On the Domain of Riesz Mean in the Space $L_s^*$

Medine Yeşilkayagil$^a$, Feyzi Başar$^b$

$^a$Department of Mathematics, Uşak University, 1 Eylul Campus, 64200 - Uşak, Turkey  
$^b$Kasımpaşa, Alin Sok. Alin Apt. No.76 63492–İstanbul, Turkey

Abstract. Let $0 < s < \infty$. In this study, we introduce the double sequence space $R^s(L_s)$ as the domain of four dimensional Riesz mean $R^s$ in the space $L_s$ of absolutely $s$-summable double sequences. Furthermore, we show that $R^s(L_s)$ is a Banach space and a barrelled space for $1 \leq s < \infty$ and is not a barrelled space for $0 < s < 1$. We determine the $\alpha$- and $\beta(\delta)$-duals of the space $L_s$ for $0 < s \leq 1$ and $\beta(bp)$-dual of the space $R^s(L_s)$ for $1 < s < \infty$, where $\delta \in \{p, bp, r\}$. Finally, we characterize the classes $(L_s : M_u)$, $(L_s : C_\delta)$, $(R^s(L_s) : M_u)$ and $(R^s(L_s) : C_\delta)$ of four dimensional matrices in the cases both $0 < s < 1$ and $1 \leq s < \infty$ together with corollaries some of them give the necessary and sufficient conditions on a four dimensional matrix in order to transform a Riesz double sequence space into another Riesz double sequence space.

1. Introduction

We denote the set of all real or complex valued double sequences by $\Omega$ which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of $\Omega$ is called as a double sequence space. A double sequence $x = (x_{mn})$ of complex numbers is said to be bounded if $\|x\|_{\infty} = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Consider the sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{mn} - l| < \varepsilon$ for all $m, n > n_0$, then we call that the double sequence $x$ is convergent in the Pringsheim’s sense to the limit $l$ and write $p - \lim_{m,n \to \infty} x_{mn} = l$; where $\mathbb{C}$ denotes the complex field. We give the set definitions of the spaces $M_p$, $C_p$ and $L_s$ of bounded, convergent in the Pringsheim’s sense and absolutely $s$-summable double sequences, respectively, as follows:

$$
M_p := \left\{ x = (x_{kl}) \in \Omega : \|x\|_{\infty} = \sup_{k,l \in \mathbb{N}} |x_{kl}| < \infty \right\},
$$

$$
C_p := \left\{ x = (x_{mn}) \in \Omega : \exists l \in \mathbb{C} \text{ such that } p - \lim_{m,n \to \infty} x_{mn} = l \right\},
$$

$$
L_s := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^p < \infty \right\}, \quad (0 < s < \infty).
$$

2010 Mathematics Subject Classification. 46A45; 40C05  

Keywords. Summability theory; double sequences, double series, double sequence spaces, alpha-, beta- and gamma-duals, 4-dimensional matrices and matrix transformations

Received: 12 December 2015; Revised: 24 June 2016; 19 September 2016; Accepted: 19 September 2016  
Communicated by Eberhard Malkowsky  
Email addresses: medine.yesilkayagil@usak.edu.tr (Medine Yeşilkayagil), feyzibasar@gmail.com (Feyzi Başar)
\( \mathcal{M}_r \) is a Banach space with the norm \( \| \cdot \|_\infty \). One can easily see that there are such sequences in the space \( \mathcal{C}_p \) but not in the space \( \mathcal{M}_r \). Indeed, if we define the sequence \( x = (x_{kl}) \) by

\[
x_{kl} := \begin{cases} 
  k & \text{if } k \in \mathbb{N}, \ l = 0, \\
  l & \text{if } l \in \mathbb{N}, \ k = 0, \\
  0 & \text{if } k, l \in \mathbb{N} \setminus \{0\}
\end{cases}
\]

for all \( k, l \in \mathbb{N} \), then it is trivial that \( x \in \mathcal{C}_p \setminus \mathcal{M}_r \), since \( p - \lim_{k,l \to \infty} x_{kl} = 0 \) but \( \|x\|_\infty = \infty \). So, we can consider the space \( \mathcal{C}_p \) of the double series which are both convergent in the Pringsheim’s sense and bounded, i.e., \( \mathcal{C}_p = \mathcal{C}_p \cap \mathcal{M}_d \). A sequence in the space \( \mathcal{C}_p \) is said to be regularly convergent if it is a single convergent sequence with respect to each index and denote the space of all such sequences by \( \mathcal{C}_r \).

