let \( s_{n}(x) \) be the Appell sequence for \( g(t) \), namely \( s_{n}(x) \sim (g(t), t) \). For \( m_1, m_2, \ldots, m_l \in \mathbb{Z}_{\geq 0}, a_1, \ldots, a_l \in \mathbb{C} \), we have

\[ \left( \frac{e^{(m_1+1)t} - 1}{t} \right) \times \cdots \times \left( \frac{e^{(m_l+1)t} - 1}{t} \right) s_{n}(x) \]

\[ = \frac{1}{(n + l)!} \sum_{i=0}^{l} (-1)^{i-l} s_{n+i} (x) (m_1 + 1) a_1, \ldots, (m_l + 1) a_l, \] (2.11)

where

\[ (n)_i = n (n - 1) \cdots (n - l + 1) = \sum_{i=0}^{n} S_l (n, l) x^i \]

\[ s_{n,i} (x_1, \ldots, x_l) = \sum_{\substack{j \in [1, l] \cap [i-l] \cap |J| = i}} s_{n} (x + x_i), \quad (0 \leq i \leq l), \]

\[ [1, l] = \{1, 2, \ldots, l\}, \quad \text{and} \quad x = \sum_{i \in J} x_i. \]

Proof. By induction on \( l \), we prove Lemma 1. For \( l = 1 \), we have

\[ \frac{e^{(m_1+1)t} - 1}{t} s_{n} (x) = \int_{x}^{x + (m_1 + 1) t} s_{n} (u) \, du \]

\[ = \frac{1}{n + 1} (s_{n+1} (x + (m_1 + 1) a_1) - s_{n+1} (x)) \]

\[ = \frac{1}{(n + 1)!} \sum_{i=0}^{1} (-1)^{i-1} s_{n+i} (x) (m_1 + 1) a_1. \] (2.12)

Assume that, for \( l > 1 \), the following holds:

\[ \left( \frac{e^{(m_{l-1}+1)t} - 1}{t} \right) \times \cdots \times \left( \frac{e^{(m_1+1)t} - 1}{t} \right) s_{n}(x) \]

\[ = \frac{1}{(n + l - 1)!} \sum_{i=0}^{l-1} (-1)^{l-1-i} s_{n+i-l} (x) (m_1 + 1) a_1, \ldots, (m_{l-1} + 1) a_{l-1}). \] (2.13)

Then the LHS of (2.11) is

\[ \frac{1}{(n + l - 1)!} \sum_{i=0}^{l-1} (-1)^{l-1-i} e^{(m_{l-1}+1)t} - 1 \]

\[ \times s_{n+l-i} (x) (m_1 + 1) a_1, \ldots, (m_{l-1} + 1) a_{l-1}) \]

\[ = \frac{1}{(n + l - 1)!} \sum_{i=0}^{l-1} (-1)^{l-1-i} \] (2.14)
Theorem 1. Let $s_n(x)$ be the Appell sequence for $g(t)$, namely $s_n(x) \sim (g(t), t)$. Then, for any integers $m_1, \ldots, m_l$ with $m_1, \ldots, m_l \geq 1$, $a_1, \ldots, a_l \neq 0$, we have

$$\sum_{\lambda_1=0}^{m_1} \cdots \sum_{\lambda_l=0}^{m_l} s_n(x + a_1 \lambda_1 + \cdots + a_l \lambda_l)$$

$$= \frac{1}{(n + l - 1)!} \sum_{i=0}^{l-1} (-1)^{l-i} \left\{ \sum_{j \in [1, l]} s_{n+i} \left( x + \sum_{j \in [1, l]} (m_j + 1) a_j \right) \right\}$$

$$+ \sum_{i=1}^{l-1} (-1)^{l-i} \sum_{j \in [1, l]} s_{n+i} \left( x + \sum_{j \in [1, l]} (m_j + 1) a_j \right) + (-1)^l s_{n+l}(x)$$

$$= \frac{1}{(n + l - 1)!} \sum_{i=0}^{l-1} (-1)^{l-i} \sum_{j \in [1, l]} s_{n+l} \left( x \right) \left( m_1 + 1 \right) a_1, \ldots, (m_l + 1) a_l.$$
Corollary 1. Let $m_1, m_2, \ldots, m_l \in \mathbb{Z}$, with $m_1, m_2, \ldots, m_l \geq 1$ and $l \in \mathbb{N}$. Then we have

