On Regularly Generated Double Sequences

Ümit Totur\textsuperscript{a}, İbrahim Çanak\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Adnan Menderes University, Aydın, Turkey
\textsuperscript{b}Department of Mathematics, Ege University, İzmir, Turkey

Abstract. In this paper, we introduce regularly generated sequences for double sequence of real numbers, and obtain some Tauberian theorems for $(C, 1, 1)$ summability method using the concept of regularly generated sequence.

1. Introduction and Definitions

A double sequence $u = (u_{mn})$ is called Pringsheim convergent (or $P$-convergent) \cite{1} to $\ell$ if for a given $\varepsilon > 0$ there exists a positive integer $N_0$ such that $|u_{mn} - \ell| < \varepsilon$ for all nonnegative integers $m, n \geq N_0$. The $(C, 1, 1)$ means of $(u_{mn})$ are defined by

$$\sigma^{(11)}_{mn}(u) = \frac{1}{(m + 1)(n + 1)} \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij}$$

for nonnegative integers $m, n$ (see \cite{2}). The sequence $(u_{mn})$ is said to be $(C, 1, 1)$ summable to a finite number $\ell$ if $\lim_{m,n \to \infty} \sigma^{(11)}_{mn}(u) = \ell$.

Every convergent double sequence in Pringsheim’s sense need not be $(C, 1, 1)$ summable. For example, the sequence $(u_{mn})$ defined by

$$u_{mn} = \begin{cases} n, & \text{if } m = 0; n = 0, 1, 2, \ldots \\ 0, & \text{otherwise,} \end{cases}$$

is convergent to 0. But, the limit

$$\lim_{m,n \to \infty} \frac{1}{(m + 1)(n + 1)} \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij} = \lim_{m,n \to \infty} \frac{n^2 + n}{2(n + 1)(m + 1)}$$

does not tend to a finite limit. Therefore, $(u_{mn})$ is not $(C, 1, 1)$ summable.

The $(C, 1, 0)$ and $(C, 0, 1)$ means of $(u_{mn})$ are defined respectively by

$$\sigma^{(10)}_{mn}(u) = \frac{1}{m + 1} \sum_{i=0}^{m} u_{im} \quad \text{and} \quad \sigma^{(01)}_{mn}(u) = \frac{1}{n + 1} \sum_{j=0}^{n} u_{mj}$$

\textsuperscript{2010 Mathematics Subject Classification}. Primary 40E05; Secondary 40G05

\textit{Keywords}. Tauberian theorems, double sequence, regularly generated sequence, one-sided condition, slowly oscillating sequence

Received: 23 January 2015; Revised 29 March 2015; Accepted: 04 April 2015

Communicated by Ljubiša D. R. Kočinac

\textit{Email addresses}: utotur@adu.edu.tr (Ümit Totur), ibrahim.canak@ege.edu.tr (İbrahim Çanak)
for nonnegative integers \( m, n \). The sequence \( u_{mn} \) is said to be \( (C,1,0) \) summable to a finite number \( \ell \) if 
\[
\lim_{m,n \to \infty} s_{mn}^{(10)}(u) = \ell.
\]
In the light of above discussion, the \( (C,0,1) \) summability is defined analogously.

A double sequence \( (u_{mn}) \) is said to be bounded if there exists a real number \( C > 0 \) such that \( |u_{mn}| \leq C \) for all nonnegative integers \( m, n \). Note that every \( P \)-convergent double sequence need not be bounded. For example, the double sequence
\[
(u_{mn}) = \begin{pmatrix}
1 & 4 & 9 & 16 & \cdots \\
2 & 0 & 0 & 0 & \cdots \\
3 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]
is \( P \)-convergent to 0, but it is not bounded.

A double sequence \( (u_{mn}) \) is said to be one-sided bounded if there exists a real number \( C > 0 \) such that \( u_{mn} \geq -C \) for all nonnegative integers \( m, n \).

Let \( N, B, \) and \( B^r \) denote the set of all double sequences which is \( P \)-converging to 0, bounded, one-sided bounded, respectively.

For a double sequence \( (u_{mn}) \), we define \( \Delta_m u_{mn} = u_{mn} - u_{m,n-1}, \Delta_n u_{mn} = u_{mn} - u_{m-1,n}, \) and \( \Delta_{m,n} u_{mn} = \Delta_m(\Delta_n u_{mn}) = \Delta_n(\Delta_m u_{mn}) \) for all integers \( m, n \geq 1 \).

We define de la Vallé Poussin means of the double sequence \( (u_{mn}) \) as follows: If \( \lambda > 1 \)
\[
\tau^>(u) = \lim_{\lambda \to 1^+} \frac{1}{(\lambda m - m)(\lambda n - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} u_{jk},
\]
and if \( 0 < \lambda < 1 \)
\[
\tau^<(u) = \lim_{\lambda \to 1^-} \frac{1}{(m - \lfloor \lambda m \rfloor)(n - \lfloor \lambda n \rfloor)} \sum_{j=\lfloor \lambda m \rfloor+1}^{m} \sum_{k=\lfloor \lambda n \rfloor+1}^{n} u_{jk},
\]
for sufficiently large nonnegative integers \( m, n \).

Now, give the concept of slow oscillation in different senses for a double sequence.

**Definition 1.1.** A double sequence \( (u_{mn}) \) is said to be slowly oscillating in sense \((1,1)\) if
\[
\lim_{\lambda \to 1^+} \lim_{m,n \to \infty} \max_{m+1 \leq j \leq \lfloor \lambda m \rfloor} \max_{n+1 \leq k \leq \lfloor \lambda n \rfloor} \left| \sum_{j=m+1}^{j} \sum_{k=n+1}^{k} \Delta_{j,k} u_{jk} \right| = 0,
\]
\( (u_{mn}) \) is said to be slowly oscillating in sense \((1,0)\) if
\[
\lim_{\lambda \to 1^+} \lim_{m,n \to \infty} \max_{m+1 \leq j \leq \lfloor \lambda m \rfloor} \left| \sum_{j=m+1}^{j} \Delta_{j,n} u_{jn} \right| = 0,
\]
\( (u_{mn}) \) is said to be slowly oscillating in sense \((0,1)\) if
\[
\lim_{\lambda \to 1^+} \lim_{m,n \to \infty} \max_{n+1 \leq k \leq \lfloor \lambda n \rfloor} \left| \sum_{k=n+1}^{k} \Delta_{j,m} u_{jk} \right| = 0.
\]
\( S_{11}, S_{10}, \) and \( S_{01} \) denote the classes of all slowly oscillating sequences in sense \((1,1)\), \((1,0)\), and \((0,1)\), respectively.

