On the Characterizations of $f$-Biharmonic Legendre Curves in Sasakian Space Forms

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Abstract. We consider $f$-biharmonic Legendre curves in Sasakian space forms. We find curvature characterizations of these types of curves in four cases.

1. Introduction

Let $(M, g)$ and $(N, h)$ be two Riemannian manifolds and $\phi : (M, g) \to (N, h)$ a smooth map. The energy functional of $\phi$ is defined by

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \upsilon_g,$$

where $\upsilon_g$ is the canonical volume form in $M$. If $\phi$ is a critical points of the energy functional $E(\phi)$, then it is called harmonic [5]. $\phi$ is called a biharmonic map if it is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \upsilon_g,$$

where $\tau(\phi)$ is the tension field of $\phi$ which is defined by $\tau(\phi) = \text{trace} \nabla d\phi$. The Euler-Lagrange equation of the bienergy functional $E_2(\phi)$ gives the biharmonic equation

$$\tau_2(\phi) = -J^0(\tau(\phi)) = -\Delta^0 \tau(\phi) - \text{trace} R^N(d\phi, \tau(\phi))d\phi = 0,$$

where $J^0$ is the Jacobi operator of $\phi$ and $\tau_2(\phi)$ is called the bitension field of $\phi$ [8].

Now, if $\phi : M \to N(c)$ is an isometric immersion from $m$-dimensional Riemannian manifold $M$ to $n$-dimensional Riemannian space form $N(c)$ of constant sectional curvature $c$, then

$$\tau(\phi) = mH$$

and

$$\tau_2(\phi) = -m\Delta^0 H + cm^2 H.$$
Thus, \( \phi \) is biharmonic if and only if

\[
\Delta^\phi H = cmH,
\]

(see [10]). In a different setting, in [4], B.Y. Chen defined a biharmonic submanifold \( M \subset \mathbb{E}^n \) of the Euclidean space as its mean curvature vector field \( H \) satisfies \( \Delta H = 0 \), where \( \Delta \) is the Laplacian. Replacing \( c = 0 \) in the above equation, we obtain Chen’s definition.

\( \phi \) is called an \( f \)-biharmonic map if it is a critical point of the \( f \)-bienergy functional

\[
E_{2,f}(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g.
\]

The Euler-Lagrange equation of this functional gives the \( f \)-biharmonic equation

\[
\tau_{2,f}(\phi) = f \tau_2(\phi) + (\Delta f) \tau(\phi) + 2\eta^{\phi}_{\text{grad}} \tau(\phi) = 0.
\]

(see [9]). It is clear that any harmonic map is biharmonic and any biharmonic map is \( f \)-biharmonic. If the map is non-harmonic biharmonic map, then it is called proper biharmonic. Likewise, if the map is non-biharmonic \( f \)-biharmonic map, then it is called proper \( f \)-biharmonic [11].

\( f \)-biharmonic maps were introduced in [9]. Ye-Lin Ou studied \( f \)-biharmonic curves in real space forms in [11]. D. Fetcu and C. Oniciuc studied biharmonic Legendre curves in Sasakian space forms in [6] and [7]. We studied biharmonic Legendre curves in generalized Sasakian space forms and \( S \)-space forms in [13] and [12], respectively. In the present paper, we consider \( f \)-biharmonic Legendre curves in Sasakian space forms. We obtain curvature equations for this kind of curves.

The paper is organized as follows. In Section 2, we give a brief introduction about Sasakian space forms. In Section 3, we obtain our main results. We also give two examples of proper \( f \)-biharmonic Legendre curves in \( \mathbb{R}^7(-3) \).

2. Sasakian Space Forms

Let \((M^{2m+1}, \varphi, \xi, \eta, g)\) be a contact metric manifold. If the Nijenhuis tensor of \( \varphi \) equals \(-2d\eta \otimes \xi\), then \((M, \varphi, \xi, \eta, g)\) is called Sasakian manifold [2]. For a Sasakian manifold, it is well-known that:

\[
(V_X \varphi)Y = g(X,Y)\xi - \eta(Y)X,
\]

\[
\nabla_X \xi = -\varphi X.
\]

(1)

(2)

(see [3]).

A plane section in \( T_pM \) is a \( \varphi \)-section if there exists a vector \( X \in T_pM \) orthogonal to \( \xi \) such that \([X, \varphi X] \) span the section. The sectional curvature of a \( \varphi \)-section is called \( \varphi \)-sectional curvature. For a Sasakian manifold of constant \( \varphi \)-sectional curvature (i.e. Sasakian space form), the curvature tensor \( R \) of \( M \) is given by

\[
R(X,Y)Z = \frac{\kappa_1}{4} \left[ g(Y,Z)X - g(X,Z)Y \right] + \frac{\kappa_2}{4} \left[ g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z \right] + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,
\]

\[
R(X,Y)Z = \frac{\kappa_1}{4} \left[ g(Y,Z)X - g(X,Z)Y \right] + \frac{\kappa_2}{4} \left[ g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z \right] + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,
\]

(3)

for all \( X,Y,Z \in TM \) [3].