(a) \[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} s_n(a_1i_1 + \cdots + a_li_l)
= \frac{1}{(n + 1)!} \sum_{i=0}^{l} (-1)^{l-i} (B(a_1, \ldots, a_l)s)_{n+i} ((m_1 + 1)a_1, \ldots, (m_l + 1)a_l),
\]

where

\[
(B(a_1, \ldots, a_l)s)_{n+i} (x_1, \ldots, x_i)
= \sum_{j \in [1, l]} (B(a_1, \ldots, a_l)s)_{n+i}(x_j), \quad (0 \leq i \leq l).
\]
(b) \[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} s_n(x + i_1 + \cdots + i_l)
= \frac{1}{(n+1)!} \sum_{i=0}^{l} (-1)^{i-1} \left( B_{m_1+i}^{(i)} \right)_{m=1, \ldots, m_l+1}.
\]

where \( B_{m}^{(i)}(x) \) are the Bernoulli polynomials of order \( l \) given by the generating function
\[
\left( \frac{t}{e^t - 1} \right)^{l} e^{nt} = \sum_{n=0}^{\infty} B_n^{(l)}(x) \frac{t^n}{n!}.
\]

(c) \[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} s_n(i_1 + \cdots + i_l)
= \frac{1}{(n+1)!} \sum_{i=0}^{l} (-1)^{i-1} \left( B_{m_1+i}^{(i)} \right)_{m=1, \ldots, m_l+1}.
\]

(d) \[
\sum_{i_1, \ldots, i_l=0}^{m} s_n(x + i_1 + \cdots + i_l)
= \frac{1}{(n+1)!} \sum_{i=0}^{l} (-1)^{i-1} \left( B_{m+i}^{l} \right)_{m=1, \ldots, m_l+1}.
\]

(e) \[
\sum_{i_1, \ldots, i_l=0}^{m} S_n(i_1 + \cdots + i_l)
= \frac{1}{(n+1)!} \sum_{i=0}^{l} (-1)^{i-1} \left( B_{m+i}^{l} \right)_{m=1, \ldots, m_l+1}.
\]

Lemma 2. Let \( s_n(x) \) be the Appell sequence for \( g(t) \), namely \( s_n(x) \sim (g(t), x) \). Then, for any \( m_1, \ldots, m_l \in \mathbb{Z}_{\geq 0}, a_1, a_2, \ldots, a_l \in \mathbb{C}, \) and \( l \in \mathbb{N} \), we have
\[
\left( (-1)^{m_l} e^{(m_1+1)a_1} + 1 \right) \cdots \left( (-1)^{m_l} e^{(m_1+1)a_l} + 1 \right) s_n(x)
= \sum_{i=0}^{l} \tilde{s}_{n,i} \left( x | (m_1+1)a_1, \ldots, (m_l+1)a_l \right).
\]
Here, for \( l \in \mathbb{N} \),
\[
\tilde{s}_{n,i} \left( x | (m_1+1)a_1, \ldots, (m_l+1)a_l \right) = \sum_{j \in [1,l]} (-1)^{m_j} s_n \left( x + ((m_1+1)a_j) \right),
\]
and

\[ m_l = \sum_{j \in J} m_{j,l}, \quad ((m + 1) a)_l = \sum_{j \in J} (m_j + 1) a_j. \] (2.21)

**Proof.** We prove Lemma 2 by induction on \( l \).

For \( l = 1 \), we have

\[ \left((-1)^{m_1} e^{(m_2+1)a_2} + 1\right)s_n(x) = (-1)^{m_1} s_n(x + (m_1 + 1) a_1) + s_n(x) \]

\[ = \sum_{i=0}^{l-1} \tilde{s}_{n,i}(x) (m_1 + 1) a_1. \]