Let us consider a double sequence \( x = (x_{mn}) \) and define the sequence \( s = (s_{mn}) \) via \( s_{mn} = \sum_{k,l=0}^{m,n} x_{kl} \) for all \( m, n \in \mathbb{N} \). Then, the pair \( (x, s) \) and the sequence \( s = (s_{mn}) \) are called as a double series and the sequence of partial sums of the double series, respectively. Here and after, unless stated otherwise we assume that \( \delta \) denotes any of the symbols \( p, bp \) or \( r \). If the double sequence \( (s_{mn}) \) is convergent in the \( \delta \)-sense, then the double series \( \sum_{k,l} x_{kl} \) is said to be convergent in the \( \delta \)-sense and it is showed that \( \delta - \lim_{k,l \to \infty} x_{kl} = \delta - \lim_{m,n \to \infty} s_{mn} \).

Also, we find some criteria about the convergence of a double series in Limaye and Zeltser [1]. Throughout the text we use the notation \( \sum_{k,l} x_{kl} \) instead of \( \sum_{k,l=0}^{\infty} x_{kl} \).

By \( \mathcal{L}_s \), we denote the space of absolutely \( s \)-summable double sequences defined by Başar and Sever [2]. Throughout the text, we assume that \( 0 < s < \infty \) and \( s' \) denotes the conjugate of \( s \), that is, \( s' = s/(s - 1) \) for \( 1 < s < \infty \), \( s' = \infty \) for \( s = 1 \) or \( s' = 1 \) for \( s = \infty \). Also, by \( \mathcal{L}_r \), we mean the space of absolutely convergent double series.

The \( \alpha \)-dual \( \lambda^\alpha \), \( \beta(\delta) \)-dual \( \lambda^{(\delta)} \) with respect to the \( \delta \)-convergence and the \( \gamma \)-dual \( \lambda^\gamma \) of a double sequence space \( \lambda \) are respectively defined by

\[
\lambda^\alpha := \left\{ (a_{kl}) \in \Omega : \sum_{k,l} |a_{kl}| x_{kl} < \infty \text{ for all } (x_{kl}) \in \lambda \right\},
\lambda^{(\delta)} := \left\{ (a_{kl}) \in \Omega : \delta - \sum_{k,l} a_{kl} x_{kl} \text{ exists for all } (x_{kl}) \in \lambda \right\},
\lambda^\gamma := \left\{ (a_{kl}) \in \Omega : \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=0}^{m,n} a_{kl} x_{kl} \right| < \infty \text{ for all } (x_{kl}) \in \lambda \right\}.
\]

It is easy to see for any two spaces \( \lambda, \mu \) of double sequences that \( \mu^\alpha \subset \lambda^\alpha \) whenever \( \lambda \subset \mu \) and \( \lambda^\alpha \subset \lambda^\gamma \). Additionally, it is known that the inclusion \( \lambda^\alpha \subset \lambda^{(\delta)} \) holds while the inclusion \( \lambda^{(\delta)} \subset \lambda^\gamma \) does not hold, since the \( \delta \)-convergence of a sequence of partial sums of a double series does not imply its boundedness.

Let \( \lambda \) and \( \mu \) be two double sequence spaces, and \( A = (a_{mnkl}) \) be any four-dimensional real or complex infinite matrix. Then, we say that \( A \) defines a matrix mapping from \( \lambda \) into \( \mu \) and we write \( A : \lambda \to \mu \), if for every sequence \( x = (x_{kl}) \in \lambda \) the \( A \)-transform \( Ax = [(Ax)_{mn}]_{m,n \in \mathbb{N}} \) of \( x \) exists and is in \( \mu \); where

\[
(Ax)_{mn} = \delta - \sum_{k,l} a_{mnkl} x_{kl} \text{ for each } m, n \in \mathbb{N}.
\]

We define the \( \delta \)-summability domain \( \lambda^{(\delta)}_A \) of \( A \) in a space \( \lambda \) of double sequences by

\[
\lambda^{(\delta)}_A := \left\{ x = (x_{kl}) \in \Omega : Ax = \left( \delta - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and is in } \lambda \right\}.
\]

We say with the notation (1) that \( A \) maps the space \( \lambda \) into the space \( \mu \) if \( \lambda \subset \mu^{(\delta)}_A \) and we denote the set of all four dimensional matrices, transforming the space \( \lambda \) into the space \( \mu \), by \( (\lambda : \mu) \). Thus, \( A = (a_{mnkl}) \in (\lambda : \mu) \) if and only if the double series on the right side of (1) converges in the sense of \( \delta \) for each \( m, n \in \mathbb{N} \), i.e,
\( A_{mn} \in \lambda^R(\delta) \) for all \( m, n \in \mathbb{N} \) and every \( x \in \lambda \), and we have \( A_{mn} = \mu \) for all \( x \in \lambda ; \) where \( A_{mn} = (a_{mnkl})_{k,l \in \mathbb{N}} \) for all \( m, n \in \mathbb{N} \). We say that a four-dimensional matrix \( A \) is \( C_\delta \)-conservative if \( C_\delta \subset (C_\delta)_L \), and is \( C_\delta \)-regular if it is \( C_\delta \)-conservative and \( \delta - \lim_{x_{mn}} x = \delta - \lim_{m,n \to \infty} (Ax)_{mn} = \delta - \lim_{m,n \to \infty} x_{mn} \), where \( x = (x_{mn}) \in C_\delta \). In this paper, we only consider bp-summability domain.