A submanifold of a Sasakian manifold is called an integral submanifold if \( \eta(X) = 0 \), for every tangent vector \( X \). A 1-dimensional integral submanifold of a Sasakian manifold \((M^{2m+1}, \varphi, \xi, \eta, g)\) is called a Legendre curve of \( M \) [3]. Hence, a curve \( \gamma : I \rightarrow M = (M^{2m+1}, \varphi, \xi, \eta, g) \) is called a Legendre curve if \( \eta(T) = 0 \), where \( T \) is the tangent vector field of \( \gamma \).
3. \( f \)-Biharmonic Legendre curves in Sasakian Space Forms

Let \( \gamma : I \rightarrow M \) be a curve parametrized by arc length in an \( n \)-dimensional Riemannian manifold \((M, g)\). If there exist orthonormal vector fields \( E_1, E_2, \ldots, E_r \) along \( \gamma \) such that

\[
\begin{align*}
    E_1 &= \gamma' = T, \\
    \nabla_T E_1 &= \kappa_1 E_2, \\
    \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\
    \ldots \\
    \nabla_T E_r &= -\kappa_{r-1} E_{r-1},
\end{align*}
\]

then \( \gamma \) is called a Frenet curve of osculating order \( r \), where \( \kappa_1, \ldots, \kappa_{r-1} \) are positive functions on \( I \) and \( 1 \leq r \leq n \).

It is well-known that a Frenet curve of osculating order 1 is a geodesic; a Frenet curve of osculating order 2 is called a circle if \( \kappa_1 \) is a non-zero positive constant; a Frenet curve of osculating order \( r \geq 3 \) is called a helix of order \( r \) if \( \kappa_1, \ldots, \kappa_{r-1} \) are non-zero positive constants; a helix of order 3 is shortly called a helix.

An arc-length parametrized curve \( \gamma : (a, b) \rightarrow (M, g) \) is called an \( f \)-biharmonic curve with a function \( f : (a, b) \rightarrow (0, \infty) \) if the following equation is satisfied [11]:

\[
f(\nabla_T \nabla_T \nabla_T T - R(T, \nabla_T T)T) + 2f' \nabla_T \nabla_T T + f'' \nabla_T T = 0.
\]

Now let \( M = (M^{2m+1}, \varphi T, \xi, \eta, g) \) be a Sasakian space form and \( \gamma : I \rightarrow M \) a Legendre Frenet curve of osculating order \( r \). Differentiating

\[
\eta(T) = 0
\]

and using (4), we get that

\[
\eta(E_2) = 0.
\]

Using (3), (4) and (7), it can be seen that

\[
\nabla_T \nabla_T \nabla_T T = -\kappa_1^2 E_1 + \kappa_1 E_2 + \kappa_1 \kappa_2 E_3,
\]

\[
\nabla_T \nabla_T \nabla_T T = -3 \kappa_1 \kappa_1' E_1 + \left( \kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2 \right) E_2 \\
+ \left( 2 \kappa_1' \kappa_2 + \kappa_1 \kappa_2' \right) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4,
\]

\[
R(T, \nabla_T T)T = -\kappa_1 \left( \frac{c + 3}{4} \right) E_2 - 3 \kappa_1 \left( \frac{c - 1}{4} \right) g(\varphi T, E_2) \varphi T,
\]

(see [7]). If we denote the left-hand side of (5) with \( F \tau_3 \), we find

\[
\tau_3 = \nabla_T \nabla_T \nabla_T T - R(T, \nabla_T T)T + 2f' \nabla_T \nabla_T T + f'' \nabla_T T \]

\[
= \left( -3 \kappa_1 \kappa_1' - 2 \kappa_1^2 \frac{f'}{f} \right) E_1 \\
+ \left( \kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2 + \kappa_1 \left( \frac{c + 3}{4} \right) + 2 \kappa_1' \frac{f'}{f} \right) E_2 \\
+ \left( 2 \kappa_1' \kappa_2 + \kappa_1 \kappa_2' \right) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4 \\
+ 3 \kappa_1 \left( \frac{c - 1}{4} \right) g(\varphi T, E_2) \varphi T.
\]

Let \( k = \min \{r, 4\} \). From (8), the curve \( \gamma \) is \( f \)-biharmonic if and only if \( \tau_3 = 0 \), that is,

(1) \( c = 1 \) or \( \varphi T \perp E_2 \) or \( \varphi T \in \text{span} \{E_2, \ldots, E_k\} \); and

(2) \( g(\tau_3, E_i) = 0 \), for all \( i = 1, \ldots, k \).