Notice that every \( P \)-convergent sequence is slowly oscillating in senses \((1,1)\), \((1,0)\), and \((0,1)\). However, the converse may not be true. The following example provides slowly oscillating sequences in senses \((1,1)\), \((1,0)\), and \((0,1)\), but they are not \( P \)-convergent.
Example 1.2.

\((u_{mn}) = (\log m \log n) \in S_{11}\).

Indeed, since

\[
\Delta_{rs} \log r \log s = \log r \log s - \log(r - 1) \log s - \log r \log(s - 1) + \log(r - 1) \log(s - 1),
\]

we have

\[
\Delta_{rs} \log r \log s = \log r \log \left( \frac{s}{s - 1} \right) - \log(r - 1) \log \left( \frac{s}{s - 1} \right).
\]

Therefore,

\[
\sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{rs} \log r \log s = \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \log \left( \frac{r}{r - 1} \right) \log \left( \frac{s}{s - 1} \right) = \log \left( \frac{j}{m} \right) \log \left( \frac{k}{n} \right).
\]

From this, we obtain

\[
\max_{m+1 \leq r \leq \lceil \lambda m \rceil} \left| \sum_{s=m+1}^{n} \sum_{n+1}^{\lfloor \lambda n \rfloor} \Delta_{rs} \log r \log s \right| = \log \left( \frac{\lceil \lambda m \rceil}{m} \right) \log \left( \frac{\lfloor \lambda n \rfloor}{n} \right).
\]

After taking lim sup of both sides as \(m, n \to \infty\), we obtain

\[
\limsup_{m,n \to \infty} \max_{m+1 \leq r \leq \lfloor \lambda m \rfloor} \left| \sum_{s=m+1}^{n} \sum_{n+1}^{\lfloor \lambda n \rfloor} \Delta_{rs} \log r \log s \right| = \log^{2} \lambda.
\]

Finally, taking the limit of both sides as \(\lambda \to 1^{+}\), we get

\[
\lim_{\lambda \to 1^{+}} \limsup_{m,n \to \infty} \max_{m+1 \leq r \leq \lfloor \lambda m \rfloor} \left| \sum_{s=m+1}^{n} \sum_{n+1}^{\lfloor \lambda n \rfloor} \Delta_{rs} u_{rs} \right| = \lim_{\lambda \to 1^{+}} \log^{2} \lambda = 0.
\]

(ii) \((u_{mn}) = (\log m) \in S_{10}\).

(iii) \((u_{mn}) = (\log n) \in S_{01}\).

The \((C, 1, 1)\) means of \((mn \Delta_{mn} u_{mn})\) is defined by

\[
V_{mn}^{(11)}(\Delta_{mn} u) := \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} ij \Delta_{ij} u_{ij}.
\]

Moreover, the \((C, 1, 0)\) means of \((m \Delta_{m} u_{mn})\) is defined by

\[
V_{mn}^{(10)}(\Delta_{m} u) := \frac{1}{m+1} \sum_{j=0}^{m} j \Delta_{mn} u_{mn},
\]

and the \((C, 0, 1)\) means of \((n \Delta_{n} u_{mn})\) is defined by

\[
V_{mn}^{(01)}(\Delta_{n} u) := \frac{1}{n+1} \sum_{j=0}^{n} j \Delta_{mn} u_{mn}.
\]

The Kronecker identity for single sequences takes the following form for double sequences (see [3]). For all nonnegative integers \(m, n\),

\[
u_{mn} - c_{mn}^{(10)}(u) - c_{mn}^{(01)}(u) + c_{mn}^{(11)}(u) = V_{mn}^{(11)}(\Delta_{mn} u).
\]
We write the following identities similar to the Kronecker identity for single sequences.

\[ u_{mn} - \sigma_{mn}^{(10)}(u) = V_{mn}^{(10)}(\Delta_m u), \]  
\[ u_{mn} - \sigma_{mn}^{(01)}(u) = V_{mn}^{(01)}(\Delta_n u). \]  

The following lemma shows the relationships between Cesàro means \( \sigma_{mn}^{(11)}(u) \), \( \sigma_{mn}^{(10)}(u) \), \( \sigma_{mn}^{(01)}(u) \) and \( V_{mn}^{(11)}(\Delta_m u) \), \( V_{mn}^{(10)}(\Delta_n u) \), \( V_{mn}^{(01)}(\Delta_n u) \), respectively.

Lemma 1.3. For a double sequence \( (u_{mn}) \) of real numbers,
\[ mn\Delta_{mn}\sigma_{mn}^{(11)}(u) = V_{mn}^{(11)}(\Delta_m u), \]  
\[ m\Delta_{mn}\sigma_{mn}^{(10)}(u) = V_{mn}^{(10)}(\Delta_n u), \]  
\[ n\Delta_{mn}\sigma_{mn}^{(01)}(u) = V_{mn}^{(01)}(\Delta_m u), \]  
for all nonnegative integers \( m, n \).

Proof. First, we prove the identity (4). We have
\[ \Delta_{mn}\sigma_{mn}^{(11)}(u) = \sigma_{mn}^{(11)}(u) - \sigma_{m,n-1}^{(11)}(u) - \sigma_{m-1,n}^{(11)}(u) + \sigma_{m-1,n-1}^{(11)}(u) \]
\[ = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij} - \frac{1}{(m+1)n} \sum_{i=0}^{m} \sum_{j=0}^{n-1} u_{ij} \]
\[ - \frac{1}{m(n+1)} \sum_{i=0}^{m-1} \sum_{j=0}^{n} u_{ij} + \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u_{ij} \]
\[ = \frac{1}{mn(m+1)(n+1)} \left( mn \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij} - m(n+1) \sum_{i=0}^{m} \sum_{j=0}^{n-1} u_{ij} \right) \]
\[ - n(m+1) \sum_{i=0}^{m-1} \sum_{j=0}^{n} u_{ij} + (m+1)(n+1) \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u_{ij} \right). \]