Using the notation of Zeltser [3], we define the double sequences \( e^{kl} = (e^{kl}_{mn}) \), \( e^l \), \( e_k \) and \( e \) by \( e^{kl}_{mn} = 1 \) if \( (k, l) = (m, n) \) and \( e^{kl}_{mn} = 0 \) otherwise, and \( e^l := \sum_k e^{kl} \), \( e_k := \sum_l e^{kl} \) and \( e := \sum_{l,j} e^{kl} \) (coordinatewise sum) for all \( k, l, m, n \in \mathbb{N} \) and we denote \( \Phi \) by \( \Phi = \text{span}(e^{kl} : k, l \in \mathbb{N}) \).

For all \( m, n, k, l \in \mathbb{N} \), we say that \( A = (a_{mnkl}) \) is a triangular matrix if \( a_{mnkl} = 0 \) for \( k > m \) or \( l > n \) or both, [4]. Following Adams [4], we also say that a triangular matrix \( A = (a_{mnkl}) \) is called a triangle if \( a_{mnmn} \neq 0 \) for all \( m, n \in \mathbb{N} \). Referring to Cooke [5, Remark (a), p. 22], one can conclude that every triangle matrix has an unique inverse which is also a triangle.

Zeltser [3] essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences in her PhD thesis. Altay and Başar [6] have defined the spaces \( B_S \) and \( C_S \) of double series whose sequence of partial sums are in the spaces \( M_\mu \), \( C_\delta \), respectively. Mursaleen and Başar [7] have introduced the spaces \( \tilde{M}_\mu \), \( C_\delta \) and \( L_\nu \) of double sequences whose Cesàro transforms are in \( M_\mu \), \( C_\delta \) and \( L_\nu \), respectively. The reader can refer to Başar [8] and Mursaleen and Mohiuddine [9] for relevant terminology and required details on the double sequences and related topics.

Following Mursaleen and Başar [7] and Alotaibi and Çakan [10], Yeşilkayagıl and Başar [11] have defined the double sequence spaces \( R^{(l)}(M_\mu), R^{(l)}(C_\delta), R^{(l)}(C_\phi) \) and \( R^{(l)}(C_\epsilon) \) as the domain of four dimensional Riesz mean \( R^{(l)} \) in the spaces \( M_\mu, C_\delta, C_\phi \) and \( C_\epsilon \), respectively. Also, they have characterized the matrix class \( (M_\mu : M_\mu) \) in [12] and have introduced the some topological property of the double spaces \( C_\phi \) and \( C_\epsilon \) of almost null and almost convergent double sequences, respectively, in [13].

In [14] Tuğ and Başar have introduced some new double sequence spaces \( B(M_\mu), B(C_\delta), \) and \( B(L_\nu) \) as the domain of four-dimensional generalized difference matrix \( B(r, s, t, u) \) in the spaces \( M_\mu, C_\delta \) and \( L_\nu \), respectively.

Let \( q = (q_k), t = (t_l) \) be two sequences of non-negative numbers which are not all zero and \( Q_m = \sum_{k=0}^{m} q_k \), \( q_0 > 0, T_n = \sum_{l=0}^{n} t_l, t_0 > 0 \). Then, the Riesz mean with respect to the sequences \( q = (q_k) \) and \( t = (t_l) \) is defined by the matrix \( R^{(l)} = (r^{(l)}_{mn}) \) as follows

\[
r^{(l)}_{mnkl} = \begin{cases} \frac{q_k t_l}{Q_m T_n} & , \quad 0 \leq k \leq m, \: 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise}, \end{cases}
\]

for all \( m, n, k, l \in \mathbb{N} \). It is known by Theorem 2.8 of Yeşilkayagıl and Başar [11] that the four dimensional Riesz mean \( R^{(l)} \) is RH-regular if and only if \( \lim_{m \to \infty} Q_m = \infty \) and \( \lim_{n \to \infty} T_n = \infty \). The Riesz transform \( R^{(l)} \) of a double sequence \( x = (x_{kl}) \) is given by

\[
y_{mn} = (R^{(l)}x)_{mn} = \frac{1}{Q_m T_n} \sum_{k,l=0}^{m,n} q_k t_l x_{kl} \tag{2}
\]

for all \( m, n \in \mathbb{N} \). Throughout the paper, we suppose that the terms of the double sequences \( x = (x_{kl}) \) and \( y = (y_{mn}) \) are connected with the relation (2) and the term with negative index is zero. If \( p - \lim (R^{(l)}x)_{mn} = s \), \( s \in C \), then the sequence \( x = (x_{kl}) \) is said to be Riesz convergent to \( s \) (see [10]). Note that in the case \( q_k = 1 \) for all \( k \) and \( t_l = 1 \) for all \( l \), the Riesz mean \( R^{(l)} \) is reduced to the four dimensional Cesàro mean \( C \) of order one.