So we can state the following theorem:
Theorem 3.1. Let $\gamma$ be a non-geodesic Legendre Frenet curve of osculating order $r$ in a Sasakian space form $(\mathcal{M}^{2m+1}, \varphi, \xi, \eta, g)$ and $k = \min \{r, 4\}$. Then $\gamma$ is $f$-biharmonic if and only if

1. $c = 1$ or $\varphi T \perp E_2$, or $\varphi T \in \text{span} \{E_2, ..., E_k\}$; and

2. the first $k$ of the following equations are satisfied (replacing $\kappa_k = 0$):

\[3\kappa_1' + 2\kappa_1 \frac{f'}{f} = 0,\]
\[\kappa_2^2 + \kappa_2^2 = 1 + \frac{\kappa_2'}{\kappa_2} + \frac{f''}{f},\]
\[\kappa_2' + 2\kappa_2 \frac{\kappa_2'}{\kappa_2} + 2\kappa_2 \frac{c_1}{\kappa_2} = 0,\]
\[\kappa_2 \kappa_3 = 0.\]

From Theorem 3.1, it can be easily seen that a curve $\gamma$ with constant geodesic curvature $\kappa_1$ is $f$-biharmonic if and only if it is biharmonic. Since Fetcu and Oniciuc studied biharmonic Legendre curves in Sasakian space forms in [7], we study curves with non-constant geodesic curvature $\kappa_1$ in this paper. If $\gamma$ is a non-biharmonic $f$-biharmonic curve, then we call it proper $f$-biharmonic.

Now we give the interpretations of Theorem 3.1.

Case I. $c = 1$.

In this case $\gamma$ is proper $f$-biharmonic if and only if

\[3\kappa_1' + 2\kappa_1 \frac{f'}{f} = 0,\]
\[\kappa_2^2 + \kappa_2^2 = 1 + \frac{\kappa_2'}{\kappa_2} + \frac{f''}{f},\]
\[\kappa_2' + 2\kappa_2 \frac{\kappa_2'}{\kappa_2} + 2\kappa_2 \frac{c_1}{\kappa_2} = 0,\]
\[\kappa_2 \kappa_3 = 0.\]

Hence, we can state the following theorem:

Theorem 3.2. Let $\gamma$ be a Legendre Frenet curve in a Sasakian space form $(\mathcal{M}^{2m+1}, \varphi, \xi, \eta, g)$, $c = 1$ and $m > 1$. Then $\gamma$ is proper $f$-biharmonic if and only if either

(i) $\gamma$ is of osculating order $r = 2$ with $f = c_1 \kappa_1^{-3/2}$ and $\kappa_1$ satisfies

\[t + \frac{1}{2} \arctan \left( \frac{2 + c_3 \kappa_1}{2\sqrt{-\kappa_1^2 - c_3 \kappa_1 - 1}} \right) + c_4 = 0,\]

where $c_1 > 0$, $c_3 < -2$ and $c_4$ are arbitrary constants, $t$ is the arc-length parameter and

\[1 \leq \sqrt{c_3^2 - 4 - c_3} < \kappa_1(t) < \frac{1}{2} \left( \sqrt{c_3^2 - 4} - c_3 \right);\]

or

(ii) $\gamma$ is of osculating order $r = 3$ with $f = c_1 \kappa_1^{-3/2}$, $\frac{c_3}{\kappa_1} = c_2$ and $\kappa_1$ satisfies

\[t + \frac{1}{2} \arctan \left( \frac{2 + c_3 \kappa_1}{2\sqrt{-(1 + c_2^2) \kappa_1^2 - c_3 \kappa_1 - 1}} \right) + c_4 = 0,\]

where $c_1 > 0$, $c_2 > 0$, $c_3 < -2 \sqrt{1 + c_2}$ and $c_4$ are arbitrary constants, $t$ is the arc-length parameter and

\[\frac{1}{2(1 + c_2^2)}(-\sqrt{c_3^2 - 4(1 + c_2^2)} - c_3) < \kappa_1(t) < \frac{1}{2(1 + c_2^2)}(\sqrt{c_3^2 - 4(1 + c_2^2)} - c_3).\]
Proof. From the first equation of (9), it is easy to see that \( f = c_1 \kappa_1^{-3/2} \) for an arbitrary constant \( c_1 > 0 \). So, we find
\[
\frac{f'}{f} = \frac{-3 \kappa_1'}{2 \kappa_1} \frac{f''}{f} = \frac{15 \left( \frac{\kappa_1'}{\kappa_1} \right)^2}{4} - \frac{3 \kappa_1''}{2 \kappa_1}.
\] (14)

