Multi-Generalized 2-Normed Space

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Abstract. In this paper, we introduce the concepts of multi-generalized 2-normed space and dual multi-generalized 2-normed space and we then investigate some results related to them. We also prove that, if \((E, \|\cdot\|)\) is a generalized 2-normed space, \(\|\cdot\|_n\) is a sequence of generalized 2-norms on \(E^2 (k \in \mathbb{N})\) such that for each \(x, y \in E, \|x, y\| = \|x\| \) and for each \(k \in \mathbb{N}\) axioms (MG1), (MG2) and (MG4) (DG4) of (dual) multi-generalized 2-normed space are true, then \((E^2, \|\cdot\|_k), k \in \mathbb{N})\) is a (dual) multi-generalized 2-normed space. Finally we deal with an application of a dual multi-generalized 2-normed space defined on a proper commutative \(H^*\)-algebra.

1. Introduction and Preliminaries

The notion of (dual) multi-normed spaces which are somewhat similar to the operator sequence spaces, was initiated by H. G. Dales and M. E. Polyakov in [5]. That provides a suitable supply for the study of multi-normed spaces together with many examples. Some results of (dual) multi-normed spaces are stable under generalized 2-normed spaces [12]. In this paper we use these properties to discover new ones for (dual) multi-generalized 2-normed spaces. In [12], Z. Lewandowska introduced a generalization of Gähler 2-normed space [7, 18], under the name of generalized 2-normed space. After that she published some papers on this issue (e.g. [9–11]). In the following lines, we present some definitions and examples which will be utilized in the sequel.

Definition 1.1. (see [12]) Let \(X\) and \(Y\) be linear spaces over the field \(K\) (\(\mathbb{C}\) or \(\mathbb{R}\)). A function \(\|\cdot, \| : X \times Y \to [0, \infty)\) is called a generalized 2-norm on \(X \times Y\) if it satisfies the following conditions,

(i) \(\|ax, y\| = |a| \|x, y\|\) for all \(a \in K\) and \(x \in X, y \in Y\);
(ii) \(\|x_1 + y_2\| \leq \|x_1, y_1\| + \|x_2, y_2\|\) for all \(x, y, y_1, y_2 \in Y\);
(iii) \(\|x_1 + x_2, y\| \leq \|x_1, y\| + \|x_2, y\|\) for all \(x_1, x_2 \in X, y \in Y\).

The pair \((X, \|\cdot, \|)\) is called a generalized 2-normed space. If \(X = Y\), then the generalized 2-normed space will be denoted by \((X, \|\cdot\|)\).

Example 1.2. (see [11]) Let \(X\) be a real linear space having two seminorms \(\|\cdot\|_1\) and \(\|\cdot\|_2\). Then \((X, \|\cdot\|)\) is a generalized 2-normed space with the generalized 2-norm defined by \(\|x, y\| = \|x\|_1 \|y\|_2\) where \(x, y \in X\).
A sequence \( \{x_n\}_n \) in a generalized 2-normed space \((X, \| \cdot \|_n)\) is said to be a 2-Cauchy sequence if \( \lim_{n \to \infty} \|x_n - x_m, u\| = 0 \) for all \( u \in X \). In addition, \( \{x_n\}_n \) is called 2-convergent if there exists \( x \in X \) such that \( \lim_{n \to \infty} \|x_n - x, u\| = 0 \) for all \( u \in X \). A generalized 2-normed space is called generalized 2-Banach space if every 2-Cauchy sequence is 2-convergent.

Since Lewandowska up to now there are many mathematicians worked on generalized 2-normed spaces and developed it in several directions, see [1, 5, 16, 17] and references cited therein.

The notion of (dual) multi-normed space first was introduced in [5]. This concept has some connections with operator spaces and Banach lattices.

Let \((E, \| \cdot \|)\) be a complex (respectively, real) normed space, and take \( n \in \mathbb{N} \) such that \( \| \cdot \|_n \) is a norm on \( E^k \) for each \( k \in \mathbb{N}_n \), such that \( \|x\|_1 = \|x\| \) for each \( x \in E \) (so that \( \| \cdot \|_1 \) is the initial norm), and such that the following axioms (MN1)-(MN4) are satisfied for each \( k \in \mathbb{N}_n \) with \( k \geq 2 \):

(MN1) for each \( \sigma \in \mathbb{C}^k \) and \( x \in E^k \), \( \|A_\sigma(x)\|_k = \|x\|_k \);

(MN2) for each \( \alpha_1, \ldots, \alpha_k \in \mathbb{C} \) (respectively, each \( \alpha_1, \ldots, \alpha_k \in \mathbb{R} \)) and \( x \in E^k \),

\[
\|M_\alpha(x)\|_k \leq (\max_{\alpha \in \mathbb{C}_1}\|x\|_k);
\]

(MN3) for each \( x_1, \ldots, x_{k-1} \in E \), \( \|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1} \);

(MN4) for each \( x_1, \ldots, x_{k-1} \in E \), \( \|(x_1, \ldots, x_{k-1}, -x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1} \).

In this case, \((E^k, \| \cdot \|_k), k \in \mathbb{N}_n \) is a multi-normed space of level \( n \). A multi-norm on \( (E^k, k \in \mathbb{N}) \) is a sequence

\[
\{\| \cdot \|_k, k \in \mathbb{N}\}
\]

such that \( \{\| \cdot \|_k, k \in \mathbb{N}_n \} \) is a multi-norm of level \( n \) for each \( n \in \mathbb{N} \). In this case, \((E^n, \| \cdot \|_n), n \in \mathbb{N} \) is a multi-normed space. Moreover, if axiom (MN4) replaced by the following axiom, then it is called a dual multi-norm and \((E^n, \| \cdot \|_n), n \in \mathbb{N}\) is called a dual multi-normed space.

(DM4) for each \( x_1, \ldots, x_{k-1} \in E \), \( \|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, 2x_{k-1})\|_{k-1} \).

Example 1.3. (see [5]) Let \((E, \| \cdot \|)\) be a normed space. For each \( k \in \mathbb{N}_n \), put

(i) \( \|(x_1, \ldots, x_k)\|^k_1 = \max_{i \in \mathbb{N}_k} \|x_i\|_k \);

(ii) \( \|(x_1, \ldots, x_k)\|^k_2 = \sum_{i=1}^{k} \|x_i\|_k \),

where \( x_1, \ldots, x_k \) are in \( E \). Then \((E^k, \| \cdot \|_1^k), k \in \mathbb{N} \) is a multi-normed space and \((E^k, \| \cdot \|_2^k), k \in \mathbb{N} \) is a dual multi-normed space.