Assume that, for \( l > 1 \), the following holds:

\[ \left((-1)^{m_{l-1}} e^{(m_{l-1}+1)a_{l-1}} + 1\right) \prod_{j=1}^{l-1} \left((-1)^{m_j} e^{(m_{j+1})a_{j+1}} + 1\right)s_n(x) \]

\[ = \sum_{i=0}^{l-1} \sum_{j \in J, |j| \leq i} (-1)^{m_{j+1}} s_n(x + ((m + 1) a)_j) \]

\[ + \sum_{j \in J, |j| = i} (-1)^{m_j} s_n(x + ((m + 1) a)_j) \]

\[ = (-1)^{m_{l-1}} s_n(x + ((m + 1) a)_{1,l}) \]

\[ + \sum_{i=1}^{l-1} \sum_{j \in J, |j| = i} (-1)^{m_{j+1}} s_n(x + ((m + 1) a)_j) \]

\[ = (-1)^{m_{l-1}} s_n(x + ((m + 1) a)_{1,l}) \]

\[ + \sum_{i=1}^{l-1} \sum_{j \in J, |j| = i} (-1)^{m_{j+1}} s_n(x + ((m + 1) a)_j) + s_n(x). \]
Theorem 2. Let \( s_n(x) \) be the Appell sequence for \( g(t) \), namely \( s_n(x) \sim (g(t), t) \). Then, for any integers \( m_1, m_2, \ldots, m_t \geq 1, a_1, \ldots, a_t \neq 0 \), we have

\[
\sum_{i=0}^{m_1} \sum_{j=0}^{m_1} (-1)^{m_1} s_n \left( x + (m_1 + 1) a_i \right) = \sum_{i=0}^{m_1} s_n(x) \left( (m_1 + 1) a_1, \ldots, (m_1 + 1) a_t \right).
\]

On one hand, (2.24) is

\[
\prod_{j=1}^{m} \left( \frac{2}{e^{x_j} + 1} \right) e^{x_i} = \sum_{n=0}^{\infty} E_n(x|a_1, \ldots, a_t) \frac{t^n}{n!}, \quad (\text{see [12]}),
\]

where \( E(a_1, \ldots, a_t) s_n(x) \) is the Barnes’ multiple Euler polynomial whose generating function is given by

\[
\prod_{j=1}^{m} \left( \frac{2}{e^{x_j} + 1} \right) e^{x_i} = \sum_{n=0}^{\infty} E_n(x|a_1, \ldots, a_t) \frac{t^n}{n!}.
\]

Proof. Now, we compute the following in two different ways:

\[
\left( \frac{(-1)^{m_1} e^{(m_1+1)x_1}}{e^{x_1} + 1} + 1 \right) \times \cdots \times \left( \frac{(-1)^{m_t} e^{(m_t+1)x_t}}{e^{x_t} + 1} + 1 \right) s_n(x).
\]

On one hand, (2.24) is

\[
\prod_{j=1}^{m} \left( \frac{2}{e^{x_j} + 1} \right) e^{x_i} = \sum_{n=0}^{\infty} E_n(x|a_1, \ldots, a_t) \frac{t^n}{n!}, \quad (\text{see [12]}),
\]

(2.24) is

\[
\prod_{j=1}^{m} \left( \frac{2}{e^{x_j} + 1} \right) e^{x_i} = \sum_{n=0}^{\infty} E_n(x|a_1, \ldots, a_t) \frac{t^n}{n!}.
\]
Corollary 2. For $m_1, \ldots, m_l \in \mathbb{Z}$ with $m_1, m_2, \ldots, m_l \geq 1$, $l \in \mathbb{N}$ and $a_1, a_2, \ldots, a_l \neq 0$, we have

(a) \[ \sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} s_n (a_1 i_1 + \cdots + a_l i_l) \]

(b) \[ \sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} s_n (x + i_1 + \cdots + i_l) \]

(c) \[ \sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} s_n (i_1 + \cdots + i_l) \]

Continuing in this manner, we obtain the expression on the LHS of (2.23).

On the other hand, (2.24) is

\[ 2^{-l} \left( (-1)^{m_1} e^{((m_1+1) \ln t + 1)} \right) \cdots \left( (-1)^{m_l} e^{((m_l+1) \ln t + 1)} \right) \prod_{j=1}^{l} \left( \frac{2}{e^{a_j t} + 1} \right) s_n (x) \]

\[ = 2^{-l} \left( (-1)^{m_1} e^{((m_1+1) \ln t + 1)} \right) \cdots \left( (-1)^{m_l} e^{((m_l+1) \ln t + 1)} \right) (\hat{E} (a_1, \ldots, a_l) s)_n (x). \]

Now, the expression on the RHS of (2.23) follows from Lemma 2. \( \square \)