Let \( I = (\delta_{mnkl}) \) is four dimensional unit matrix, that is, \( \delta_{mnkl} = \begin{cases} 1, & (m, n) = (k, l), \\ 0, & \text{otherwise}. \end{cases} \) Using the equality \( \delta_{mnkl} = \sum_{i,j} r_{mnij} d_{ijkl} = \frac{1}{Q_m T_n} \sum_{i,j} q_i t_l d_{ijkl} \), one can obtain by a straightforward calculation that the inverse \( (R^{(l)})^{-1} = (d_{mnkl}) \) of the triangle matrix \( R^{(l)} \) is given, as follows:

\[
d_{mnkl} = \begin{cases} (-1)^{m+n-(k+l)} \frac{Q_m T_n}{q_k t_l} & , \quad m-1 \leq k \leq m, \: n-1 \leq l \leq n, \\ 0 & , \quad \text{otherwise}, \end{cases}
\]
for all $m, n, k, l \in \mathbb{N}$.

In the present paper, referring Başar and Sever [2] we introduce the new space $R^d(\mathcal{L}_s)$ defined by

$$R^d(\mathcal{L}_s) := \{x = (x_{kl}) \in \Omega : [(R^d x)_{mn}] \in \mathcal{L}_s\}, \quad (0 < s < \infty).$$

2. The Space $R^d(\mathcal{L}_s)$ of Double Sequences

In this section, we give some results on the space $R^d(\mathcal{L}_s)$.

**Theorem 2.1.** The set $R^d(\mathcal{L}_s)$ is the linear space with the coordinatewise addition and scalar multiplication, and the following statements hold:

(i) If $0 < s < 1$, then $R^d(\mathcal{L}_s)$ is a complete $s$-normed space with

$$\|x\|_s = \left(\sum_{m,n} \left(\frac{1}{Q_m T_n} \sum_{k,l=0}^{m,n} q_k l x_{kl}\right)^s\right)^{1/s}$$

which is $s$-norm isomorphic to the space $\mathcal{L}_s$.

(ii) If $1 \leq s < \infty$, then $R^d(\mathcal{L}_s)$ is a Banach space with

$$\|x\|_s = \left(\sum_{m,n} \left(\frac{1}{Q_m T_n} \sum_{k,l=0}^{m,n} q_k l x_{kl}\right)^s\right)^{1/s}$$

which is norm isomorphic to the space $\mathcal{L}_s$.

**Proof.** Since Part (i) can be proved in the similar way, we give the proof only for Part (ii).

The first part is a routine verification and so we omit it.

To prove the fact $R^d(\mathcal{L}_s)$ is norm isomorphic to the space $\mathcal{L}_s$, we should show the existence of a linear bijection between the spaces $R^d(\mathcal{L}_s)$ and $\mathcal{L}_s$. Consider the transformation $U$ defined from $R^d(\mathcal{L}_s)$ to $\mathcal{L}_s$ by $x \mapsto Ux = [(R^d x)_{mn}]$. It is trivial that $U$ is linear. We get from the equation

$$Ux = \begin{bmatrix}
    x_{00} & \frac{t_1 x_{00} + t_0 x_{10}}{1} & \frac{t_2 x_{00} + t_1 x_{01} + t_0 x_{11}}{t_1} & \cdots \\
    \frac{t_1 x_{00} + t_0 x_{10}}{Q_1} & \frac{t_2 x_{00} + t_1 x_{01} + t_0 x_{11}}{Q_1} & \frac{t_2 x_{00} + t_1 x_{01} + t_0 x_{11}}{Q_2} & \cdots \\
    \frac{t_1 x_{00} + t_0 x_{10}}{Q_2} & \frac{t_2 x_{00} + t_1 x_{01} + t_0 x_{11}}{Q_2} & \frac{t_2 x_{00} + t_1 x_{01} + t_0 x_{11}}{Q_2} & \cdots \\
    \vdots & \vdots & \vdots & \ddots \\
    \frac{t_1 x_{00} + t_0 x_{10}}{Q_n} & \frac{t_2 x_{00} + t_1 x_{01} + t_0 x_{11}}{Q_n} & \frac{t_2 x_{00} + t_1 x_{01} + t_0 x_{11}}{Q_n} & \cdots \\
    \vdots & \vdots & \vdots & \ddots \\
    \vdots & \vdots & \vdots & \ddots
\end{bmatrix} = \theta$$

that $x = \theta$ whenever $Ux = \theta$, where $\theta$ denotes the zero vector. This shows that $U$ is injective.