Suppose that \((E^k, \| \cdot \|_1^k), k \in \mathbb{N} \) is a (dual) multi-normed space. The following property is almost immediate consequence of the axioms.

\[
\max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\| \ (x_1, \ldots, x_k \in E).
\]

It follows from the above assertion that, if \((E, \| \cdot \|_1)\) is a Banach space, then \((E^k, \| \cdot \|_1^k)\) is a Banach space for each \( k = 2,3, \ldots \), in this case, \((E^k, \| \cdot \|_1^k), k \in \mathbb{N} \) is called a (dual) multi-Banach space.
By now, many authors have already contributed to the theoretical development of the theory of multi-
normed spaces (e.g. see [6, 13–15]). In the present work we demonstrate the concept of (dual) multi-
normed space in the framework of generalized 2-normed spaces. We also provide many examples together with
an application of a dual multi-generalized 2-normed space defined on a proper commutative
space. Throughout this paper, we mean by (dual) multi-normed
By now, many authors have already contributed to the theoretical development of the theory of multi-
normed spaces (e.g. see [2, 4, 8, 19]). We will describe
and for each
space, 
\[ \| x \|_k = \| x \| \]
for each \( x \) \( \in \mathbb{E} \), \( y \) \( \in \mathbb{E} \), \( \| x \|_k = \| x \| \)
and for each \( k \in \mathbb{N} \) axioms (MG1), (MG2) and (MG4) (DG4) of (dual) multi-generalized 2-normed space
are true, then \( \{ \mathbb{E}, \| \cdot \|_k \}, k \in \mathbb{N} \) is a (dual) multi-generalized 2-normed space. In section 4, we give an
application of a dual multi-generalized 2-normed space. Throughout this paper, we mean by \( T \) and by \( S \) the
unit ball and the closed unit ball of \( \mathbb{C} \) respectively, more precisely \( T = \{ \alpha \in \mathbb{C}, |\alpha| = 1 \} \) and \( S = \{ \alpha \in \mathbb{C}, |\alpha| \leq 1 \} \).

2. (Dual) Multi-Generalized 2-Normed Space

In this section we introduce a (dual) multi-generalized 2-normed space and investigate some properties
of it. For this, we need the following definition.

**Definition 2.1.** Let \( \{ \mathbb{E}, \| \cdot \|_k \} \) be a generalized 2-normed space (over the field \( \mathbb{K} \)). A special generalized 2-norm on
\( \{ \mathbb{E}, k \in \mathbb{N} \} \) is a sequence \( \{ \| \cdot \|_k \}_{k \in \mathbb{N}} \) such that for each \( k \in \mathbb{N} \), \( \| \cdot \|_k \) is a generalized 2-norm on \( \mathbb{E} \), \( \| x, y \|_1 = \| x \| \)
for each \( x, y \in \mathbb{E} \) and the following axioms (MG1)-(MG3) are satisfied for each \( k \in \mathbb{N} \) with \( k \geq 2 \):

(\text{MG1}) for each \( \sigma \in \mathbb{C}_k \) and \( x, y \in \mathbb{E} \), \( \| A_\sigma(x), A_\sigma(y) \|_k = \| x, y \|_k \);

(\text{MG2}) for each \( \alpha_1, \ldots, \alpha_k \in \mathbb{K} \) and \( x, y \in \mathbb{E} \), \( \| M_\alpha(x), y \|_k = \| x, M_\alpha(y) \|_k \leq (\max_{1 \leq i \leq k} |\alpha_i|) \| x, y \|_k \);

(\text{MG3}) for each \( x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1} \in \mathbb{E} \),
\[ \| (x_1, \ldots, x_{k-1}, 0), (y_1, \ldots, y_{k-1}, 0) \|_k = \| (x_1, \ldots, x_{k-1}), (y_1, \ldots, y_{k-1}) \|_{k-1}. \]

Now consider two following more axioms.

(\text{MG4}) for each \( x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1} \in \mathbb{E} \),
\[ \| (x_1, \ldots, x_{k-1}, x_{k-1}), (y_1, \ldots, y_{k-1}, y_{k-1}) \|_k = \| (x_1, \ldots, x_{k-1}), (y_1, \ldots, y_{k-1}) \|_{k-1}. \]

(\text{DG4}) for each \( x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1} \in \mathbb{E} \),
\[ \| (x_1, \ldots, x_{k-1}, x_{k-1}), (y_1, \ldots, y_{k-1}, y_{k-1}) \|_k = \| (x_1, \ldots, y_{2k-1}), (y_1, \ldots, y_{2k-1}) \|_{k-1}. \]

A special generalized 2-norm is said to be a (dual) multi-generalized 2-norm if it is equipped with the axiom (MG4)
(\text{DG4}). In this case, \( \{ \mathbb{E}, \| \cdot \|_k \}, k \in \mathbb{N} \) is called a (dual) multi-generalized 2-normed space.

We give the definition in the case where the index set is \( \mathbb{N} \). If the index set is \( \mathbb{N}_k \) \( (k \in \mathbb{N}) \), then special, multi-and dual multi-generalized 2-normed spaces are of level \( k \).

**Remark 2.2.** It is readily verified from the axioms (MG2) and (MG3), that
\[ \| (x_1, \ldots, x_k, 0), (y_1, \ldots, y_k, y_{k+1}) \|_{k+1} = \| (x_1, \ldots, x_k), (y_1, \ldots, y_k) \|_k, \]
where \( x_1, \ldots, x_k, y_1, \ldots, y_{k+1} \in \mathbb{E} \). Indeed, we have
\[ \| (x_1, \ldots, x_k, 0), (y_1, \ldots, y_k, y_{k+1}) \|_{k+1} = \| M_{(1, \ldots, 1, 0)}(x_1, \ldots, x_k, 0), (y_1, \ldots, y_k, y_{k+1}) \|_{k+1} \]
\[ = \| (x_1, \ldots, x_k, 0), M_{(1, \ldots, 1, 0)}(y_1, \ldots, y_k, y_{k+1}) \|_{k+1} \]
\[ = \| (x_1, \ldots, x_k, 0), (y_1, \ldots, y_k, 0) \|_{k+1} \]
\[ = \| (x_1, \ldots, x_k), (y_1, \ldots, y_k) \|_k. \]
Example 2.3. Let \((E, ||, ||)\) be a non-zero generalized 2-normed space. For each \(k \in \mathbb{N}\), set

\[
(i) \quad ||(x_1, \ldots, x_k), (y_1, \ldots, y_k)||^k = \max\{||x_1, y_1||, \ldots, ||x_k, y_k||\},
\]

\[
(ii) \quad ||(x_1, \ldots, x_k), (y_1, \ldots, y_k)||^k = \frac{1}{k} \sum_{i=1}^{k} ||x_i, y_i||,
\]

where \((x_1, \ldots, x_k), (y_1, \ldots, y_k) \in E^k\). Then, \((\mathbb{E}^k, ||, ||)^k, k \in \mathbb{N}\) is a multi-generalized 2-normed space and \((\mathbb{E}^k, ||, ||, ||^k)\), \(k \in \mathbb{N}\) is a dual multi-generalized 2-normed space.