From these lines we deduce that
\[ mn\Delta_{mn}\sigma_{mn}^{(11)}(u) = \frac{1}{(m+1)(n+1)} \left( mn \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u_{ij} + mn \sum_{i=0}^{m-1} u_{mj} + mn \sum_{j=0}^{n-1} u_{mj} \right) \]
\[ + mn u_{mn} - m(n+1) \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u_{ij} - m(n+1) \sum_{j=0}^{n-1} u_{mj} \]
\[ - (m+1) n \sum_{i=0}^{m-1} \sum_{j=0}^{n} u_{ij} - (m+1)n \sum_{i=0}^{m-1} u_{in} + (m+1)(n+1) \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u_{ij} \right). \]

We finally obtain
\[ mn\Delta_{mn}\sigma_{mn}^{(11)}(u) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (u_{mn} - u_{in} - u_{mj} + u_{ij}) \]
\[ = \frac{1}{(m+1)(n+1)} \sum_{i=1}^{m} \sum_{j=1}^{n} ij \Delta_{ij} u_{ij} \]
\[ = V_{mn}^{(11)}(\Delta_m u). \]
Now, we prove the identity (5). We have
\[
\Delta_m \sigma^{(10)}_{mn}(u) = \sigma^{(10)}_{mn}(u) - \sigma^{(10)}_{m-1,m}(u)
\]
\[
= \frac{1}{m+1} \sum_{i=0}^{m} u_{in} - \frac{1}{m} \sum_{i=0}^{m-1} u_{in}
\]
\[
= \frac{1}{m(m+1)} \left( m \sum_{i=0}^{m} u_{in} - (m+1) \sum_{i=0}^{m-1} u_{in} \right).
\]
From these lines we obtain
\[
m \Delta_m \sigma^{(10)}_{mn}(u) = \frac{1}{m+1} \left( m \sum_{i=0}^{m} u_{in} - \sum_{i=0}^{m-1} u_{in} \right)
\]
\[
= \frac{1}{m+1} \left( m u_{mn} - \sum_{i=0}^{m-1} u_{in} \right) = \frac{1}{m+1} \sum_{i=0}^{m} (u_{mn} - u_{in})
\]
\[
= \frac{1}{m+1} \sum_{i=0}^{m} i \Delta_i u_{in} = V^{(10)}_{mn}(\Delta_m u).
\]
The identity (6) can be similarly showed. □

2. Regularly Generated Double Sequences

The idea of regularly generated sequence for single sequences has been introduced by Dik et al. [4]. Using the concept of regularly generated sequence, some Tauberian theorems for Abel summability methods have been obtained by many authors (see [5–7]). In the light of this information, we introduce the concept of regularly generated sequence for double sequences.

Let \( L \) be any linear space of real double sequences and \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) be subclasses of \( L \). If

\[
u_{mn} = \xi_{mn} + v_{mn} + \sum_{i=1}^{n} \sum_{j=1}^{m} \eta_{ij} \eta_{mn},
\]

for some \((\xi_{mn}), (v_{mn}), (\eta_{mn}) \in \mathcal{A}\), we say that the double sequence \((u_{mn})\) is regularly generated by the double sequences \((\xi_{mn}), (v_{mn}), (\eta_{mn})\) and the double sequences \((\xi_{mn}), (v_{mn}), (\eta_{mn})\) are called the generators of \((u_{mn})\).

The classes of all sequences regularly generated by \(\xi = (\xi_{mn}), v = (v_{mn}), \eta = (\eta_{mn})\) are denoted by \(U_1(\xi, v, \eta)\).

If

\[
u_{mn} = \xi_{mn} + \sum_{i=1}^{m} \frac{\xi_{in}}{i},
\]

for some \((\xi_{mn}) \in \mathcal{B}\), we say that the double sequence \((u_{mn})\) is regularly generated by the double sequence \((\xi_{mn})\) and the double sequence \((\xi_{mn})\) is called a generator of \((u_{mn})\). The classes of all sequences regularly generated by \(\xi = (\xi_{mn})\) are denoted by \(U_2(\xi)\).

If

\[
u_{mn} = v_{mn} + \sum_{j=1}^{n} \frac{v_{mj}}{j},
\]

for some \((v_{mn}) \in \mathcal{C}\), we say that the double sequence \((u_{mn})\) is regularly generated by the double sequence \((v_{mn})\) and the double sequence \((v_{mn})\) is called a generator of \((u_{mn})\). The classes of all sequences regularly generated by \(v = (v_{mn})\) are denoted by \(U_3(v)\).
Example 2.1.
(a) If \( S_{11} \) is the class of slowly oscillating sequences in sense (1, 1), then \( U_1(S_{11}, S_{11}, B) \) is the class of sequence.
(b) If \( SB \) is the class of all bounded and slowly oscillating sequences in sense (1, 0), then \( U_2(SB) \) is the class of all slowly oscillating sequences in sense (1, 0).

For a double sequence \( (u_{mn}) \) of real numbers, \( a_m^{(11)}(u) = \sum_{i=1}^m \sum_{j=1}^n \frac{V^{(11)}_{ij}(\Delta_n u)}{ij} \), \( a_m^{(10)}(u) = \sum_{i=1}^m \frac{V^{(10)}_{ij}(\Delta_n u)}{ij} \) and \( a_m^{(01)}(u) = \sum_{j=1}^n \frac{V^{(01)}_{ij}(\Delta_n u)}{ij} \) by Lemma 1.3.

Since \( (u_{mn}) \) can be expressed as
\[
u_{mn} = V_{mn}^{(10)}(\Delta_n u) + \sum_{i=1}^m \frac{V_{mn}^{(11)}(\Delta_i u)}{i} - V_{mn}^{(11)}(\Delta_{mn} u)
\]
the sequences \( V_{mn}^{(10)}(\Delta_n u) \), \( V_{mn}^{(01)}(\Delta_n u) \), and \( V_{mn}^{(11)}(\Delta_{mn} u) \) are generators of \( (u_{mn}) \).

In addition, the sequence \( (u_{mn}) \) can also be represented as
\[
u_{mn} = V_{mn}^{(10)}(\Delta_n u) + \sum_{j=1}^n \frac{V_{mn}^{(01)}(\Delta_j u)}{j}
\]
or
\[
u_{mn} = V_{mn}^{(01)}(\Delta_n u) + \frac{\sum_{i=1}^m \frac{V_{mn}^{(11)}(\Delta_i u)}{i}}{i}
\]
We can say that both \( V_{mn}^{(10)}(\Delta_n u) \) and \( V_{mn}^{(01)}(\Delta_n u) \) are generators of \( (u_{mn}) \).