**Corollary 2.** For $m_1, \ldots, m_l \in \mathbb{Z}$ with $m_1, m_2, \ldots, m_l \geq 1$, $l \in \mathbb{N}$ and $a_1, a_2, \ldots, a_l \neq 0$, we have
(d)
\[
\sum_{i_1,\ldots,i_l=0}^{m} (-1)^{i_1+\cdots+i_l} s_n (x + i_1 + \cdots + i_l) \\
= 2^{-l} \sum_{i=0}^{l} \binom{l}{i} (-1)^{mi} \left( \prod_{i}^{s} (x + (m + 1) i) \right).
\]

(e)
\[
\sum_{i_1,\ldots,i_l=0}^{m} (-1)^{i_1+\cdots+i_l} s_n (i_1 + \cdots + i_l) \\
= 2^{-l} \sum_{i=0}^{l} \binom{l}{i} (-1)^{mi} \left( \prod_{i=0}^{s} ((m + 1) i) \right).
\]

3. Examples on Theorem 1

1. From Corollary 1 (e) with \( s_n (x) = x^n \sim (1, t) \) and \( l = 1 \), we can derive the classical power sum identity as follows:
\[
\sum_{i=0}^{m} t^i = \frac{1}{n+1} \left( B_{n+1} (m+1) - B_{n+1} \right).
\]
However, it is instructive to repeat our proof for this simplest possible case:
\[
\sum_{i=0}^{m} t^i = \left. \left( e^{(m+1)t} - 1 \right) \right|_{e^t - 1} \]
\[
= \left. \left( e^{(m+1)t} - 1 \right) \frac{t}{e^t - 1} \right|_{x^n} \]
\[
= \left. \left( e^{(m+1)t} - 1 \right) \frac{t}{e^t - 1} \right|_{B_n (x)} \]
\[
= \int_{0}^{m+1} B_n (u) du \]
\[
= \frac{1}{n+1} \left( B_{n+1} (m+1) - B_{n+1} \right).
\]

More generally, from Corollary 1 (c), we have
\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (i_1 + \cdots + i_l)^n \\
= \sum_{i=0}^{m_{1+\cdots+m_l}} p_i (m_1, \ldots, m_l) i^n \\
= \frac{1}{(n+1)} \sum_{i=0}^{l} (-1)^{i-1} B_{n+1}^{(i)} (m_1 + 1, \ldots, m_l + 1),
\]
2. Apply Theorem 1 to \( s_k \) in the expansion of \( \prod_{i=1}^{m} \left( 1 - t^i \right) \) where

\[
\begin{align*}
p_t (m_1, \ldots, m_l) &= \# \left\{ (i_1, \ldots, i_l) \in \mathbb{Z}^l \mid 0 \leq i_j \leq m_j \ (1 \leq j \leq l), \ i_1 + \cdots + i_l = i \right\} \\
&= \text{the coefficient of } t^i \text{ in the expansion of } \prod_{i=1}^{m} \left( 1 - t^i \right) \times \cdots \times \prod_{i=1}^{m} \left( 1 - t^i \right).
\end{align*}
\]

2. Applying Theorem 1 to the poly-Bernoulli polynomial of index \( s_n \) where

\[
\begin{align*}
\left\{ \begin{array}{l}
\Sigma_{j=0}^{n} B_n (x + a_1, \ldots, a_l) = \left( 1 - e^{-x} \right) \left( \frac{L_n}{L_k} \right), \\
\Sigma_{j=0}^{n} B_n (x + (m + 1) a_j) = \left( 1 - e^{-x} \right) \left( \frac{L_n}{L_k} \right).
\end{array} \right.
\end{align*}
\]

3. Applying Theorem 1 to the poly-Bernoulli polynomial of index \( k \)

\[
s_n (x) = B_n^{(k)} (x) \sim \left( \frac{1 - e^{-x}}{L_k (1 - e^{-x})} \right)^k,
\]

we obtain

\[
\begin{align*}
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} B_n^{(k)} (x + a_1 i_1 + \cdots + a_l i_l) &= \frac{1}{(n + l)!} \sum_{j=0}^{l} (-1)^{j-l} \sum_{i_j=1}^{n+1} B_n^{(k)} (x + (m + 1) a_j) \\
&= \frac{1}{(n + l)!} \sum_{j=0}^{l} (-1)^{j-l} \sum_{i_j=1}^{n+1} \left( x + ((m + 1) a_j) \right) B_n^{(k)} (x + a_1, \ldots, a_l).
\end{align*}
\]