Example 2.4. Let \((E, ||, ||)\) be an \(H^\star\)-algebra (for the definition see section 4). Define a generalized 2-norm on \(E^k\) \((k \in \mathbb{N})\) by setting \(||(x_1, \ldots, x_k), (y_1, \ldots, y_k)||_k = \sum_{i=1}^{k} ||x_i, y_i||\) then \((\mathbb{E}^k, ||, ||, ||_k)\), \(k \in \mathbb{N}\) is a dual multi-generalized 2-normed space.

Example 2.5. (see [5]) Let \((\mathbb{E}^k, ||, ||, ||^k), k \in \mathbb{N}\), be a family of (dual) multi-generalized 2-normed spaces. For each \(k \in \mathbb{N}\) and \(x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{E}\), define

\[
||(x_1, \ldots, x_k), (y_1, \ldots, y_k)||_k = \sup_a \|(x_1, \ldots, x_k), (y_1, \ldots, y_k)||_k^a.
\]

Then \((\mathbb{E}^k, ||, ||, ||_k)\), \(k \in \mathbb{N}\) is a (dual) multi-generalized 2-normed space, too.

Inspired by the examples of [5] we give some examples show that axioms (MG1)-(MG4) ( - (DG4)) are independent of each other.

Example 2.6. Let \((E, ||, ||)\) be a non-zero generalized 2-normed space. Set \(||x, y||_1 = ||x, y||, (x, y) \in \mathbb{E}\).

\[
(I) \quad \text{For each } k \in \mathbb{N} - \{1\}, \text{ set } ||(x_1, \ldots, x_k), (y_1, \ldots, y_k)||_k = \max\{||x_1, y_1||, ||x_2, y_2||, \ldots, ||x_k, y_k||\}, \text{ where } (x_1, \ldots, x_k), (y_1, \ldots, y_k) \in \mathbb{E}^k.
\]

\[
\text{Then it is immediately checked that } ||, ||_k \text{ is a generalized 2-norm on } \mathbb{E}^k \text{ and that } ||, ||_k \in \mathbb{N}
\]

\[
\text{satisfies (MG2), (MG3) and (MG4). However, take } x, y \in \mathbb{E} \text{ with } ||x, y|| = 1. \text{ Then } ||(2x, 3x), (2y, 4y)||_2 = 6, \text{ but}
\]

\[
||\mathbb{E}^6, ||, ||, ||_2 \text{ does not satisfy axiom (MG1).}
\]

\[
(II) \quad \text{Set } ||(x_1, x_2), (y_1, y_2)||_2 = \max\{||x_1, y_1||, ||x_2, y_2||\}, \text{ where } (x_1, x_2), (y_1, y_2) \in \mathbb{E}^2.
\]

\[
\text{Then it is immediately checked that } ||, ||_2 \text{ is a generalized 2-norm on } \mathbb{E}^2 \text{ and that } ||, ||_2 \text{ satisfies (MG2), (MG3) and (DG4). However, we claim that}
\]

\[
||, ||_2 \text{ does not satisfy axiom (MG1). For this, similar previous part take } x, y \in \mathbb{E} \text{ with } ||x, y|| = 1. \text{ Then}
\]

\[
||\mathbb{E}^3, 2x, 3x), (2y, 4y)||_2 = 12, \text{ but } ||\mathbb{E}^6, 2x, 3x), (2y, 4y)||_2 = 24, \text{ and so } ||, ||_2 \text{ does not satisfy axiom (MG1).}
\]

Example 2.7. (III) Let \(E = \mathbb{R}\) and \(k \in \mathbb{N}\). Define \(||(x_1, \ldots, x_k), (y_1, \ldots, y_k)||_k = \max\{||x_i - x_j||, ||y_i - y_j||, \ i, j \in \mathbb{N}_k \cup \{0\}, x_0, y_0 = 0\}\), where \(x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{E}\). We observe that \(||\mathbb{E}^k, ||, ||, ||_k\), \(k \in \mathbb{N}\) is a sequence of generalized 2-normed spaces, and (MG1), (MG3) and (MG4) are true. However we claim that (MG2) does not hold, because obviously \(||M_\alpha(x), y||_k \neq ||x, M_\alpha(y)||_k\), \((x, y) \in \mathbb{E}^2, \alpha \in \mathbb{R}^2\) and moreover, \(4 = ||(1, -1), (-1, 1)||_2 \neq ||(1, 1), (-1, 1)||_2 = 1\) giving the claim.

(IV) Let \((E, ||, ||)\) be a non-zero complex generalized 2-normed space. For each \(k \in \mathbb{N}\), \(x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{E}\), define \(||(x_1, \ldots, x_k), (y_1, \ldots, y_k)||_k = \max\{||q_\epsilon x, \epsilon y||, \ i, j \in \mathbb{N}_k, \eta, \epsilon_j \in T\}\). Clearly, \(||, ||, ||_k\) is a generalized 2-norm on \(E^k\) and axioms (MG1), (MG3) and (MG4) hold. Also \(||M_\alpha(x), y||_k\), \(|x, M_\alpha(y)||_k \leq (\max_{\alpha \in \mathbb{R}^k})||x, y||_k\) for each \(\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k\) and \(x, y \in \mathbb{E}^2\), but evidently \(||M_\alpha(x), y||_k \neq ||x, M_\alpha(y)||_k\).