Lemma 2.2. Let \( (u_{mn}) \in L \) and \( B, C \subseteq L \).

(i) If \( (u_{mn}) \in U_2(B) \), then \( V_{mn}^{(10)}(\Delta_n u) \in B \).

(ii) If \( (u_{mn}) \in U_3(C) \), then \( V_{mn}^{(01)}(\Delta_n u) \in C \).

Proof. (i) Since \( (u_{mn}) \in U_2(B) \), then
\[
u_{mn} = \xi_{mn} + \sum_{i=1}^m \frac{\xi_{in}}{i}
\]
for some \( (\xi_{mn}) \in B \). Hence, we have \( \Delta_m u_{mn} = \Delta_m \xi_{mn} + \frac{\xi_{mn}}{m} \), and \( m \Delta_n u_{mn} = m \Delta_n \xi_{mn} + \xi_{mn} \). Therefore, taking \( (C, 1, 0) \) means of both sides, we get \( V_{mn}^{(10)}(\Delta_n u) = V_{mn}^{(10)}(\Delta_n \xi) + \sigma_{mn}^{(10)}(\xi) \). It follows from the Kronecker identity that \( V_{mn}^{(10)}(\Delta_n u) = \xi_{mn} \). This completes the proof.

(ii) The proof of (ii) is similar to that of (i).

Lemma 2.3. [8] Let \( (u_{mn}) \) be a double sequence of real numbers. For sufficiently large integers \( m, n \):

(i) If \( \lambda > 1 \)
\[
u_{mn} - a_m^{(11)}(u) = a_m^{(11)}(u) = \frac{[\lambda m] + 1}{[\lambda m]} \frac{[\lambda n] + 1}{[\lambda n]} \left( a_m^{(11)}(u) - a_m^{(11)}(u) \right) + \sum_{j=m+1}^{[\lambda m]} \left( a_m^{(11)}(u) - a_m^{(11)}(u) \right) - \frac{1}{[\lambda m] - m} \frac{[\lambda n] - n}{[\lambda n] - n} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn}).
\]
(ii) if \( 0 < \lambda < 1 \)

\[
\begin{align*}
\sigma_{mn}^{(10)}(u) &= \frac{[\lambda m] + 1}{m - [\lambda n]} \left( \sigma_{m,[\lambda n]}^{(10)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) + \sigma_{m,[\lambda n]}^{(11)}(u) \right) \\
+ \frac{[\lambda m] + 1}{m - [\lambda n]} \left( \sigma_{mn}^{(10)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) \right) + \frac{\lambda n}{n - [\lambda n]} \left( \sigma_{mn}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) \right) \\
+ \frac{1}{(m - [\lambda m])(n - [\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn} - u_{jk}),
\end{align*}
\]

where \([\lambda n]\) and \([\lambda m]\) denote the integer part of \(\lambda n\) and \(\lambda m\), respectively.

**Remark 2.4.** In analogy to Lemma 2.3, we have the following identities.

(i) For \(\lambda > 1\),

\[
\sigma_{mn}^{(10)}(u) = \frac{[\lambda m] + 1}{m - [\lambda n]} \left( \sigma_{m,[\lambda n]}^{(10)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) + \sigma_{m,[\lambda n]}^{(11)}(u) \right) - \frac{[\lambda m] + 1}{[\lambda m] - m} \sum_{j=[\lambda m]+1}^{m} (u_{mn} - u_{mn}).
\]

(ii) For \( 0 < \lambda < 1 \),

\[
\sigma_{mn}^{(10)}(u) = \frac{[\lambda m]^2 + 1}{[\lambda m] - m} \left( \sigma_{m,[\lambda n]}^{(10)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) + \sigma_{m,[\lambda n]}^{(11)}(u) \right) + \frac{[\lambda m] + 1}{m - [\lambda n]} \sum_{j=[\lambda m]+1}^{m} (u_{mn} - u_{mn}).
\]

We can show the identities as in the proof of the corresponding lemma for single sequence in [9]. We do not give details.

Moreover, we note that we can similarly represent the difference \(u_{mn} - \sigma_{mn}^{(10)}(u)\) in two different ways as in Remark 2.4.

**Lemma 2.5.**

(i) If \((u_{mn})\) is \((C, 1, 0)\) summable to \(\ell\), and the condition \(m \Delta_n u_{mn} \geq -C\) is satisfied for some \(C > 0\) and large enough \(m, n\), then \((u_{mn})\) is \(P\)-convergent to \(\ell\).

(ii) If \((u_{mn})\) is \((C, 0, 1)\) summable to \(\ell\), and the condition \(n \Delta_n u_{mn} \geq -C\) is satisfied for some \(C > 0\) and large enough \(m, n\), then \((u_{mn})\) is \(P\)-convergent to \(\ell\).

*Proof.* The proof of lemma is done step by step using the identities in Remark 2.4 as in the proof of the one-sidedly Tauberian theorem for single sequence.

**Lemma 2.6.**

(i) If \((u_{mn})\) is \((C, 1, 0)\) summable to \(\ell\), and \((u_{mn})\) is slowly oscillating in sense \((1, 0)\), then \((u_{mn})\) is \(P\)-convergent to \(\ell\).

(ii) If \((u_{mn})\) is \((C, 0, 1)\) summable to \(\ell\), and \((u_{mn})\) is slowly oscillating in sense \((0, 1)\), then \((u_{mn})\) is \(P\)-convergent to \(\ell\).

*Proof.* The proof of lemma is done step by step using the identities in Remark 2.4 as in the proof of the generalized littlewood theorem for single sequence.
3. Some Tauberian Theorems for Regularly Generated Double Sequences

If a double sequence is $P$-convergent to $\ell$, then it is $(C, 1, 1)$ summable to $\ell$ provided that it is bounded [10]. However the converse is not necessarily true. Namely, a double sequence which is bounded and $(C, 1, 1)$ summable may not be $P$-convergent.

We can recover $P$-convergence of a double sequence from its $(C, 1, 1)$ summability under some suitable conditions. Such a condition is called a Tauberian condition and the resulting theorem is called a Tauberian theorem.

Now, let us give some classical type Tauberian theorems, which are called Landau’s theorem and generalized Littlewood theorem for $(C, 1, 1)$ summability method of a double sequence, respectively (see [2]).