If \( \kappa_2 = 0 \), then \( \gamma \) is of osculating order \( r = 2 \) and the first two equations of (9) must be satisfied. Hence the second equation and (14) give us the ODE
\[
3(\kappa_1')^2 - 2 \kappa_1 \kappa_1'' = 4 \kappa_1^2 (\kappa_1^2 - 1).
\] (15)

Let \( \kappa_1 = \kappa_1(t) \), where \( t \) denotes the arc-length parameter. If we solve (15), we find (10). Since (10) must be well-defined, \(-\kappa_1^2 - c_3 \kappa_1 - 1 > 0 \). Since \( \kappa_1 > 0 \), we have \( c_3 < -2 \) and (11).

If \( \kappa_2 = \text{constant} \neq 0 \), we find \( f \) is a constant. Hence \( \gamma \) is not proper \( f \)-biharmonic in this case. Let \( \kappa_2 = \text{constant} \). From the fourth equation of (9), we have \( \kappa_3 = 0 \). So, \( \gamma \) is of osculating order \( r = 3 \). The third equation of (9) gives us \( \frac{\kappa_3}{c_3} = c_2 \), where \( c_2 > 0 \) is a constant. Replacing in the second equation of (9), we have the ODE
\[
3(\kappa_1')^2 - 2 \kappa_1 \kappa_1'' = 4 \kappa_1^2 [(1 + c_2^2) \kappa_1^2 - 1]
\]
which has the general solution (12) under the condition \( c_3 < -2 \sqrt{(1 + c_2^2)} \). (13) must be also satisfied. \( \square \)

Remark 3.3. If \( m = 1 \), then \( M \) is a 3-dimensional Sasakian space form. Since a Legendre curve in a Sasakian 3-manifold has torsion 1 (see [1]), we can write \( \kappa_1 > 0 \) and \( \kappa_2 = 1 \). The first and the third equations of (9) give us \( f \) is a constant. Hence \( \gamma \) cannot be proper \( f \)-biharmonic.

Case II. \( \gamma \neq 1 \), \( qT \perp E_2 \).

In this case, \( g(qT, E_2) = 0 \). From Theorem 3.1, we obtain
\[
3 \kappa_1' + 2 \kappa_1 \frac{f'}{f} = 0,
\]
\[
\kappa_1^2 + \kappa_2^2 = \frac{c_3^3}{4} + \frac{\kappa_1''}{\kappa_1} + \frac{f''}{f} + 2 \frac{\kappa_1'}{\kappa_1} \frac{f'}{f},
\]
\[
\kappa_2' = 2 \kappa_2 \frac{f'}{f} + 2 \kappa_2 \frac{\kappa_1'}{\kappa_1} = 0,
\]
\[
k_2 \kappa_3 = 0.
\] (16)

Firstly, we need the following proposition from [7]:

Proposition 3.4. [7] Let \( \gamma \) be a Legendre Frenet curve of osculating order 3 in a Sasakian space form \( \mathcal{M}^{2m+1}, q, \xi, \eta, g \) and \( qT \perp E_2 \). Then \( \{T = E_1, E_2, E_3, qT, V_T qT, \xi, \zeta\} \) is linearly independent at any point of \( \gamma \). Therefore \( m \geq 3 \).

Now we can state the following Theorem:

Theorem 3.5. Let \( \gamma \) be a Legendre Frenet curve in a Sasakian space form \( \mathcal{M}^{2m+1}, q, \xi, \eta, g \), \( c \neq 1 \) and \( qT \perp E_2 \). Then \( \gamma \) is proper biharmonic if and only if

(1) \( \gamma \) is of osculating order \( r = 2 \) with \( f = c_1 \kappa_1^{-3/2} \), \( m \geq 2 \), \( \{T = E_1, E_2, qT, V_T qT, \xi, \zeta\} \) is linearly independent and

(a) if \( c > -3 \), then \( \kappa_1 \) satisfies
\[
t \pm \frac{1}{\sqrt{c + 3}} \arctan \left( \frac{c + 3 + 2 c_3 \kappa_1}{\sqrt{c + 3} \sqrt{-4 c_1^2 - 4 c_3 \kappa_1 - c - 3}} \right) + c_4 = 0,
\]

(b) if \( c = -3 \), then \( \kappa_1 \) satisfies
\[
t \pm \frac{\sqrt{-c_1^2 \kappa_1 + c_2}}{c_3 \kappa_1} + c_4 = 0,
\]
Theorem 3.6. There does not exist any proper \(f\)-biharmonic Legendre curve in a Sasakian space form.

Proof. The proof is similar to the proof of Theorem 3.2. □

Case III. \(c \neq 1\), \(qT \parallel E_2\).