(V) Let \((E, ||, ||)\) be a non-zero generalized 2-normed space. For each \(k \in \mathbb{N}\), \(x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{E}\), define

\[
||(x_1, \ldots, x_k), (y_1, \ldots, y_k)||_k = \sup\{\max_\eta \{\sum_{j=1}^{k} \eta_j x_j, y_j\}, \eta_1, \ldots, \eta_k \in S\}.
\]

Clearly, \(||, ||, ||_k\) is a generalized 2-norm on \(E^k\) and (MG1), (MG3) and (DG4) hold. For (DG4), we have
\[(x_1, \ldots, x_k, y_1, \ldots, y_k) \|_{E} \]

Further for nonzero \(\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k\),

\[
\frac{1}{\max_{i \in \{1,\ldots,K\}} |\alpha_i|} \|\|M_{\alpha}(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_{E}\|
\]

where \(\eta_j = \frac{1}{\max_{i \in \{1,\ldots,K\}} |\alpha_i|} \eta_i \alpha_j\) (j \(\in\) \(\mathbb{N}_k\)). This equality gives us the second part of (MG2). Similarly one can quickly checked that

\[
\|\|M_{\alpha}(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_{E}\| \leq \max_{1 \leq i \leq k} \|\|\alpha_i\|\|(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_{E}\|,
\]

but trivially \(\|\|M_{\alpha}(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_{E}\| \neq \|\|(x_1, \ldots, x_k)\|_{E}\|\) and so (MG2) does not hold in general.

(VII) Suppose that \(E = \mathbb{C}\) and \(\|z_1, z_2\| = 2|z_1z_2|\) (\((z_1, z_2) \in E\)). Then \((E, \|\|, \|\|)\) is a generalized 2-normed space.
Assume that \(\|(z_1, z_2), (w_1, w_2)\| = |z_1w_1 + z_2w_2|\), where \((z_1, z_2), (w_1, w_2) \in E^2\). It is a generalized 2-norm on \(E^2\) such that satisfies in the axioms (MG1), (MG3), (DG4) and for each \((x_1, x_2) \in \mathbb{C}^2\), \(\|M_{\alpha}(z_1, z_2), (w_1, w_2)\| = \|\|z_1, z_2\|_{E}\|\)\). However the second part of axiom (MG2) does not hold. For instance, we have \(4 = \|\|(1, 0), (1, -i)\|\|_2 \not\leq \|\|(1, 1), (0, 0)\|\|_2 = 2 \sqrt{2}\).

Example 2.8. (VII) Let \(E = \mathbb{C}\), \(|x, y| = |xy|\) and \(\|\|(x_1, x_2), (y_1, y_2)\| = \frac{1}{2}|x_1 y_1 + x_2 y_2|\), where \(x, y, x_1, x_2, y_1, y_2 \in E\).
It is not hard to see that \((E, \|\|, \|\|)\) and \((E^2, \|\|, \|\|)\) are generalized 2-normed spaces and (MG1), (MG2), (MG4) are true but (MG3) is not.

(VIII) Suppose that \(E = \mathbb{R}^2\) and \(\|\|(x_1, y_1), (x_2, y_2)\| = |x_1 y_2 - y_1 x_2|\), then \((E, \|\|, \|\|)\) is a generalized 2-normed space (see [18]). Define

\[
\|\|(x_1, y_1), (z_1, z_2), (x_2, y_2), (z_2, w_2)\| = 2 \max\{\|\|(x_1, y_1), (z_1, w_1)\|, \|\|(x_2, y_2), (z_2, w_2)\|\}.\]

We observe that \((E^2, \|\|, \|\|)\) is a generalized 2-normed space and axioms (MG1), (MG2) are true. The calculation \(\|\|(x_1, y_1), (x_1, y_1), (z_1, w_1), (z_1, w_1)\| = 2\|\|(x_1, y_1), (z_1, w_1)\| = \|\|(x_1, y_1), (z_1, w_1)\|\) shows that (DG4) is also valid.
On the other hand (MG3) does not hold, since \(\{||(1, 1), (0, 0), ((-1, 1), (0, 0))\| = 4\) but \(\|\|(1, 1), (1, -i)\|\|_2 = 2 \sqrt{2}\).

Example 2.9. (IX) Let \(E = \mathbb{C}\), \(|x, y| = |xy|\) and \(\|\|(x_1, x_2), (y_1, y_2)\| = |x_1 y_1| + |x_2 y_2|\), where \(x, y, x_1, x_2, y_1, y_2 \in E\).
It is immediately verified that \((E, \|\|, \|\|)\) and \((E^2, \|\|, \|\|)\) are generalized 2-normed spaces and (MG1), (MG2), (MG3) are true but (MG4) is not.

(X) Let \(E = \mathbb{R}\). For \(k \in \mathbb{N}\), and \(x_1, \ldots, x_k, y_1, \ldots, y_k \in E\), define \(\|\|(x_1, \ldots, x_k), (y_1, \ldots, y_k)\|_{E} = \sum_{i=1}^{k} |x_i y_i|^2\). Then \(\|\|, \|\|, k \in \mathbb{N}\) is a special generalized 2-norm on \([E^k, k \in \mathbb{N}]\), but both of axioms (MG4) and (DG4) are not true.

The four presented examples in the above are just in level 2. In the following lemma we assume \((E, \|\|, \|\|)\) is a generalized 2-normed space and \((E^k, \|\|, \|\|, k \in \mathbb{N})\) is a special generalized 2-normed space with \(\|x, y\|_1 = |x, y|\) for all \(x, y \in E\). The proof is trivial and so is omitted (see [5, pp. 44-47]).
Lemma 2.10. Let \( j, k \in \mathbb{N}, x_1, \ldots, x_{j+k}, y_1, \ldots, y_{j+k} \in E \) and \( \eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_k \in T \). Then

(i) \( \| (\eta_1 x_1, \ldots, \eta_k x_k, (\xi_1 y_1, \ldots, \xi_k y_k)) \|_k = \| (x_1, \ldots, x_k, (y_1, \ldots, y_k)) \|_k \).

(ii) \( \| (x_1, \ldots, x_k, (y_1, \ldots, y_k)) \|_k \leq \| (x_1, \ldots, x_k, x_{k+1}) \|_1 \), \( (y_1, \ldots, y_k, y_{k+1}) \|_1 \).

(iii) \( \| (x_1, \ldots, x_k, x_{k+j}, y_1, \ldots, y_j, \ldots, y_{j+k}) \|_{j+k} \leq \| (x_1, \ldots, x_k, y_1, \ldots, y_j) \|_k + \| (x_{k+j}, \ldots, x_k, y_j, \ldots, y_{j+k}) \|_k \).