3. Applying Theorem 1 to the poly-Bernoulli polynomial of index \( k \)

\[
s_n (x) = B_n^{(k)} (x) \sim \left( \frac{1 - e^{-x}}{L_k (1 - e^{-x})} \right)^k,
\]

we obtain

\[
\begin{align*}
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} B_n^{(k)} (x + a_1 i_1 + \cdots + a_l i_l) &= \frac{1}{(n + l)!} \sum_{j=0}^{l} (-1)^{j-l} \sum_{i_j=1}^{n+1} B_n^{(k)} (x + (m + 1) a_j) \\
&= \frac{1}{(n + l)!} \sum_{j=0}^{l} (-1)^{j-l} \sum_{i_j=1}^{n+1} \left( x + (m + 1) a_j \right) B_n^{(k)} (x + a_1, \ldots, a_l).
\end{align*}
\]

Here \( B_n^{(k)} (x|a_1, \ldots, a_l) \) are the Barnes’ multiple Bernoulli and poly-Bernoulli mixed-type polynomials studied in [4] whose generating function is given by

\[
\left( \frac{t^l}{\prod_{j=1}^{l} (e^{t^j} - 1)} \right) \times \left( \frac{L_k (1 - e^{-x})}{L_k (1 - e^{-x})} \right)^k = \sum_{n=0}^{\infty} \frac{S_n^{(k)} (x|a_1, \ldots, a_l) \left( \frac{t^n}{n!} \right)}{n!}.
\]
In the special case of $a_1 = a_2 = \cdots = a_t = 1$, and $m_1 = \cdots = m_t = m$, we get

$$\sum_{i_1=0}^{m_1} \cdots \sum_{i_t=0}^{m_t} B_n^{(0)} (x + a_1 i_1 + \cdots + a_t i_t) = \frac{1}{(n+1)!} \sum_{i=0}^{m} (-1)^{i+\ell} \binom{n+1-i}{i} S_{n+1}^{(0)} (x + (m+1)i) ,$$

where $S_{n}^{(k)} (x)$ is the higher-order Bernoulli and poly-Bernoulli mixed-type polynomials studied in [7] whose generating function is given by

$$\left( \frac{t}{e^t - 1} \right)^{l} \frac{\ln (1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} S_n^{(k)} (x) \frac{t^n}{n!}.$$ 

For $l = 1$ and $x = 0$, we have

$$\sum_{i=0}^{m} B_n^{(0)} (i) = \frac{1}{n+1} \left( c_{n+1}^{(1,1)} (m+1) - S_{n+1}^{(1,1)} \right).$$

In [9], it is known that

$$B_n^{(0)} (x) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \sum_{j=0}^{l} \frac{(-1)^{n-j-l}}{(j+1)!} S_2 (n-l,j) \right) x^l,$$

where $S_2 (n,l)$ is the Stirling number of the second kind. Thus, we get

$$\sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \sum_{j=0}^{l} \frac{(-1)^{n-j-l}}{(j+1)!} S_2 (n-l,j) \right) \frac{1}{l+1} (B_{l+1} (m+1) - B_{l+1})$$

$$= \frac{1}{n+1} \left( c_{n+1}^{(1,1)} (m+1) - S_{n+1}^{(1,1)} \right).$$

4. We now apply Theorem 1 to the generalized Barnes’ multiple Frobenius-Euler polynomial

$$s_n (x) = H_n (x|b; \lambda; \mu)$$

$$= H_n (x|b_1, \ldots, b_l; \lambda_1, \ldots, \lambda_l; \mu_1, \ldots, \mu_l),$$

and obtain

$$\sum_{i_1=0}^{m_1} \cdots \sum_{i_t=0}^{m_t} H_n (x + a_1 i_1 + \cdots + a_t i_t|b; \lambda; \mu)$$

$$= \frac{1}{(n+1)!} \sum_{i=0}^{m} (-1)^{i+\ell} \binom{n+1-i}{i} \cdot (m_1 + 1) a_1, \ldots, (m_t + 1) a_t)$$