**Theorem 3.1.** If $(u_{mn})$ is $(C, 1, 1)$ summable to $\ell$, and

$$
(mm\Delta_{m,n} u_{mn}) \in B^\prime, (m\Delta_m u_{mn}) \in B^\prime, \text{ and } (n\Delta_n u_{mn}) \in B^\prime,
$$

then $(u_{mn})$ is $P$-convergent to $\ell$.

Note that Stadtmüller [11] indicated that the condition $(mm\Delta_{m,n} u_{mn}) \in B^\prime$ in the Theorem 3.1 is superflous.

**Theorem 3.2.** If $(u_{mn})$ is $(C, 1, 1)$ summable to $\ell$, and

$$
(u_{mn}) \in S_{11}, (u_{mn}) \in S_{10}, \text{ and } (u_{mn}) \in S_{01}
$$

then $(u_{mn})$ is $P$-convergent to $\ell$.

Note that Stadtmüller [11] indicated that the condition $(u_{mn}) \in S_{11}$ in the Theorem 3.2 is superflous.

Now, we should mention the main goal of the present paper. Certain conditions on the double sequence $(u_{mn})$ or the sequence $(V^{(11)}_{mn}(\Delta_m u))$ in a class of sequence which is regularly generated sequences are sufficient conditions for $(C, 1, 1)$ summable sequence to be $P$-convergent. Furthermore, we extended some classical type Tauberian theorems for $(C, 1, 1)$ summability method.

**Theorem 3.3.** If $(u_{mn})$ is $(C, 1, 1)$ summable to $\ell$, and

$$
(u_{mn}) \in U_1(N, N, N), (u_{mn}) \in U_2(N), \text{ and } (u_{mn}) \in U_3(N),
$$

then $(u_{mn})$ is $P$-convergent to $\ell$.

**Proof.** Since $(u_{mn}) \in U_2(N)$, then

$$
V^{(10)}_{mn}(\Delta_m u) \in N,
$$

by Lemma 2.2 (i). On the other hand, since $(u_{mn}) \in U_3(N)$, then

$$
V^{(01)}_{mn}(\Delta_n u) \in N,
$$

from Lemma 2.2 (ii). By the hypothesis $(u_{mn}) \in U_1(N, N, N)$, it follows

$$
u_{mn} = \xi_{mn} + \nu_{mn} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\eta_{ij}}{ij} - \eta_{mn},
$$

where $(\xi_{mn}) \in N$, $(\nu_{mn}) \in N$, and $(\eta_{mn}) \in N$. From this, we get

$$
\Delta_{m,n} U_{mn} = \Delta_{m,n} \xi_{mn} + \Delta_{m,n} \nu_{mn} + \frac{\eta_{mn}}{mn} - \Delta_{m,n} \eta_{mn},
$$

and

$$
mm\Delta_{m,n} u_{mn} = mm\Delta_{m,n} \xi_{mn} + mm\Delta_{m,n} \nu_{mn} + \eta_{mn} - mm\Delta_{m,n} \eta_{mn}.
$$
Therefore, taking \((C, 1, 1)\) means of both sides of the last identity, we get
\[
V_{mn}^{(1)}(\Delta_{m,n}u) = V_{mn}^{(1)}(\Delta_{m,n}c) + V_{mn}^{(1)}(\Delta_{m,n}v) + \sigma_{mn}^{(1)}(\eta) - V_{mn}^{(1)}(\Delta_{m,n}\eta).
\] (12)

Applying identities (1), (2), (3) to sequences \((c_{mn}), (v_{mn}),\) and \((\eta_{mn})\), respectively, we obtain \((V_{mn}^{(1)}(\Delta_{m,n}c)) \in \mathcal{N}, \ (V_{mn}^{(1)}(\Delta_{m,n}v)) \in \mathcal{N}, \ (V_{mn}^{(1)}(\Delta_{m,n}\eta)) \in \mathcal{N},\) and \((\sigma_{mn}^{(1)}(\eta)) \in \mathcal{N}.\) Therefore, we have
\[
(V_{mn}^{(1)}(\Delta_{m,n}u)) \in \mathcal{N}.
\] (13)

By the identity (1), the proof is completed. \(\Box\)

**Remark 3.4.** If the double sequence \((u_{mn})\) is in \(\mathcal{B}\), then the condition \((u_{mn}) \in \mathcal{U}_1(\mathcal{N}, \mathcal{N}, \mathcal{N})\) is omitted. Indeed, it follows from the identity
\[
V_{mn}^{(10)}(\Delta_{m,n}u) - \sigma_{mn}^{(01)}(V_{mn}^{(10)}(\Delta_{m,n}u)) = V_{mn}^{(11)}(\Delta_{m,n}u),
\]
and \((u_{mn}) \in \mathcal{B}\) that
\[
(V_{mn}^{(11)}(\Delta_{m,n}u)) \in \mathcal{N} \Rightarrow (\sigma_{mn}^{(01)}(V_{mn}^{(10)}(\Delta_{m,n}u))) \in \mathcal{N}.
\]

Therefore, we obtain \((V_{mn}^{(11)}(\Delta_{m,n}u)) \in \mathcal{N}.\)

**Theorem 3.5.** Let the double sequence \((u_{mn})\) be bounded. If \((u_{mn})\) is \((C, 1, 1)\) summable to \(\ell,\) and
\[
(V_{mn}^{(10)}(\Delta_{m,n}u)) \in \mathcal{U}_2(\mathcal{S}_{10}), (V_{mn}^{(10)}(\Delta_{m,n}u)) \in \mathcal{U}_3(\mathcal{S}_{01}),
\] (14)

then \((u_{mn})\) is \(P\)-convergent to \(\ell.\)

**Proof.** Since \((V_{mn}^{(01)}(\Delta_{m,n}u)) \in \mathcal{U}_2(\mathcal{S}_{10})\) and \((V_{mn}^{(10)}(\Delta_{m,n}u)) \in \mathcal{U}_3(\mathcal{S}_{01}),\) then
\[
(V_{mn}^{(11)}(\Delta_{m,n}u)) \in \mathcal{S}_{10},
\] (15)
\[
(V_{mn}^{(11)}(\Delta_{m,n}u)) \in \mathcal{S}_{01},
\] (16)

by Lemma 2.2.