(iv) \( \max_{i \in \mathbb{N}_k} \| x_i, y_i \| \leq \| (x_1, \ldots, x_k, (y_1, \ldots, y_k)) \|_k \leq \sum_{i=1}^k \| x_i, y_i \| \leq k \max_{i \in \mathbb{N}_k} \| x_i, y_i \| \).

The last part of the above lemma guides us to the following result.

**Corollary 2.11.** Suppose that \( \| \cdot \|_{k} \) is a family of (dual) multi-generalized 2-norms on \( E^k, k \in \mathbb{N} \), and \( (E, \| \cdot \|_1) \) is a generalized 2-Banach space. Then for each \( k \in \mathbb{N}, (E^k, \| \cdot \|_k) \) is a generalized 2-Banach space, too.

In this case, \((E^k, \| \cdot \|_k), k \in \mathbb{N}\) is called a (dual) multi-generalized 2-Banach space.

Lemma 2.12. Let \((E^k, \| \cdot \|_k) , k \in \mathbb{N}\) be a multi-generalized 2-normed space and \( x_1, \ldots, x_{k-2}, x', x'', y_1, \ldots, y_{k-2}, y', y'' \in E \). Then

\[
\| (X, x', x''), (Y, y', y'') \|_k \leq \| (X, x', x'') \|_k + \| (Y, y', y'') \|_k,
\]

where \( X = x_1, \ldots, x_{k-2}, Y = y_1, \ldots, y_{k-2} \).

**Proof.** Applying Lemma 2.10 and axiom (MG1), we deduce that

\[
\| (X, x', x''), (Y, y', y'') \|_k \leq \| (X, x', x'') \|_k + \| (Y, y', y'') \|_k \\
\leq \| (X, x', x''), (Y, y', y'') \|_k + \| x', x'' \|_k + \| y', y'' \|_k,
\]

Therefore we get the desired result. 

The following lemma is a version of [5, Lemma 2.16] in the framework of multi-generalized 2-normed spaces.

Lemma 2.13. Let \((E^k, \| \cdot \|_k) , k \in \mathbb{N}\) be a multi-generalized 2-normed space, \( x = (x_1, \ldots, x_k) \) and \( y = (y_1, \ldots, y_k) \) be in \( E^k, x_{k+1}, x_{k+2}, y_{k+1}, y_{k+2} \) be in \( E \) and \( a, b, p, q \in [0, 1] \) with \( a + b = 1, p + q = 1 \). Then

\[
\| (x, ax_{k+1} + bx_{k+2}, ax_{k+1} + bx_{k+2}, (y, p y_{k+1} + q y_{k+2}, p y_{k+1} + q y_{k+2}) \|_k \\
\leq \| X, (y, y_{k+1}, y_{k+1}) \|_k + \| X, (y, y_{k+2}, y_{k+2}) \|_k,
\]

where \( X = (x_1, \ldots, x_{k+2}) \).

**Proof.** We have \((x, ax_{k+1} + bx_{k+2}, ax_{k+1} + bx_{k+2}) = a^2(x, x_{k+1}, x_{k+1}) + ab(x, x_{k+1}, x_{k+2}) + ab(x, x_{k+2}, x_{k+1}) + b^2(x, x_{k+2}, x_{k+2})\). Similar relation holds when \(x, x_{k+1}, x_{k+2}, a, b \) substitute with \( y, y_{k+1}, y_{k+2}, p, q \), respectively. Applying Lemmata 2.10 and 2.12 and also axiom (MG1), it follows that

\[
\| (x, ax_{k+1} + bx_{k+2}, ax_{k+1} + bx_{k+2}, (y, p y_{k+1} + q y_{k+2}, p y_{k+1} + q y_{k+2}) \|_k \\
\leq \| (x, x_{k+1}, x_{k+2}) \|_k + \| (y, p y_{k+1} + q y_{k+2}, p y_{k+1} + q y_{k+2}) \|_k \\
= \| X, p y_{k+1} + q y_{k+2}, p y_{k+1} + q y_{k+2} \|_k \\
\leq \| X, (y, y_{k+1}, y_{k+1}) \|_k + 2pq \| X, (y, y_{k+1}, y_{k+2}) \|_k \\
\leq \| X, (y, y_{k+2}, y_{k+2}) \|_k + 2pq \| X, (y, y_{k+2}, y_{k+2}) \|_k \\
= \| X, (y, y_{k+1}, y_{k+1}) \|_k + 2pq \| X, (y, y_{k+2}, y_{k+2}) \|_k \\
\leq \| X, (y, y_{k+1}, y_{k+1}) \|_k + 2pq \| X, (y, y_{k+2}, y_{k+2}) \|_k \\
= \| X, (y, y_{k+1}, y_{k+1}) \|_k + 2pq \| X, (y, y_{k+2}, y_{k+2}) \|_k \\
= \| X, (y, y_{k+1}, y_{k+1}) \|_k + 2pq \| X, (y, y_{k+2}, y_{k+2}) \|_k.
\]

Note that the second inequality in the above relation holds by Lemma 2.12. So the proof is complete. 

By slightly modification in the proof of [5, Lemmata 2.19, 2.22], and using Lemma 2.10, one gets the following proposition.

**Proposition 2.14.** Let \((E^k, \|\cdot\|, \|\cdot\|_k)\), \(k \in \mathbb{N}\) be a dual multi-generalized 2-normed space and \(k'\) and \(n\) be arbitrary fixed elements in \(\mathbb{N}\). Then for each \(x_1, \ldots, x_{k+n}, y_1, \ldots, y_{k+1} \in E\), we have

(i) \(\|x_1, \ldots, x_k, x_{k+1} + x_{k+2} + \ldots + x_{k+n}, (y_1, \ldots, y_k, y_{k+1}, \ldots, y_{k+1})\|_k\)

(ii) \(\|x_1, \ldots, x_{k-2}, x_{k-1} + x_k, (y_1, \ldots, y_{k-2}, y_{k-1} + y_k)\|_k\)

(iii) \(\|x_1, \ldots, x_k, x_{k+1}, (y_1, \ldots, y_{k-1} + y_k)\|_k\)

(iv) \(\|x_1, \ldots, x_k, (y_1, \ldots, y_k)\|_k = \sum_{i=1}^k |\alpha_i||x, y\|_k\)

where \(\alpha_i, \beta_i \geq 0\) and \(\alpha_i + \beta_i = 1\) for each \(i \in \mathbb{N}_{k-2}\).