Let $y = (y_{kl}) \in \mathcal{L}_s$ and define the sequence $x = (x_{kl})$ via $y$ by

$$x_{kl} = \frac{1}{q_k l} (Q_l T_j y_{kl} - Q_{k-1} T_j y_{k-1,j} - Q_k T_{j-1} y_{k,j-1} + Q_{k-1} T_{j-1} y_{k-1,j-1})$$

(4)
for all \(k, l \in \mathbb{N}\). Then, we have

\[
Q_m T_n (R^{kl} x)_{mn} = \sum_{k,l=0}^{m,n} (Q_k T_l y_{kl} - Q_k T_{l-1} y_{k,l-1} + Q_k T_l y_{k,l-1}) \\
= Q_0 \sum_{l=0}^{n} (T_1 y_{0l} - T_{l-1} y_{0,l-1}) + \sum_{l=0}^{n} (Q_1 T_l y_{1l} - Q_1 T_{l-1} y_{l,l-1} - Q_0 T_{l-1} y_{0,l}) \\
+ \sum_{l=0}^{n} (Q_2 T_l y_{2l} - Q_2 T_{l-1} y_{2,l-1} - Q_1 T_{l-1} y_{1,l-1}) + \ldots + \\
+ \sum_{l=0}^{n} (Q_m T_l y_{ml} - Q_m T_{l-1} y_{m,l-1} - Q_{m-1} T_{l-1} y_{m-1,l-1} - Q_{m-2} T_{l-1} y_{m-2,l-1}) \\
+ \sum_{l=0}^{n} (Q_{m-1} T_l y_{m-1,l} - Q_{m-1} T_{l-1} y_{m-1,l-1} - Q_{m-2} T_{l-1} y_{m-2,l-1} - Q_{m-3} T_{l-1} y_{m-3,l-1}) \\
= Q_m \sum_{l=0}^{n} (T_l y_{ml} - T_{l-1} y_{m,l-1}) = Q_m T_n y_{mn}
\]

and so

\[
\|(R^{kl} x)_{mn}\| = |y_{mn}|
\]

which yields that

\[
\sum_{m,n} |(R^{kl} x)_{mn}|^p = \sum_{m,n} |y_{mn}|^p. \tag{5}
\]

Since \(y \in \mathcal{L}_s\), we have \(x \in R^{kl}(\mathcal{L}_s)\), that is, \(U\) is surjective. Also, we see from (5) that \(U\) is a norm preserving transformation.

Now, we can show that \(R^{kl}(\mathcal{L}_s)\) is a Banach space with the norm \(\|\cdot\|\) defined by (3). To prove this, we use Part (b) of Corollary 6.3.41 in [15] which says that "Let \((X, p)\) and \((Y, q)\) be semi-normed spaces and \(U : (X, p) \to (Y, q)\) be an isometric isomorphism. Then, \((X, p)\) is complete if and only if \((Y, q)\) is complete. In particular, \((X, p)\) is a Banach space if and only if \((Y, q)\) is a Banach space." Since the transformation \(U\) defined above from \(R^{kl}(\mathcal{L}_s)\) to \(\mathcal{L}_s\) is an isometric isomorphism and the space \(\mathcal{L}_s\) is a Banach space from Theorem 2.1 in [2], we conclude that the space \(R^{kl}(\mathcal{L}_s)\) is a Banach space. This step completes the proof. \qed

A non-empty subset \(S\) of a locally convex space \(X\) is called fundamental if the closure of the linear span of \(S\) equals \(X\), [15]. Using this definition, we define the set \(S \subset \mathcal{L}_s\) as \(S := \{x^{kl} : k, l \in \mathbb{N}\}\). Then, we have \(\Phi = \text{span} S\). We shall show that \(\Phi\) is dense in \(\mathcal{L}_s\), that is, \(c\Phi = \mathcal{L}_s\). Let the relation \(c\Phi = \mathcal{L}_s\) does not hold. Hence, there exists a ball in \(\mathcal{L}_s\), no matter how small, does not contain any points of \(\Phi\), i.e, there does not exist a \(y \in \Phi\) such that

\[
\|x - y\| \neq \varepsilon^s \tag{6}
\]

for a point \(x \in \mathcal{L}_s\). Then, by (6) we have that

\[
\|x - y\| = \sum_{i,j} |x_{ij} - e_{ij}^{kl}|^s = |x^{kl} - 1|^s \neq \varepsilon^s,
\]

that is, \(|x^{kl} - 1|^s \geq \varepsilon^s\). Choose \(\varepsilon = 1/2\). Then, we have either \(x^{kl} \leq 1/2\) or \(3/2 \leq x^{kl}\) for all \(k, l \in \mathbb{N}\). For both statement, we can find \(x \notin \mathcal{L}_s\), a contradiction. Since \(x \in \mathcal{L}_s\) is arbitrary, every ball in \(\mathcal{L}_s\) contains a point of
\( \Phi, \text{ i.e. } \Phi \text{ is dense in } \mathcal{L}_s. \) Therefore, \( S \) is fundamental set of \( \mathcal{L}_s. \) Using this fact, we define the double sequence \( b^{(kl)} = \left(b^{(kl)}_{mn}\right) \) by