On the other hand, since \((u_{mn})\) is bounded and \((C, 1, 1)\) summable to \(\ell,\) \((\sigma_{mn}^{(01)}(u))\) is \(P\)-convergent to \(\ell.\)

We know that the \((C, 1, 1),\) \((C, 1, 0)\) and \((C, 0, 1)\) summability methods are regular; so \((\sigma_{mn}^{(11)}(u))\) is \((C, 1, 1)\) summable to \(\ell,\) \((\sigma_{mn}^{(01)}(u))\) is \((C, 1, 1)\) summable to \(\ell\) and \((\sigma_{mn}^{(01)}(u))\) is \((C, 1, 1)\) summable to \(\ell.\) It follows from the identity (1) that \((V_{mn}^{(11)}(\Delta_{m,n}u))\) is \((C, 1, 1)\) summable to 0. If we replace \((u_{mn})\) by \((V_{mn}^{(11)}(\Delta_{m,n}u))\) in Lemma 2.3 (i), we obtain
\[
V_{mn}^{(11)}(\Delta_{m,n}u) - \sigma_{mn}^{(11)}(V_{mn}^{(11)}(\Delta_{m,n}u)) = \frac{([\lambda m] + 1)([\lambda n] + 1)}{([\lambda m] - m)([\lambda n] - n)} \left( \sigma_{mn}^{(11)}(V_{mn}^{(11)}(\Delta_{m,n}u)) - \sigma_{mn}^{(11)}(V_{mn}^{(11)}(\Delta_{m,n}u)) \right)
\]
\[
+ \frac{[\lambda m] + 1}{[\lambda m] - m} \left( \sigma_{mn}^{(11)}(V_{mn}^{(11)}(\Delta_{m,n}u)) - \sigma_{m,n}^{(11)}(V_{mn}^{(11)}(\Delta_{m,n}u)) \right)
\]
\[
+ \frac{[\lambda n] + 1}{[\lambda n] - n} \left( \sigma_{mn}^{(11)}(V_{mn}^{(11)}(\Delta_{m,n}u)) - \sigma_{m,n}^{(11)}(V_{mn}^{(11)}(\Delta_{m,n}u)) \right)
\]
\[
- \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (V_{jk}^{(11)} \Delta_{jk} u) - V_{mn}^{(11)}(\Delta_{m,n}u))
\]
for $\lambda > 1$. From this, we get

$$
|V_{mn}^{(1)}(\Delta_{m,n}u) - \sigma_{mn}^{(1)}(V^{(1)})(\Delta_{m,n}u)| \leq \left| \frac{\lfloor \lambda m \rfloor + 1}{\lfloor \lambda m \rfloor} + \frac{\lfloor \lambda n \rfloor + 1}{\lfloor \lambda n \rfloor} \right| \left( \sigma_{\lfloor \lambda n \rfloor, \lfloor \lambda m \rfloor}^{(1)}(V^{(1)})(\Delta_{m,n}u) - \sigma_{\lfloor \lambda n \rfloor, \lfloor \lambda m \rfloor}^{(1)}(V^{(1)})(\Delta_{m,n}u) \right) + \left| \frac{\lfloor \lambda n \rfloor + 1}{\lfloor \lambda n \rfloor - m} \right| \left( \sigma_{\lfloor \lambda n \rfloor, \lambda n}^{(1)}(V^{(1)})(\Delta_{m,n}u) - \sigma_{\lfloor \lambda n \rfloor, \lambda n}^{(1)}(V^{(1)})(\Delta_{m,n}u) \right)
$$

From the last term on the right-hand side of the inequality (17), we have

$$
\left| \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} (V_{jk}^{(1)})(\Delta_{j,k}u) - V_{ms}^{(1)}(\Delta_{m,n}u) \right| \leq \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \left( \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \Delta_{r} V_{rk}^{(1)}(\Delta_{j,k}u) + \sum_{n+1}^{\lfloor \lambda n \rfloor} \Delta_{s} V_{ms}^{(1)}(\Delta_{m,n}u) \right),
$$

and then

$$
\left| \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} (V_{jk}^{(1)})(\Delta_{j,k}u) - V_{ms}^{(1)}(\Delta_{m,n}u) \right| \leq \max_{m+1 \leq j \leq \lfloor \lambda m \rfloor} \left| \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \Delta_{r} V_{rk}^{(1)}(\Delta_{j,k}u) \right| + \max_{n+1 \leq k \leq \lfloor \lambda n \rfloor} \left| \sum_{n+1}^{\lfloor \lambda n \rfloor} \Delta_{s} V_{ms}^{(1)}(\Delta_{m,n}u) \right|.
$$

Taking $\limsup$ of both sides of the inequality (17) as $m,n \to \infty$, then we have

$$
\limsup_{m,n \to \infty} \left| V_{mn}^{(1)}(\Delta_{m,n}u) - \sigma_{mn}^{(1)}(V^{(1)})(\Delta_{m,n}u) \right| \leq \frac{\lambda^2}{(\lambda - 1)^2} \limsup_{m,n \to \infty} \sigma_{\lfloor \lambda n \rfloor, \lfloor \lambda m \rfloor}^{(1)}(V^{(1)})(\Delta_{m,n}u)
$$

Since the sequence $\sigma_{mn}^{(1)}(V^{(1)})(\Delta_{m,n}u)$ is $P$-convergent, then the terms on the right-hand side of the last inequality vanish. Therefore, taking the limit of both sides as $\lambda \to 1^+$, we obtain...
Lemma 2.6(ii).

oscillating in sense (0, 0).

Similarly, since the sequence \((V_{mn}(\Delta_{mn}u))\) is slowly oscillating in senses (1, 0) and (0, 1), we get

\[
\limsup_{m,n \to \infty} |V_{mn}^{(11)}(\Delta_{mn}u) - \sigma_{mn}^{(11)}(V_{mn}^{(11)}(\Delta_{mn}u))| \leq 0
\]

by (15) and (16). Hence, we obtain

\[
V_{mn}^{(11)}(\Delta_{mn}u) = o(1).
\]

On the other hand, since \((u_{mn})\) is \((C, 1, 1)\) summable to \(\ell\), then \((\sigma_{mn}^{(11)}(u))\) is \((C, 1, 0)\) summable to \(\ell\). Moreover, \((\sigma_{mn}^{(10)}(u))\) is \((C, 0, 1)\) summable to \(\ell\). Therefore, we get \((\sigma_{mn}^{(11)}(V_{mn}^{(11)}(\Delta_{mn}u)))\) is \((C, 1, 0)\) summable to 0 by the identity (2), and \((\sigma_{mn}^{(10)}(V_{mn}^{(10)}(\Delta_{mn}u)))\) is \((C, 0, 1)\) summable to 0 by the identity (3).