**Theorem 3.1.** Let \((E, \|\cdot\|, \|\cdot\|_k)\) be a generalized 2-normed space. Let \(\{\|\cdot\|, \|\cdot\|_k\}_{k \in \mathbb{N}}\) be a sequence such that \(\|\cdot\|, \|\cdot\|_k\) is a generalized 2-norm on \(E^k\) for each \(k \in \mathbb{N}\) and \(\|x, y\|_k = \|x, y\|\) for all \(x, y \in E\). Also axioms (MG1), (MG2) and (MG4) are satisfied for each \(k \in \mathbb{N}\). Then \(\{\|\cdot\|, \|\cdot\|_k\}_{k \in \mathbb{N}}\) is a multi-generalized 2-norm on \((E^k, \|\cdot\|_k)\).

**Proof.** By Definition 2.1, it is enough to show that axiom (MG3) holds. For, let \(k \in \mathbb{N}\), \(x = (x_1, \ldots, x_k)\) and \(y = (y_1, \ldots, y_k)\) be in \(E^k\) such that \(\|x, y\|_k = 1\). Set \(\alpha = \|(x_1, \ldots, x_k, 0), (y_1, \ldots, y_k, 0)\|_{k+1}\), so that \(\alpha \leq 1\). Indeed, by axioms (MG2) and (MG4), we have

\[
\begin{align*}
\alpha &= \|M_{1,1,1,0}(x_1, \ldots, x_k, y_1, \ldots, y_k, y_1)\|_{k+1} \\
&\leq \|(x_1, \ldots, x_k, y_1), (y_1, \ldots, y_k, y_1)\|_{k+1} \\
&= \|(x_1, \ldots, x_k, y_1), (y_1, \ldots, y_k)\|_k \\
&= 1.
\end{align*}
\]

Let \(n\) be any arbitrary fixed element in \(\mathbb{N}\), take \(x^{[n+2]}, y^{[n+2]} \in E^{(n+2)k}\), by (MG1), (MG4), \(\|x^{[n+2]}, y^{[n+2]}\|_{(n+2)k} = \|x, y\|_k = 1\) (1). For \(1 \leq i \leq n + 2\), let \(B_i\) be the subset \(\{i-1)k+1, \ldots, ik\} \subseteq \mathbb{N}_{(n+2)k}\), and let \(Q_{B_i}\) be a projection onto the complement of \(B_i\). We thus find that \(\|Q_{B_i}(x^{[n+2]}), Q_{B_i}(y^{[n+2]})\|_{(n+2)k} = \|Q_{B_i \cup i}(x^{[n+2]}), Q_{B_i \cup i}(y^{[n+2]})\|_{i+2k}\) (2), by (MG2). Applying again axioms (MG1) and (MG4) we deduce that (2) is equal to \(\alpha\). Further,

\[
\sum_{i=1}^{n+2} Q_{B_i}(x^{[n+2]}) = (n + 1)x^{[n+2]} \quad \text{and} \quad \sum_{i=1}^{n+2} Q_{B_i}(y^{[n+2]}) = (n + 1)y^{[n+2]}\]

and it follows from (1) that

\[
\begin{align*}
(n + 1)^2 &= \|Q_{B_i}(x^{[n+2]}), Q_{B_i}(y^{[n+2]})\|_{(n+2)k} \\
&= \|Q_{B_i}(x^{[n+2]}), (n + 1)y^{[n+2]}\|_{(n+2)k} \\
&\leq \sum_{i=1}^{n+2} \|Q_{B_i}(x^{[n+2]}), Q_{B_i}(y^{[n+2]})\|_{(n+2)k} \\
&= (n + 2)^2 \alpha.
\end{align*}
\]
Therefore $\alpha \geq \frac{(n-1)^2}{2^n \beta}$. Letting $n$ tends to infinity, we obtain that $\alpha = 1$ and our goal is achieved. \(\square\)

**Theorem 3.2.** Let $(E, \|\cdot\|)$ be a generalized 2-normed space, $\{\|\cdot\|_k\}_{k \in \mathbb{N}}$ be a sequence such that $\|\cdot\|_k$ be a generalized 2-norm on $E_k$ for each $k \in \mathbb{N}$ and $\|x, y\| = \|x, y\|_k$ for each $x, y \in E$. Also (MG1), (MG2) and (DG4) are satisfied for each $k \in \mathbb{N}$. Then $\{\|\cdot\|_k\}_{k \in \mathbb{N}}$ is a dual multi-generalized 2-norm on $\{E_k, k \in \mathbb{N}\}$.

**Proof.** Let $k \in \mathbb{N}$, and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_k)$ be in $E_k$. For convenience, by $\beta$ we denote the real number $\|\{(x_1, \ldots, x_k), (y_1, \ldots, y_k)\}\|$ and by $\alpha$ the real number $\|\{(x_1, \ldots, x_k), (y_1, \ldots, y_k)\}\|_{k+1}$. If $\beta = 0$, then

$$0 \leq \alpha = \|\{(x_1, \ldots, x_k), (y_1, \ldots, y_k)\}\|_{k+1} = \|\{(x_1, \ldots, x_k), (y_1, \ldots, y_k)\}\|_{k+1} \leq 2\|\{(x_1, \ldots, x_k), (y_1, \ldots, y_k)\}\|_k \leq 2\|\{(x_1, \ldots, x_k), (y_1, \ldots, y_k)\}\|_k \leq 2\beta = 0.$$ 

It forces that $\alpha = 0$ too. Now assume that $\beta$ is nonzero and $n$ is an arbitrary fixed element of $\mathbb{N}$, then $x^{(2)}$, $y^{(2)}$ are in $E^{(2n)}$ and so by axioms (MG1) and (DG4), $\|x^{(2)}(1), y^{(2)}(1)\|_{2^n} = 2^n \beta$ (3). For $i = 1, \ldots, 2^n$, let $B_i$ be the subset $(i-1)k+1, \ldots, ik$ of $\mathbb{N}_{2^n}$, and let $Q_{B_i}$ be a projection onto the complement of $B_i$. From (MG2), if yields that $\|Q_{B_i}(x^{(2)}), Q_{B_i}(y^{(2)}(1))\|_{2^n} = \|Q_{B_i \cup B_j}(x^{(2)}), Q_{B_i \cup B_j}(y^{(2)}(1))\|_{2^n}$ (4).