$$= \frac{1}{(n+1)!} \sum_{i=0}^{m} (-1)^{i+\ell} \sum_{j=1, \ldots, m} B H_{n+i} (x + ((m+1) a_j)|b; \lambda; \mu).$$
As a very special case, we have
\[ BH (a; b; \lambda; \mu) (x) \]
where
\[ BH_n (x; a; b; \lambda; \mu) \]
are given by the generating function
\[
\prod_{j=1}^{l} \left( \frac{t}{e^{\lambda j} - 1} \right) \prod_{j=1}^{\nu} \left( \frac{1 - \lambda_j}{e^{\lambda_j} - \lambda_j} \right)^{n_j} e^{xt}
\]
\[
= \sum_{n=0}^{\infty} BH_n (x; a; b; \lambda; \mu) \frac{t^n}{n!},
\]
and are called the Barnes’ multiple Bernoulli and generalized Barnes’ multiple Frobenius-Euler mixed-type polynomials. These are studied in [8].

5. We apply Theorem 1 to the Hermite polynomial \( s_n (x) = H^{(\nu)}_n (x) \sim \left( e^{xt}, t \right) \). Then we have
\[
\sum_{i=0}^{m_0} \cdots \sum_{i=0}^{m_l} H^{(\nu)}_n (x + a_1 t_1 + \cdots + a_l t_l)
\]
\[
= \frac{1}{(n + 1)!} \sum_{i=0}^{l} (-1)^{l-i} \left( BH^{(\nu)}_n (a_1, \ldots, a_l) \right)_{n+l} (x) (m_1 + 1) a_1, \ldots, (m_l + 1) a_l
\]
\[
= \frac{1}{(n + 1)!} \sum_{i=0}^{l} (-1)^{l-i} \sum_{|J|=i} BH^{(\nu)}_n (x + ((m + 1) a)_J, a_1, \ldots, a_l).
\]

Here \( BH^{(\nu)}_n (a_1, \ldots, a_l) (x) = BH^{(\nu)}_n (x; a_1, \ldots, a_l) \) are given by the generating function to be
\[
\prod_{j=1}^{l} \left( \frac{t}{e^{\lambda_j} - 1} \right) e^{-x\lambda_j} e^{xt} = \sum_{n=0}^{\infty} BH^{(\nu)}_n (x; a_1, \ldots, a_l) \frac{t^n}{n!}
\]
and are called the Barnes’ multiple Bernoulli and Hermite mixed-type polynomials. These mixed-type polynomials are studied in [10].

As a very special case, we have
\[
\sum_{i=0}^{m} H^{(\nu)}_n (i) = \frac{1}{n + 1} \left( BH^{(\nu)}_{n+1} (m + 1) - BH^{(\nu)}_{n+1} \right),
\]
where \( BH^{(\nu)}_n (x) \) are the polynomials given by
\[
\frac{t}{e^{\lambda} - 1} e^{xt-\lambda x} = \sum_{n=0}^{\infty} BH^{(\nu)}_n (x) \frac{t^n}{n!}.
\]
An explicit expression for the Hermite polynomials is given in [16] by
\[
H^{(\nu)}_n (x) = \sum_{j=0}^{\frac{n}{2}} \frac{(-1)^j (n/2)_j}{j!} x^{n-2j}.
\]
Thus, we see that

\[
\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \frac{(-\nu)^{2j}}{j!} \left( \frac{1}{n-2j+1} (B_{n-2j+1} (m+1) - B_{n-2j+1}^{(0)}) \right)
\]

\[
= \frac{1}{n+1} (BH_{n+1}^{(\nu)} (m+1) - BH_{n+1}^{(\nu)}).
\]

4. Examples on Theorem 2

1. From Corollary 2 (e) with \( s_n (x) = x^n \sim (1, t) \) and \( l = 1 \), we have the classical alternating power sum identity:

\[
\sum_{i=0}^{m} (-1)^i i^n = \frac{1}{2} \left( (-1)^m E_n (m+1) + E_n \right).
\]

But it is instructive to go over our proof in this simplest possible case.

\[
\sum_{i=0}^{m} (-1)^i i^n = \left( \frac{(-1)^m e^{(m+1)i} + 1}{e^i + 1} \right) x^n
\]

\[
= \left( \frac{(-1)^m e^{(m+1)i} + 1}{2} \right) x^n
\]

\[
= \left( \frac{(-1)^m e^{(m+1)i} + 1}{2} E_n (x) \right)
\]

\[
= \frac{1}{2} \left( (-1)^m E_n (m+1) + E_n \right).
\]