In this case, \(qT = \pm E_2\), \(g(qT, E_2) = \pm 1\), \(g(qT, E_3) = g(\pm E_2, E_3) = 0\) and \(g(qT, E_4) = g(\pm E_2, E_4) = 0\). From Theorem 3.1, \(\gamma\) is biharmonic if and only if

\[
\begin{align*}
3k_1' + 2k_2' &= 0, \\
k'_2 + k^2 &= c + \frac{k_1'}{k_1} + \frac{\kappa}{k_1} + 2 \frac{k_1'}{k_1}, \\
k'_2 + 2k_2' &= 0, \\
k_2k_3 &= 0.
\end{align*}
\]

Since \(qT \parallel E_2\), it is easily proved that \(k_2 = 1\). Then, the first and the third equations of (17) give us \(f\) is a constant. Thus, we give the following Theorem:

Theorem 3.6. There does not exist any proper \(f\)-biharmonic Legendre curve in a Sasakian space form \((M^{2m+1}, \varphi, \xi, \eta, g)\) with \(c \neq 1\) and \(qT \parallel E_2\).

Case IV. \(c \neq 1\) and \(g(qT, E_2)\) is not constant 0, 1 or \(-1\).

Now, let \((M^{2m+1}, \varphi, \xi, \eta, g)\) be a Sasakian space form and \(\gamma : I \rightarrow M\) a Legendre curve of osculating order \(r\), where \(4 \leq r \leq 2m + 1 \) and \(m \geq 2\). If \(\gamma\) is \(f\)-biharmonic, then \(qT \in \text{span}\{E_2, E_3, E_4\}\). Let \(\theta(t)\) denote the angle function between \(qT\) and \(E_2\), that is, \(g(qT, E_2) = \cos \theta(t)\). Differentiating \(g(qT, E_2)\) along \(\gamma\) and using (1) and (4), we find

\[
-\theta'(t) \sin \theta(t) = V_\gamma g(qT, E_2) = g(V_\gamma qT, E_2) + g(qT, V_\gamma E_2)
= g(\xi + k_1qE_2, E_2) + g(qT, -k_1T + k_2E_3)
= k_2 g(qT, E_3).
\]

(c) if \(c < -3\), then \(k_1\) satisfies

\[
t \pm \frac{1}{\sqrt{-c - 3}} \ln \left(\frac{c + 3 + 2c_3k_1 - \sqrt{-c - 3}\sqrt{-4c_3^2k_1^2 - 4c_3k_1 - c - 3}}{(c + 3)k_1} + c_4 = 0\right)
\]

(2) \(\gamma\) is of osculating order \(r \leq 3\) with \(f = c_1k_1^{-3/2}, c_2 = \text{constant} > 0\), \(m \geq 3\), \(\{T = E_1, E_2, E_3, qT, V_\gamma qT, \xi, \eta\}\) is linearly independent and 

(a) if \(c > -3\), then \(k_1\) satisfies

\[
t \pm \frac{1}{\sqrt{-c + 3}} \arctan \left(\frac{c + 3 + 2c_3k_1}{\sqrt{c + 3}\sqrt{-4(1 + c_2^2)k_1^2 - 4c_3k_1 - c - 3}}\right) + c_4 = 0
\]

(b) if \(c = -3\), then \(k_1\) satisfies

\[
t \pm \frac{\sqrt{-k_1}}{c_3k_1} \left(1 + c_2^2k_1 + c_3\right) + c_4 = 0
\]

(c) if \(c < -3\), then \(k_1\) satisfies

\[
t \pm \frac{1}{\sqrt{-c - 3}} \ln \left(\frac{c + 3 + 2c_3k_1 - \sqrt{-c - 3}\sqrt{-4(1 + c_2^2)k_1^2 - 4c_3k_1 - c - 3}}{(c + 3)k_1} + c_4 = 0\right)
\]

where \(c_1 > 0\), \(c_2 > 0\), \(c_3\) and \(c_4\) are convenient arbitrary constants, \(t\) is the arc-length parameter and \(k_1(t)\) is in convenient open interval.
If we write \( \phi T = \varphi(qT, E_2)E_2 + \varphi(qT, E_3)E_3 + \varphi(qT, E_4)E_4 \), Theorem 3.1 gives us

\[
3\kappa_1' + 2\kappa_1'' = 0, \quad (19)
\]

\[
\kappa_1^2 + \kappa_2^2 = \frac{c + 3}{4} + \frac{3(c - 1)}{4} \cos^2 \theta + \frac{\kappa_1''}{\kappa_1} \frac{f''}{f} + \frac{2}{\kappa_1} \frac{\kappa_1'}{\kappa_1}, \quad (20)
\]

\[
\kappa_2 + \frac{3(c - 1)}{4} \cos \theta \varphi(qT, E_3) + 2\kappa_2 f' + 2\kappa_2' = 0, \quad (21)
\]

\[
\kappa_2 \kappa_3 + \frac{3(c - 1)}{4} \cos \theta \varphi(qT, E_4) = 0. \quad (22)
\]