Using (MG1), (MG2) and (DG4) we deduce that the equality (4) is less than or equal to $2^n \alpha$. Further,

$$\sum_{i=1}^{2^n} Q_{B_i}(x^{(2)}) = (2^n - 1)x^{(2)} + 1 \quad \text{and} \quad \sum_{j=1}^{2^n} Q_{B_j}(y^{(2)}) = (2^n - 1)y^{(2)} + 1 \quad \text{and it follows from (3) that}$$

$$\sum_{i=1}^{2^n} Q_{B_i}(x^{(2)}) = \frac{(2^n - 1)^2 \|x^{(2)}(1), y^{(2)}(1)\|_{2^n}}{2^n \beta} = \frac{\|x^{(2)}(1), y^{(2)}(1)\|_{2^n}}{2^n \beta} \leq \frac{\sum_{i,j=1}^{2^n} \|Q_{B_i}(x^{(2)}), Q_{B_j}(y^{(2)}(1))\|_{2^n}}{2^n \beta} \leq \frac{(2^n)^2 \|x^{(2)}(1), y^{(2)}(1)\|_{2^n}}{2^n \beta} = \frac{(2^n)^2 \alpha}{2^n \beta}.$$ 

Therefore $\alpha \geq \frac{(2^n-1)^2 \beta}{2^n \beta}$. Since this is true for any $n$, so letting $n \to \infty$, then $\alpha \geq \beta$.

For the reverse direction assume that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then $\|x^{(2)}(1), y^{(2)}(1)\|_{2^n} = 2^n \alpha$. For $i = 1, \ldots, 2^n$, let $C_i = \{(k+1)k, \ldots, (k+1)k\}$ and let $Q_{C_i}$ be a projection onto the complement of $C_i$. Next, put

$$X_1 = (x_1, x_2, \ldots, x_k, 0, \ldots, 0), \quad Y_1 = (y_1, y_2, \ldots, y_k, 0, \ldots, 0).$$

where the number of repetitions of each item $x_i$ and $y_i$ is $2^n - 2$ and also zero has repeated $(2^n - 2) + 2(k + 1)$ times in each of $X_i$ and $Y_i$. 


\[ X_2 = (x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k, 0, \ldots, 0), \]
\[ Y_2 = (y_1, \ldots, y_k, \ldots, y_1, \ldots, y_k, 0, \ldots, 0), \]
where the number of repetitions of each item \( x_i \) and \( y_i \), \( i = 1, \ldots, k \) is \( 2^n - 2 \) and also zero has repeated \( 2k \) times in each of \( X_2 \) and \( Y_2 \).

Finally, set \( y = (1, \ldots, 1, 0, \ldots, 0) \), where 1 has repeated \( (2^n - 2)k \) times and zero has repeated \( 2k \) times. Then
\[
\|Q_{C}(x^{2^j}), Q_{C}(y^{2^j})\|_{2^j(k+1)} = \|Q_{C}\cup_{C}(x^{2^j}), Q_{C}\cup_{C}(y^{2^j})\|_{2^j(k+1)} = \|X_1, Y_1\|_{2^j(k+1)} = \|X_2, Y_2\|_{2^j(k+1)} = \|M_x, x^{2^j}, M_y, y^{2^j}\|_{2^j(k)} \leq 2^n\beta. \text{(by(MG2))}
\]

It is easily verified that \( \sum_{j=1}^{2^n} Q_{C}(x^{2^j}) = (2^n - 1)x^{2^j} \) and \( \sum_{j=1}^{2^n} Q_{C}(y^{2^j}) = (2^n - 1)y^{2^j} \). It follows that
\[
(2^n - 1)^2 = \frac{\|((2^n - 1)x^{2^j}, (2^n - 1)y^{2^j})\|_{2^j(k+1)}}{2^n\alpha} = \frac{\|\sum_{j=1}^{2^n} Q_{C}(x^{2^j}), \sum_{j=1}^{2^n} Q_{C}(y^{2^j})\|_{2^j(k+1)}}{2^n\alpha} \leq \frac{\sum_{i,j=1}^{2^n} \|Q_{C}(x^{2^j}), Q_{C}(y^{2^j})\|_{2^j(k+1)}}{2^n\alpha} \leq \frac{(2^n)^22^n\beta}{2^n\alpha}.
\]

Hence, \( \alpha \leq \frac{(2^n)^22^n\beta}{2^n\alpha} \). Letting \( n \to \infty \), we conclude that \( \alpha \leq \beta \). Therefore \( \alpha = \beta \) and so we get our desired result. \( \Box \)

4. Application

In this section we give an application of multi-generalized 2-normed spaces. For this purpose, it is convenient to make a few observation about H*-algebras (see [2]).

Definition 4.1. An H*-algebra, introduced by W. Ambrose [2] in the associaticve case, is a Banach algebra \( A \), satisfying the following conditions:
(i) \( A \) is itself a Hilbert space under an inner product \( \langle ., . \rangle \);
(ii) For each \( a \) in \( A \) there is an element \( a' \) in \( A \), the so-called adjoint of \( a \), such that we have both \( \langle ab, c \rangle = \langle b, a'c \rangle \) and \( \langle ab, c \rangle = \langle a, cb' \rangle \) for all \( b, c \) in \( A \). Recall that \( A_0 = \{ a \in A, aA = \{0\} \} = \{ a \in A : Aa = \{0\} \} \) is called the annihilator ideal of \( A \). A proper H*-algebra is an H*-algebra with zero annihilator ideal. Ambrose proved that an H*-algebra is proper if and only if every element has a unique adjoint. The trace-class \( \tau(A) \) of \( A \) is defined by the set \( \tau(A) = \{ ab, a, b \in A \} \). The trace functional \( tr \) on \( \tau(A) \) is defined by \( tr(ab) = \langle a, b' \rangle = \langle b, a' \rangle = tr(ba) \) for each \( a, b \in A \), in particular \( tr(aa') = \langle a, a \rangle = \|a\|^2 \), for all \( a \in A \). A nonzero element \( e \) in \( A \) is called a projection, if it is self-adjoint and idempotent. In addition, if \( eA = Ce \), then it is called a minimal projection. For example each simple H*-algebra (an H*-algebra without nontrivial closed two-sided ideals) contains minimal projections. Two idempotents \( e \) and \( e' \) are doubly orthogonal if \( \langle e, e' \rangle = 0 \) and \( e'e = e'e = 0 \). Suppose that \( e \) is a minimal projection in a commutative, proper H*-algebra \( A \), then \( Ae = eA = Ce \). Recall that if \( [e]_{\text{max}} \) is a maximal family of doubly orthogonal minimal projections in a proper H*-algebra \( A \), then \( A \) is the direct sum of the minimal left ideals \( Ae \), or the minimal right ideals \( eA \) [2, Theorem 4.1]. If \( M \) is a subset of an H*-algebra \( A \), then we mean by \( M^\perp \) the orthogonal complement of \( M \). For more details on H*-algebras, see [4, 19] and references cited therein.
Example 4.2. Let $(E, \|\cdot\|)$ be an $H^*$-algebra. We know that $E_k$ $(k \in \mathbb{N})$ is an $H^*$-algebra where the linear operations are considered componentwise and moreover $\langle (x_1, \ldots, x_k), (y_1, \ldots, y_k) \rangle = \sum_{i=1}^{k} \langle x_i, y_i \rangle$. Define a generalized 2-norm on $E_k$ by setting