\(P\)-convergence of the sequence \((V_{mn}^{(11)}(\Delta_{mn}u))\) implies the slow oscillation in sense (1, 0) of \((V_{mn}^{(11)}(\Delta_{mn}u))\) by Lemma 2.2. Therefore, we obtain

\[
V_{mn}^{(10)}(\Delta_{mn}u) = o(1),
\]

by Lemma 2.6 (i). Similarly, since the sequence \((V_{mn}^{(11)}(\Delta_{mn}u))\) is \(P\)-convergent, then \((V_{mn}^{(10)}(\Delta_{mn}u))\) is slowly oscillating in sense \((0, 1)\) by Lemma 2.2. Hence, we obtain

\[
V_{mn}^{(10)}(\Delta_{mn}u) = o(1)
\]

by Lemma 2.6(ii).

Taking (18), (19), and (20) into consideration completes the proof by identity (12).

**Theorem 3.6.** Let the double sequence \((u_{mn})\) be bounded. If \((u_{mn})\) is \((C, 1, 1)\) summable to \(\ell\), and

\[
(m\Delta_{mn} u_{mn}) \in U_2(B^\gamma), \text{ and } (n\Delta_{mn} u_{mn}) \in U_3(B^\gamma),
\]

\[
(n\Delta_{mn} V_{mn}^{(11)}(\Delta_{mn}u)) \in U_2(B^\gamma), \text{ and } (m\Delta_{mn} V_{mn}^{(10)}(\Delta_{mn}u)) \in U_3(B^\gamma),
\]

then \((u_{mn})\) is \(P\)-convergent to \(\ell\).

**Proof.** Since \((n\Delta_{mn} V_{mn}^{(11)}(\Delta_{mn}u)) \in U_2(B^\gamma)\) and \((m\Delta_{mn} V_{mn}^{(10)}(\Delta_{mn}u)) \in U_3(B^\gamma)\), then

\[
(n\Delta_{mn} V_{mn}^{(11)}(\Delta_{mn}u)) \in B^\gamma,
\]

\[
(m\Delta_{mn} V_{mn}^{(11)}(\Delta_{mn}u)) \in B^\gamma,
\]

by Lemma 2.2.

Since \((u_{mn})\) is bounded and \((C, 1, 1)\) summable to \(\ell\), then it can be satisfied exactly in the same way as in Theorem 3.6 in order to prove the \((C, 1, 1)\) summability of \((V_{mn}^{(11)}(\Delta_{mn}u))\) to 0.
For \( \lambda > 1 \), if we replace \((u_{mn})\) by \((V^{(1)}_{mn}(\Delta_{m,n}u))\) in Lemma 2.3 (i), we have

\[
V^{(1)}_{mn}(\Delta_{m,n}u) - \sigma^{(11)}_{mn}(V^{(1)}(\Delta_{m,n}u)) = \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \left( \frac{\lambda^2}{(\lambda - 1)^2} - \frac{\lambda}{\lambda - 1} \right) \Delta_j V^{(1)}(\Delta_{jk}u) + \sum_{\substack{i \geq m+1 \, k \geq n+1}} \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} V^{(1)}(\Delta_{mk}u) - V^{(1)}_{mn}(\Delta_{m,n}u)).
\]

Taking lim sup of both sides of the previous equation as \( m, n \to \infty \), we get

\[
\limsup_{m,n \to \infty} \left( V^{(1)}_{mn}(\Delta_{m,n}u) - \sigma^{(11)}_{mn}(V^{(1)}(\Delta_{m,n}u)) \right) \leq \frac{\lambda^2}{(\lambda - 1)^2} \limsup_{m,n \to \infty} \left( \sigma^{(11)}_{\lfloor \lambda m \rfloor, \lfloor \lambda n \rfloor}(V^{(1)}(\Delta_{m,n}u)) - \sigma^{(11)}_{\lfloor \lambda m \rfloor, \lambda n}(V^{(1)}(\Delta_{m,n}u)) \right) - \sigma^{(11)}_{\lfloor \lambda m \rfloor, \lambda n}(V^{(1)}(\Delta_{m,n}u)) + \sigma^{(11)}_{m,n}(V^{(1)}(\Delta_{m,n}u)))
\]

\[
+ \frac{\lambda}{\lambda - 1} \limsup_{m,n \to \infty} \left( \sigma^{(11)}_{\lfloor \lambda m \rfloor, \lambda n}(V^{(1)}(\Delta_{m,n}u)) - \sigma^{(11)}_{m,n}(V^{(1)}(\Delta_{m,n}u)) \right)
\]

\[
+ \frac{\lambda}{\lambda - 1} \limsup_{m,n \to \infty} \left( \sigma^{(11)}_{\lfloor \lambda m \rfloor, \lfloor \lambda n \rfloor}(V^{(1)}(\Delta_{m,n}u)) - \sigma^{(11)}_{m,n}(V^{(1)}(\Delta_{m,n}u)) \right)
\]

\[
+ \limsup_{m,n \to \infty} \left( \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \left( \frac{\lambda^2}{(\lambda - 1)^2} - \frac{\lambda}{\lambda - 1} \right) \Delta_j V^{(1)}(\Delta_{jk}u) + \sum_{\substack{i \geq m+1 \, k \geq n+1}} \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} V^{(1)}(\Delta_{mk}u) - V^{(1)}_{mn}(\Delta_{m,n}u)).
\]

From this, we have

\[
\limsup_{m,n \to \infty} (V^{(1)}_{mn}(\Delta_{m,n}u) - \sigma^{(11)}_{mn}(V^{(1)}(\Delta_{m,n}u))) \leq \frac{\lambda^2}{(\lambda - 1)^2} \limsup_{m,n \to \infty} \sigma^{(11)}_{\lfloor \lambda m \rfloor, \lambda n}(V^{(1)}(\Delta_{m,n}u))
\]

\[
- \frac{\lambda}{\lambda - 1} \liminf_{m,n \to \infty} \sigma^{(11)}_{\lfloor \lambda m \rfloor, \lambda n}(V^{(1)}(\Delta_{m,n}u))
\]

\[
- \frac{\lambda^2}{(\lambda - 1)^2} \limsup_{m,n \to \infty} \sigma^{(11)}_{\lfloor \lambda m \rfloor, \lambda n}(V^{(1)}(\Delta_{m,n}u))
\]

\[
+ \limsup_{m,n \to \infty} \left( \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \left( \frac{\lambda^2}{(\lambda - 1)^2} - \frac{\lambda}{\lambda - 1} \right) \Delta_j V^{(1)}(\Delta_{jk}u) + \sum_{\substack{i \geq m+1 \, k \geq n+1}} \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} V^{(1)}(\Delta_{mk}u) - V^{(1)}_{mn}(\Delta_{m,n}u)).
\]