If we put (14) in (20) and (21) respectively, we obtain

\[
\kappa_1^2 + \kappa_2^2 = \frac{c + 3}{4} + \frac{3(c - 1)}{4} \cos^2 \theta - \frac{\kappa_1'}{2\kappa_1} + \frac{3}{4} \left(\frac{\kappa_1'}{\kappa_1}\right)^2, \quad (23)
\]

\[
\kappa_2' - \frac{\kappa_1'}{\kappa_1} \kappa_2 + \frac{3(c - 1)}{4} \cos \theta \varphi(qT, E_3) = 0. \quad (24)
\]

If we multiply (24) with \(2\kappa_2\), using (18), we find

\[
2\kappa_2' = -2\frac{\kappa_1'}{\kappa_1} \kappa_2^2 + \frac{3(c - 1)}{4} (-2\theta' \cos \theta \sin \theta) = 0. \quad (25)
\]

Let us denote \( v(t) = \kappa_2^2(t) \), where \( t \) is the arc-length parameter. Then (25) becomes

\[
v' - 2\frac{\kappa_1'}{\kappa_1} v = -\frac{3(c - 1)}{4} (-2\theta' \cos \theta \sin \theta), \quad (26)
\]

which is a linear ODE. If we solve (26), we obtain the following results:

i) If \( \theta \) is a constant, then

\[
\frac{\kappa_2}{\kappa_1} = c_2, \quad (27)
\]

where \( c_2 > 0 \) is an arbitrary constant. From (18), we find \( \varphi(qT, E_3) = 0 \). Since \( \|\varphi T\| = 1 \) and \( \varphi T = \cos \theta E_2 + \varphi(qT, E_4)E_4 \), we get \( \varphi(qT, E_4) = \pm \sin \theta \). By the use of (20) and (27), we find

\[
3(\kappa_1')^2 - 2\kappa_1' \kappa_2' = 4\kappa_1^2 [(1 + c_2^2)\kappa_1^2 - \frac{c + 3 + 3(c - 1) \cos^2 \theta}{4}]. \quad (28)
\]

ii) If \( \theta = \theta(t) \) is a non-constant function, then

\[
\kappa_2^2 = -\frac{3(c - 1)}{4} \cos^2 \theta + \lambda(t)\kappa_1^2, \quad (28)
\]

where

\[
\lambda(t) = -\frac{3(c - 1)}{2} \int \frac{\cos^2 \theta \kappa_1'}{\kappa_1'} dt. \quad (29)
\]

If we write (28) in (23), we have

\[
[1 + \lambda(t)] \kappa_1^2 = \frac{c + 3 + 6(c - 1) \cos^2 \theta}{4} - \frac{\kappa_1''}{2\kappa_1} + \frac{3}{4} \left(\frac{\kappa_1'}{\kappa_1}\right)^2.
\]

Now we can state the following Theorem:
Theorem 3.7. Let \( \gamma : I \to M \) be a Legendre curve of osculating order \( r \) in a Sasakian space form \((M^{2m+1}, \varphi, \xi, \eta, g)\), where \( r \geq 4, m \geq 2, c \neq 1 \), \( g(\varphi T, E_2) = \cos \theta(t) \) is not constant \( 0, 1 \) or \(-1\). Then \( \gamma \) is proper \( f \)-biharmonic if and only if \( f = c_1 \kappa_1^{-3/2} \) and

(i) if \( \theta \) is a constant,

\[
\frac{\kappa_2}{\kappa_1} = c_2,
\]

\[
3(\kappa_1')^2 - 2\kappa_1 \kappa_1'' = 4\kappa_1^2 [(1 + c_2^2)\kappa_1^2 - \frac{c + 3 + 3(c - 1)\cos^2 \theta}{4} ],
\]

\[
\kappa_2 \kappa_3 = \pm \frac{3(c - 1) \sin 2\theta}{8},
\]