\[
 b^{(kl)}_{mn} := \begin{cases} 
 \frac{Q_lT_i}{q_{kl}T_i}, & m = k, n = l, \\
 \frac{Q_{l+1}T_i}{q_{kl}T_i}, & m = k, n = l + 1 \\
 \frac{Q_lT_i}{q_{k+1}T_i}, & m = k + 1, n = l, \\
 \frac{q_{k+1}T_i}{q_{kl}T_i}, & m = k + 1, n = l + 1, \\
 0 & \text{otherwise} 
\end{cases}
\]

(7)

for all \( k, l, m, n \in \mathbb{N}. \) Then, \( \{b^{(kl)}; k, l \in \mathbb{N}\} \) is the fundamental set of the space \( R^q(\mathcal{L}_s); \) since \( R^q b^{(kl)} = e^{kl} \) with \( 0 < s < \infty. \)

**Theorem 2.2.** If \( \left( \frac{1}{q_{kl}T_i} \right) \notin \mathcal{L}_s, \) then \( \mathcal{L}_s \notin R^q(\mathcal{L}_s) \) holds.

**Proof.** Let \( \left( \frac{1}{q_{kl}T_i} \right) \notin \mathcal{L}_s. \) We take the sequence \( e^{00}. \) Obviously, \( e^{00} \in \mathcal{L}_s. \) For all \( m, n \in \mathbb{N}, \) we have that

\[
(R^q e^{00})_{mn} = \frac{q_{00}}{Q_m T_n}.
\]

Since \( \left( \frac{1}{q_{kl}T_i} \right) \notin \mathcal{L}_s, \) \( R^q e^{00} \notin \mathcal{L}_s. \) So, \( e^{00} \notin R^q(\mathcal{L}_s), \) as desired. □

**Theorem 2.3.** Let \( 1 < s < r < \infty. \) Then, the inclusion \( R^q(\mathcal{L}_s) \subset R^q(\mathcal{L}_r) \) strictly holds.

**Proof.** Let \( 1 < s < r < \infty \) and \( x = (x_{kl}) \in R^q(\mathcal{L}_s). \) Then, the following inequality

\[
\left( \sum_{i,j} \left( \frac{1}{Q_m T_n} \sum_{k,l=0}^{m,n} q_{kl} x_{kl} \right)^r \right)^{1/r} \leq \left( \sum_{i,j} \left( \frac{1}{Q_m T_n} \sum_{k,l=0}^{m,n} q_{kl} x_{kl} \right)^s \right)^{1/s}
\]

(8)

holds by Jensen’s inequality. Therefore, one can see by applying \( p \)-limit to (8), as \( i, j \to \infty \) that \( \|x\|_s < \|x\|_r < \infty \) which means that \( x \in R^q(\mathcal{L}_s), \) as desired.

Now, consider the sequence \( x = (x_{kl}) \) defined by

\[
x_{kl} = \frac{1}{q_{kl}T_i} \left( \frac{Q_l T_i}{(k+2)(l+2)} - \frac{Q_{l+1} T_i}{(k+1)(l+2)} + \frac{Q_{l+1} T_{i+1}}{(k+2)(l+1)} + \frac{Q_{l+1} T_i}{(k+1)(l+1)} \right)
\]

(9)

for all \( k, l \in \mathbb{N}. \) Using (9), we have

\[
\|R^q x\|_{Q_n} = \frac{1}{(m+2)(n+2)}
\]

and so

\[
\sum_{i,j} \|R^q x\|_{Q_n}^r = \sum_{i,j} \left( \frac{1}{(m+2)(n+2)} \right)^{1/r} = \sum_{i,j} \frac{1}{(m+2)(n+2)} = \infty,
\]

that is, \( x \notin R^q(\mathcal{L}_s) \). Since \( 1 < s < r < \infty, 1 < r/s \). So, we have

\[
\sum_{i,j} \|R^q x\|_{Q_n}^r = \sum_{i,j} \left( \frac{1}{(m+2)(n+2)} \right)^{1/s} < \infty,
\]

that is, \( x \in R^q(\mathcal{L}_r). \) This step completes the proof. □

Let \( \lambda \) be a locally convex space. Then, a subset is called \textit{barrel} if it is absolutely convex, absorbing and closed in \( \lambda. \) Moreover, \( \lambda \) is called a \textit{barrelled space} if each barrel is a neighborhood of zero; [15, p. 336].