More generally, from Corollary 2 (c), we can derive

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} (i_1 + i_2 + \cdots + i_l)^n
\]

\[
= \sum_{j=0}^{m_1+\cdots+m_l} \sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^j j^n
\]

\[
= 2^{-l} \sum_{j=0}^{l} \sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{m_1} \cdots (-1)^{m_l} E_n^{(j)} ((m+1)_{i_1})
\]

2. Applying Theorem 2 to the poly-Bernoulli polynomial of index \( k \)

\[
s_n (x) = B_n^{(k)} (x) \sim \left( \frac{1 - e^{-t}}{L_k (1 - e^{-t})} \right) t.
\]
we have
\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} B_n^{(k)} (x + a_1 i_1 + \cdots + a_l i_l)
= 2^{1-l} \sum_{j=0}^{l} \left( \sum_{i\in[1,l]} \sum_{|j|=i} (-1)^{m_l} \left( x + ((m + 1) a_l | a_1, \ldots, a_l) \right) \right),
\]
where \( T_n^{(\lambda_k)} (x | a_1, \ldots, a_l; \lambda_1, \ldots, \lambda_l) \) are the special \( \lambda_1 = \cdots = \lambda_l = -1 \) case of the Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials \( T_n^{(\lambda_k)} (a_1, \ldots, a_l; \lambda_1, \ldots, \lambda_l) \) studied in [11] whose generating function is given by
\[
\prod_{j=1}^{l} \left( \frac{1 - \lambda_j}{e^{\lambda_j} - 1} \right) \frac{\ln (1 - e^{-t})}{1 - e^{-t}} e^{\lambda t} = \sum_{n=0}^{\infty} T_n^{(\lambda_k)} (x | a_1, \ldots, a_l; \lambda_1, \ldots, \lambda_l) \frac{t^n}{n!}.
\]
In the special case of \( m_1 = \cdots = m_l = m \) and \( a_1 = \cdots = a_l = 1 \), we have
\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_1+\cdots+i_l} B_n^{(k)} (x + i_1 + \cdots + i_l)
= 2^{1-l} \sum_{j=0}^{l} (-1)^{m_l} \left( T_n^{(\lambda_k)} (x + (m + 1) i) \right),
\]
where \( T_n^{(\lambda_k)} (x) = T_n^{(\lambda_k)} (x - 1) \) are the special \( \lambda = -1 \) case of the higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials \( T_n^{(\lambda_k)} (x | \lambda) \) studied in [6] whose generating function is defined by
\[
\left( \frac{1 - \lambda}{e^{\lambda} - 1} \right) \frac{\ln (1 - e^{-t})}{1 - e^{-t}} e^{\lambda t} = \sum_{n=0}^{\infty} T_n^{(\lambda_k)} (x | \lambda) \frac{t^n}{n!}.
\]
For \( l = 1 \) and \( x = 0 \), we get
\[
\sum_{j=0}^{n} (-1)^{j} B_n^{(k)} (j) = \frac{1}{2} \left\{ (-1)^{m} T_n^{(1,k)} (m + 1) + T_n^{(1,k)} \right\}.
\]
Again, using
\[
B_n^{(k)} (x) = \sum_{l=0}^{n} \left( \sum_{j=0}^{n-l} (-1)^{n-j-l} \frac{j!}{(j+1)^k} S_2 (n-l, j) \right) x^j,
\]
we obtain
\[
\sum_{l=0}^{n} \left( \sum_{j=0}^{n-l} (-1)^{n-j-l} \frac{j!}{(j+1)^k} S_2 (n-l, j) \right) \frac{1}{2} \left\{ (-1)^{m} E_l (m + 1) + E_l \right\} = \frac{1}{2} \left\{ (-1)^{m} T_n^{(1,k)} (m + 1) + T_n^{(1,k)} \right\}.
\]
3. We now apply Theorem 2 to the Barnes’ multiple Euler polynomial \( s_n(x) = E_n(x; b_1, \ldots, b_k) \). Then, we obtain

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_l=0}^{m_l} (-1)^{i_l} E_n(x + a_1 i_1 + \cdots + a_l i_l; b_1, \ldots, b_k) = 2^{-l} \sum_{i=0}^{l} \sum_{(j|\{1, \ldots, l\})} (-1)^m E_n \left( x + ((m + 1) a_1, \ldots, a_l, b_1, \ldots, b_l) \right).
\]

References