(ii) if \( \theta \) is a non-constant function,

\[
\kappa_2^2 = -\frac{3(c - 1)}{4} \cos^2 \theta + \lambda(t) \kappa_1^2,
\]

\[
3(\kappa_1')^2 - 2\kappa_1 \kappa_1'' = 4\kappa_1^2 [(1 + \lambda(t))\kappa_1^2 - \frac{c + 3 + 6(c - 1)\cos^2 \theta}{4} ],
\]

\[
\kappa_2 \kappa_3 = \pm \frac{3(c - 1) \sin 2\theta \sin \omega}{8},
\]

where \( c_1 \) and \( c_2 \) are positive constants, \( q \varphi T = \cos \theta E_2 \pm \sin \theta \cos \omega E_3 \pm \sin \theta \sin \omega E_4 \), \( \omega \) is the angle function between \( E_3 \) and the orthogonal projection of \( q \varphi T \) onto \( \text{span} \{E_3, E_4\} \). \( \omega \) is related to \( \theta \) by \( \cos \omega = \frac{\kappa_2}{\kappa_1} \) and \( \lambda(t) \) is given by

\[
\lambda(t) = -\frac{3(c - 1)}{2} \int \frac{\cos^2 \theta \kappa_1''}{\kappa_1^4} dt.
\]

We can give the following direct corollary of Theorem 3.7:

Corollary 3.8. Let \( \gamma : I \to M \) be a Legendre curve of osculating order \( r \) in a Sasakian space form \((M^{2m+1}, \varphi, \xi, \eta, g)\), where \( r \geq 4, m \geq 2, c \neq 1 \), \( g(\varphi T, E_2) = \cos \theta \) is a constant and \( \theta \in (0, 2\pi) \setminus \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} \). Then \( \gamma \) is proper \( f \)-biharmonic if and only if \( f = c_1 \kappa_1^{-3/2} \), \( \kappa_2 \kappa_3 = c_2 = \text{constant} > 0 \),

\[
\kappa_2 \kappa_3 = \pm \frac{3(c - 1) \sin 2\theta}{8},
\]

\[
\kappa_4 = \pm \frac{\eta(E_5) + g(\varphi E_2, E_5) \kappa_1}{\sin \theta} \quad (\text{if } r > 4); \text{ and}
\]

(i) if \( a > 0 \), then \( \kappa_1 \) satisfies

\[
t \pm \frac{1}{2 \sqrt{a}} \arctan \left( \frac{1}{2 \sqrt{a}} \frac{2a + c_3 \kappa_1}{\sqrt{-(1 + c_2^2)\kappa_1^2 - c_3 \kappa_1 - a}} \right) + c_4 = 0,
\]

(ii) if \( a = 0 \), then \( \kappa_1 \) satisfies

\[
t \pm \sqrt{-\kappa_1 [(1 + c_2^2)\kappa_1 + c_3]} \frac{c_3 \kappa_1}{c_2 \kappa_1} + c_4 = 0,
\]
(iii) if \( a < 0 \), then \( \kappa_1 \) satisfies
\[
t = \frac{1}{2} \ln \left( \frac{2a + c_3 \kappa_1 - 2 \sqrt{-a}}{2a \kappa_1} \right) + c_4 = 0,
\]
where \( a = \left[ c + 3 + 3(c - 1) \cos^2 \theta \right] / 4 \). \( \phi T = \cos \theta E_2 \pm \sin \theta E_4 \), \( c_1 > 0 \), \( c_2 > 0 \), \( c_3 \) and \( c_4 \) are convenient arbitrary constants, \( t \) is the arc-length parameter and \( \kappa_1(t) \) is in convenient open interval.

In order to obtain explicit examples, we will first need to recall some notions about the Sasakian space form \( \mathbb{R}^{2m+1}(-3) \) [3]:

Let us consider \( M = \mathbb{R}^{2m+1} \) with the standard coordinate functions \((x_1, \ldots, x_m, y_1, \ldots, y_m, z)\), the contact structure \( \eta = \frac{1}{2} (dz - \sum_{i=1}^{m} y_i dx_i) \), the characteristic vector field \( \xi = 2 \frac{\partial}{\partial z} \) and the tensor field \( \phi \) given by
\[
\phi = \begin{bmatrix}
0 & \delta_{ij} & 0 \\
-\delta_{ij} & 0 & 0 \\
0 & y_j & 0
\end{bmatrix}.
\]
The associated Riemannian metric is \( g = \eta \otimes \eta + \frac{1}{2} \sum_{i=1}^{m} (dx_i)^2 + (dy_i)^2 \). Then \((M, \phi, \xi, \eta, g)\) is a Sasakian space form with constant \( \phi \)-sectional curvature \( c = -3 \) and it is denoted by \( \mathbb{R}^{2m+1}(-3) \). The vector fields
\[
X_i = 2 \frac{\partial}{\partial y_i}, \quad X_{m+i} = \phi X_i = 2(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}), \quad i = 1, m, \quad \xi = 2 \frac{\partial}{\partial z}
\]
form a \( \phi \)-orthonormal basis and the Levi-Civita connection is calculated as
\[
\nabla_{X_i} X_j = \nabla_{X_j} X_i = 0, \quad \nabla_{X_i} X_{m+j} = \delta_{ij} \xi, \quad \nabla_{X_m} X_j = -\delta_{ij} \xi,
\]
\[
\nabla_{X_m} \xi = -X_{m+i}, \quad \nabla_{X_m} \xi = \xi, \quad \nabla_{X_{m+i}} X_i = X_i
\]
(see [3]).