**Lemma 2.4.** [17] Every Banach space and every Fréchet space is a barrelled space.
**Theorem 2.5.** The following statements hold:

(i) Let $1 \leq s < \infty$. Then, $R^s(\mathcal{L}_s)$ is a barrelled space.

(ii) Let $0 < s < 1$. Then, $R^s(\mathcal{L}_s)$ is not a barrelled space.

**Proof.**

(i) By Lemma 2.4 and Part (ii) of Theorem 2.1, we say that $R^s(\mathcal{L}_s)$ is a barrelled space for $1 \leq s < \infty$.

(ii) We show that the space $\mathcal{L}_s$ is not a locally convex space for $0 < s < 1$. Let $\mathcal{U} := \{x : \|x\|_s \leq 1\}$. We shall show that $\mathcal{U}$ includes no convex neighborhood of 0. Let $\mathcal{V}$ be a convex neighborhood of 0. For some $\varepsilon > 0$, $\mathcal{V} \supset \{x : \|x\|_s \leq \varepsilon\}$. In particular, $\varepsilon^{1/2}e^{\delta_1} \in \mathcal{V}$ for each $k, l \in \mathbb{N}$. Choose integers $m, n > \frac{1}{\varepsilon^{1/2}}$ and define the sequence $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} \frac{\varepsilon^{1/2}}{m+1(n+1)} & 0 \leq k \leq m 	ext{ and } 0 \leq l \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by choosing of $\varepsilon$ we see that $x \in \mathcal{V}$ and

$$\|x\|_s = \sum_{k=0}^{m} \sum_{l=0}^{n} \left| \frac{\varepsilon^{1/2}}{(m+1)(n+1)} \right| = \varepsilon \sum_{k=0}^{m} \sum_{l=0}^{n} 1 = \varepsilon \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m} \sum_{l=0}^{n} 1 = \varepsilon \frac{1}{(m+1)(n+1)} = 1.$$

So, $\mathcal{V} \notin \mathcal{U}$. Since the space $\mathcal{L}_s$ is not a locally convex space for $0 < s < 1$, the space $R^s(\mathcal{L}_s)$ is not, too. Therefore, the space $R^s(\mathcal{L}_s)$ is not a barrelled space.

A double sequence space $\lambda$ is said to be solid if and only if

$$\lambda := \{(u_{kl}) \in \Omega : \exists(x_{kl}) \in \lambda \text{ such that } |u_{kl}| \leq |x_{kl}| \text{ for all } k, l \in \mathbb{N} \subset \lambda,$$

[2, p. 153]. A double sequence space $\lambda$ is said to be monotone if $xu = (x_{kl}u_{kl}) \in \lambda$ for every $x = (x_{kl}) \in \lambda$ and $u = (u_{kl}) \in \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$, where $\{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ denotes the set of all double sequences of zeros and ones. If $\lambda$ is monotone, then $\lambda^\uparrow = \lambda^{\rho(0)}$, [3, p. 36] and $\lambda$ is monotone whenever $\lambda$ is solid.

**Theorem 2.6.** Let $0 < s < \infty$. Then, the space $\mathcal{L}_s$ is monotone.

**Proof.** Let $0 < s < \infty$, $x = (x_{kl}) \in \mathcal{L}_s$ and $u = (u_{kl}) \in \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$. Then, we have $|x_{kl}u_{kl}|^s = |x_{kl}|^s|u_{kl}|^s \leq |x_{kl}|^s$, for each $k, l \in \mathbb{N}$. So, we have that $\sum_{k,l} |x_{kl}u_{kl}|^s \leq \sum_{k,l} |x_{kl}|^s$, that is, $xu \in \mathcal{L}_s$.

**Theorem 2.7.** Let $0 < s < \infty$. If $\left(\frac{1}{Q_{m,l}}\right) \notin \mathcal{L}_s$, then the space $R^s(\mathcal{L}_s)$ is not monotone.

**Proof.** Let $0 < s < \infty$ and $\left(\frac{1}{Q_{m,l}}\right) \notin \mathcal{L}_s$. Choose the sequence $x = (x_{kl}) \in R^s(\mathcal{L}_s)$ such that $x_{00} \neq 0$ and take the sequence $u = (u_{kl}) = e^{00} \in \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$. Hence, for the sequence $z = ux = e^{00}x$ we derive that

$$(R^sz)_{mn} = \frac{1}{Q_{m,l}} q_{0l} x_{00}.$$

Since $\left(\frac{1}{Q_{m,l}}\right) \notin \mathcal{L}_s$, $R^s z \notin \mathcal{L}_s$. So, $z \notin R^s(\mathcal{L}_s)$, as desired.
3. Dual Spaces

In this section, we determine the $\alpha$- and $\beta(\mathfrak{s})$- duals of the space $L_s$ in the case $0 < s \leq 1$ and $\beta(bp)$- dual of the space $R'_{\mathfrak{s}}(L_s)$ for $1 < s < \infty$.