$$\| (x_1, \ldots, x_k), (y_1, \ldots, y_k) \| = \sum_{i=1}^{k} |\langle x_i, y_i \rangle|.$$

Then $\{ (E_k, \|\cdot\|), k \in \mathbb{N} \}$ is a dual multi-generalized 2-normed space.

Furthermore we can improve the axiom (MG3) as follow:

(MG3) Let $(E, \|\cdot\|)$ be a proper commutative $H^*$-algebra, $\{e_i\}_{i \in N}$ be a maximal family of doubly orthogonal minimal projections in $E$, and $\{ (E_k, \|\cdot\|), k \in \mathbb{N} \}$ be the dual multi-generalized 2-normed space as the above example. For each $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k) \in E_k$, if $x_i y_i = 0$, then

$$\| (x_1, \ldots, x_k), (y_1, \ldots, y_k) \| = \| (x_1, \ldots, x_{k-1}), (y_1, \ldots, y_{k-1}) \|.$$ 

The last equality is true by the definition of $\|\cdot\|$ and the equality $|\langle x_i, y_i \rangle| = \text{tr}(x_i y_i^*) = 0$. Note that if $y_i = \sum_{i} \lambda_i e_i$ ($\lambda_i \in \mathbb{C}$), then $y_i^* = \sum_{i} \lambda_i^* e_i$. By virtue of this fact one can see that $x_i y_i^* = 0$ too.

Definition 4.3. Let $(E, \|\cdot\|)$ be a proper commutative $H^*$-algebra, $\{e_i\}_{i \in N}$ be a maximal family of doubly orthogonal minimal projections in $E$, and $x$ be an arbitrary element of $E$. The least ideal of $E$ containing $x$, is called $x$-ideal of $E$ and it is denoted by $I_x$. Now if $x = \sum_{i} \lambda_i e_i$, for some $\lambda_i \in \mathbb{C}$, then clearly $I_x$ generated by $e_i$’s with nonzero coefficients which appear in the expansion of $x$ in terms of $\{e_i\}_{i \in N}$.

Theorem 4.4. Suppose that $(E, \|\cdot\|)$ is a commutative proper $H^*$-algebra, $\{ (E_k, \|\cdot\|), k \in \mathbb{N} \}$ is the dual multi-generalized 2-normed space as Example 4.2, and $k \in \mathbb{N}$. Let $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k) \in E_k$.

(i) If there is at least $i \in N_k$ in which $x_i y_i \neq 0$ and $I_x$ or $I_y$ is not the whole of $E$, then there exists $k_0 \in N_k$ and a nonzero element $z = (z_1, \ldots, z_k) \in E_k$ with $z_i \neq x_i, y_i$ $(i = 1, \ldots, k_0)$ and $\| (x_1 z_1, \ldots, x_k z_k), (y_1, \ldots, y_k) \|_k = \| (x_1, \ldots, x_k), (y_1, \ldots, y_k) \|_k = 0$ (5).

(ii) If $I_{\sum_{i=1}^{k} x_i}$ or $I_{\sum_{i=1}^{k} y_i}$ are not equal whole of $E$, then we can select equal components for $z$ in the preceding part.

Proof. (i) By (MG1) and (MG3), there exists $k_0 \in N_k$ such that $\| (x_1, \ldots, x_{k_0}), (y_1, \ldots, y_{k_0}) \|_k = \| (x_1, \ldots, x_k), (y_1, \ldots, y_k) \|_k$ and $x_i y_i \neq 0$ $(i = 1, \ldots, k_0)$. Now by assumption $I_{x_i^+} \cup I_{y_i^+} \neq \{0\}$ for some $i = 1, \ldots, k_0$, then it suffices to take $z_i$ any nonzero element of this set, otherwise put $z_i = 0$. Clearly in the first case $|\langle x_i z_i, y_i \rangle| = 0$, since $z_i \neq x_i$, then $x_i z_i \in I_{x_i^+} \cap I_{y_i^+} = \{0\}$ and if $z_i \in I_{x_i^+}$ then $\langle x_i z_i, y_i \rangle = 0$. The last equality holds by virtue of the fact that $I_{x_i^+}$ is a self adjoint ideal and $y_i z_i^* \in I_{x_i^+} \cap I_{y_i^+} = \{0\}$. Take $z = (z_1, \ldots, z_k) \in E_k$, by the above results $z$ is nonzero and also fulfills condition (5). Next we are going to show the $z_i \neq x_i, y_i$ for $i = 1, \ldots, k_0$. This is obvious if $z_i = 0$ (note that $x_i$ and $y_i$ are nonzero for each $i = 1, \ldots, k_0$). The case that $z_i$ is nonzero, first let $z_i \in I_{x_i^+}^\perp$. Then $z_i \neq x_i$ and $x_i y_i = 0$ implies that $y_i$ does not belong to $I_{x_i^+}^\perp$, so $z_i \neq y_i$. A similar argument shows that $z_i \neq x_i, y_i$ if $z_i \in I_{y_i^+}^\perp$.

(ii) It is enough to get $z_i^\perp$ $(i = 1, \ldots, k_0)$ equal to an arbitrary element of $(I_{\sum_{i=1}^{k} x_i})^\perp \cup (I_{\sum_{i=1}^{k} y_i})^\perp$. Evidently $I_{\sum_{i=1}^{k} x_i}$ is the ideal generated by all minimal projections $e_i$’s that appear in the expansion $x_i z_i^\perp$ $(i = 1, \ldots, k)$ with nonzero coefficients. This fact causes that $I_{x_i} \subseteq I_{\sum_{i=1}^{k} x_i}$. Thus the result follows by the preceding part.

References