Since the sequence \((\sigma^{(11)}_{mn}(V^{(1)}(\Delta_{m,n}u)))\) is \(P\)-convergent, then the terms on the right-hand side of the last
inequality vanish. Hence, we obtain by the conditions (23) and (24)

$$\limsup_{m,n \to \infty} (V_{mn}^{(1)}(\Delta_{mn}u) - \sigma^{(1)}_{mn}(V^{(1)}(\Delta_{mn}u))) \leq \limsup_{m,n \to \infty} \left( \frac{1}{[\lambda m] - m} \sum_{j=n+1}^{[\lambda n]} \sum_{k=n+1}^{[\lambda n]} \times \left( \sum_{r=m+1}^{j} \frac{C}{r} + \sum_{s=n+1}^{k} \frac{C}{s} \right) \right)$$

$$\leq \limsup_{m,n \to \infty} \left( \frac{C}{[\lambda m] - m} \sum_{j=n+1}^{[\lambda n]} \sum_{k=n+1}^{[\lambda n]} \times \left( \log \left( \frac{j}{m} \right) + \log \left( \frac{k}{n} \right) \right) \right)$$

$$\leq \limsup_{m,n \to \infty} \left( C_1 \log \left( \frac{[\lambda m]}{m} \right) + C_2 \log \left( \frac{[\lambda n]}{n} \right) \right),$$

for some $C_1, C_2 > 0$. Therefore, we get

$$\limsup_{m,n \to \infty} (V_{mn}^{(1)}(\Delta_{mn}u) - \sigma^{(1)}_{mn}(V^{(1)}(\Delta_{mn}u))) \leq C_3 \log \lambda,$$

for some $C_3 > 0$. Taking the limit of both sides as $\lambda \to 1^+$, we have

$$\limsup_{m,n \to \infty} (V_{mn}^{(1)}(\Delta_{mn}u) - \sigma^{(1)}_{mn}(V^{(1)}(\Delta_{mn}u))) \leq 0. \tag{25}$$

For $0 < \lambda < 1$, in a similar way using Lemma 2.3 (ii) we have

$$\liminf_{m,n \to \infty} (V_{mn}^{(1)}(\Delta_{mn}u) - \sigma^{(1)}_{mn}(V^{(1)}(\Delta_{mn}u))) \geq 0. \tag{26}$$

By the inequalities (25) and (26), we obtain

$$V_{mn}^{(1)}(\Delta_{mn}u) = o(1). \tag{27}$$

On the other hand, by hypothesis, since $(m\Delta_{mn}u_{mn}) \in U_2(B^\circ)$ and $(n\Delta_{mn}u_{mn}) \in U_3(B^\circ)$, then

$$(m\Delta_{mn}V_{mn}^{(10)}(\Delta_{mn}u)) \in B^\circ, \tag{28}$$

$$(n\Delta_{mn}V_{mn}^{(01)}(\Delta_{mn}u)) \in B^\circ, \tag{29}$$

by Lemma 2.2.

Since $(u_{mn})$ is $(C,1,1)$ summable to $\ell$, then $(\sigma^{(1)}_{mn}(u))$ is $(C,1,0)$ summable to $\ell$. Moreover, $(\sigma^{(1)}_{mn}(u))$ is $(C,0,1)$ summable to $\ell$. As a result, we get $(\sigma^{(1)}_{mn}(V^{(10)}(\Delta_{mn}u)))$ is $(C,1,0)$ summable to 0 by the identity (3), and $\sigma^{(10)}_{mn}(V^{(10)}(\Delta_{mn}u))$ is $(C,0,1)$ integrable to 0 by the identity (2).

Using the identity (2), we have $m\Delta_{mn}V_{mn}^{(01)}(\Delta_{mn}u) - m\Delta_{mn}\sigma_{mn}^{(01)}(V^{(10)}(\Delta_{mn}u)) = m\Delta_{mn}V_{mn}^{(11)}(\Delta_{mn}u)$. By (27) and Lemma 1.3, it follows that

$$m\Delta_{mn}V_{mn}^{(01)}(\Delta_{mn}u) \geq -C, \tag{30}$$

for some $C > 0$. Moreover, $m\Delta_{mn}V_{mn}^{(01)}(\Delta_{mn}u) \geq -C$, for some $C > 0$. Since the sequence $(\sigma_{mn}^{(01)}(V^{(01)}(\Delta_{mn}u)))$ is $(C,1,0)$ summable to 0, then we get $(\sigma_{mn}^{(01)}(V^{(01)}(\Delta_{mn}u)))$ is $P$-convergent to 0 from Lemma 2.5(i). Therefore, we obtain that $(V_{mn}^{(01)}(\Delta u))$ is $(C,0,1)$ summable to 0. By the condition (3) and Lemma 2.5(ii), we have

$$V_{mn}^{(01)}(\Delta_{mn}u) = o(1). \tag{31}$$
Similarly, from (3), (27), and Lemma 1.3, we obtain
\[ n\Delta_n V_{mn}^{(10)}(\Delta_n u) \geq -C, \]  
for some $C > 0$. Moreover, $n\Delta_n v_{mn}^{(10)}(V_{mn}^{(10)}(\Delta_n u)) \geq -C$, for some $C > 0$. Since the sequence $(\sigma_{mn}^{(10)}(V_{mn}^{(10)}(\Delta_n u)))$ is $(C,1,0)$ summable to 0, then we have $(\sigma_{mn}^{(10)}(V_{mn}^{(10)}(\Delta_n u)))$ is $P$-convergent to 0 by Lemma 2.5 (ii). Hence, we deduce that $(V_{mn}^{(10)}(\Delta_m u))$ is $(C,1,0)$ summable to 0. By the condition (3) and Lemma 2.5 (i), we have
\[ V_{mn}^{(10)}(\Delta_m u) = o(1). \]  
(33)

The proof is completed by using (27), (31), and (33) in the identity (1).

References