Now, let us produce examples of proper \( f \)-biharmonic Legendre curves in \( \mathbb{R}^7(-3) \):

Let \( \gamma = (\gamma_1, \ldots, \gamma_7) \) be a unit speed curve in \( \mathbb{R}^7(-3) \). The tangent vector field of \( \gamma \) is
\[
T = \frac{1}{2} \left[ (\gamma'_4)^2 X_1 + (\gamma'_5)^2 X_2 + (\gamma'_6)^2 X_3 + (\gamma'_4)^2 X_4 + (\gamma'_5)^2 X_5 + (\gamma'_6)^2 X_6 + (\gamma'_7 - \gamma'_4 \gamma_4 - \gamma'_5 \gamma_5 - \gamma'_6 \gamma_6) \xi \right].
\]
Thus, \( \gamma \) is a unit speed Legendre curve if and only if \( \eta(T) = 0 \) and \( g(T, T) = 1 \), that is,
\[
\gamma'_7 = \gamma'_4 \gamma'_4 + \gamma'_5 \gamma'_5 + \gamma'_6 \gamma'_6
\]
and
\[
(\gamma'_4)^2 + \ldots + (\gamma'_7)^2 = 4.
\]
For a Legendre curve, we can use the Levi-Civita connection and (30) to write
\[
\nabla_T T = \frac{1}{2} \left( (\gamma''_4)^2 X_1 + (\gamma''_5)^2 X_2 + (\gamma''_6)^2 X_3 + (\gamma''_4)^2 X_4 + (\gamma''_5)^2 X_5 + (\gamma''_6)^2 X_6 \right),
\]
\[
\phi T = \frac{1}{2} \left( -\gamma'_4 X_1 - \gamma'_5 X_2 - \gamma'_6 X_3 + \gamma'_4 X_4 + \gamma'_5 X_5 + \gamma'_6 X_6 \right).
\]
From (31) and (32), \( \phi T \perp E_2 \) if and only if
\[
\gamma''_1 \gamma'_4 + \gamma''_2 \gamma'_5 + \gamma''_3 \gamma'_6 = \gamma'_1 \gamma'_4' + \gamma'_2 \gamma'_5' + \gamma'_3 \gamma'_6'.
\]
Finally, we can give the following explicit examples:
**Example 3.9.** Let us take $\gamma(t) = (2 \sinh^{-1}(t), \sqrt{1+t^2}, \sqrt{3} \sqrt{1+t^2}, 0, 0, 0, 1)$ in $\mathbb{R}^7(-3)$. Using the above equations and Theorem 3.5, $\gamma$ is a proper $f$-biharmonic Legendre curve with osculating order $r = 2$, $\kappa_1 = \frac{1}{1+t^2}$, $f = c_1(1+t^2)^{3/2}$ where $c_1 > 0$ is a constant. We can easily check that the conditions of Theorem 3.5 (i.e. $c \neq 1, \varphi^T \perp E_2$) are verified, where $c_3 = -1$ and $c_4 = 0$.

**Example 3.10.** Let $\gamma(t) = (a_1, a_2, a_3, \sqrt{2} t, 2 \sinh^{-1}(\frac{t}{\sqrt{2}}), \sqrt{2} \sqrt{1+t^2}, a_4)$ be a curve in $\mathbb{R}^7(-3)$, where $a_i \in \mathbb{R}$, $i = 1,4$. Then we calculate

$$T = \frac{\sqrt{2}}{2} X_1 + \frac{1}{\sqrt{2}+t^2} X_2 + \frac{\sqrt{2}t}{2 \sqrt{2}+t^2} X_3,$$

$$E_2 = \frac{-t}{\sqrt{2}+t^2} X_2 + \frac{\sqrt{2}}{\sqrt{2}+t^2} X_3,$$

$$E_3 = \frac{\sqrt{2}}{2} X_1 - \frac{1}{\sqrt{2}+t^2} X_2 - \frac{\sqrt{2}t}{2 \sqrt{2}+t^2} X_3,$$

$$\kappa_1 = \kappa_2 = \frac{1}{2 + t^2}, \quad r = 3.$$

From Theorem 3.5, it follows that $\gamma$ is proper $f$-biharmonic with $f = c_1(2 + t^2)^{3/2}$, where $c_1 > 0$, $c_2 = 1$, $c_3 = -1$ and $c_4 = 0$.

### References