**Theorem 3.1.** Let $0 < s \leq 1$. Then, the $\alpha$-dual of the space $L_s$ is the space $M_u$.

*Proof.* Since $L_u \subset M_u$, $L_u^\alpha = M_u$ and $L_s \subset L_u$ for $0 < s \leq 1$, we have that $M_u \subset L_u^\alpha$.

Conversely, suppose that $z = (z_{kl}) \in L_u^\alpha \setminus M_u$. Then, $\sum_{k,l} |z_{kl}x_{kl}| < \infty$ for all $x = (x_{kl}) \in L_s$ and $\operatorname{sup}_{k,l \in \mathbb{N}} |z_{kl}| = \infty$. Hence, there exist sequences $(k_m)$ and $(l_m)$ such that at least one is strictly increasing for $m \in \mathbb{N}$. So, we can take $z_{k_m,l_m} > (m + 1)^{2/3}$. If we define $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} (m + 1)^{-2/3}, & k = k_m \text{ and } l = l_m, \\ 0, & k \neq k_m \text{ or } l \neq l_m \end{cases}$$

for all $k, l, m \in \mathbb{N}$, then we have $x \in L_s$. But $\sum_{k,l} |z_{kl}x_{kl}| = \sum_m |z_{k_m,l_m}x_{k_m,l_m}| > \sum_m 1 = \infty$, that is, $z \notin L_u^\alpha$, a contradiction. Therefore, the inclusion $L_u^\alpha \subset M_u$ holds.

By combining the inclusions $M_u \subset L_u^\alpha$ and $L_u^\alpha \subset M_u$, we get $L_u^\alpha = M_u$, as desired. □

**Corollary 3.2.** Let $0 < s \leq 1$. Then, the $\beta(\mathfrak{s})$-dual of the space $L_s$ is the space $M_u$.

**Theorem 3.3.** Let $0 < s \leq 1$. Then, the inclusion $(R'_{\mathfrak{s}}(L_s))^\alpha \subset M_u$ holds.

*Proof.* Suppose that $z = (z_{kl}) \in (R'_{\mathfrak{s}}(L_s))^\alpha \setminus M_u$. Then, $zx \in L_u$ for all $x \in R'_{\mathfrak{s}}(L_s)$. We take the sequence $b^{(kl)}$ as in (7). So, we have $\sum_{m,n} |(R'_{\mathfrak{s}}b^{(km)})^m| = \sum_{m,n} |b^{kl}|^m = 1$ for all $k, l \in \mathbb{N}$. Hence, $b^{(kl)} \in R'_{\mathfrak{s}}(L_s)$ and so $zx = (z_{kl}b^{(kl)}) \in L_u$.

With some calculation, we have following five cases:

- **Case 1.** $z_{ij}b^{(kl)} = z_{kl} \frac{Q_T}{q_{kl}}$, for $(i,j) = (k,l)$.

- **Case 2.** $z_{ij}b^{(kl)} = -z_{k,l+1} \frac{Q_T}{q_{kl}}$, for $(i,j) = (k,l+1)$.

- **Case 3.** $z_{ij}b^{(kl)} = z_{k+1,l} \frac{Q_T}{q_{kl}}$, for $(i,j) = (k+1,l)$.

- **Case 4.** $z_{ij}b^{(kl)} = z_{k+1,l+1} \frac{Q_T}{q_{kl}}$, for $(i,j) = (k+1,l+1)$.

- **Case 5.** $z_{ij}b^{(kl)} = 0$ for otherwise.

For example, in case 1, we write that $(z_{kl} \frac{Q_T}{q_{kl}}) \in L_u$ so is in $M_u$. But, we know that $(Q_T)$ (or $(T_l)$) is a positive increasing sequence, that is, it is not bounded. Therefore, $(z_{kl}) \in M_u$, a contradiction. Hence, the inclusion $(R'_{\mathfrak{s}}(L_s))^\alpha \subset M_u$ holds, as desired. □

**Theorem 3.4.** Let $1 < s < \infty$ and define the sets $d_1$, $d_2$ and $d_3$, as follows:

$$d_1 = \left\{ a = (a_{kl}) \in \Omega : \sum_{k,l} \left| Q_kT_{k,l} \Delta_{k,l}\left( \frac{a_{kl}}{q_{kl}} \right) \right|^p < \infty \right\},$$

$$d_2 = \left\{ a = (a_{kl}) \in \Omega : \sup_{n \in \mathbb{N}} \sum_{k,l} \left| Q_kT_{k,l} \Delta_{k,l}\left( \frac{a_{kl}}{q_{kl}l_n} \right) \right|^p < \infty \right\},$$

$$d_3 = \left\{ a = (a_{kl}) \in \Omega : \sup_{m \in \mathbb{N}} \sum_{k,l} \left| Q_mT_{k,l} \Delta_{k,l}\left( \frac{a_{kl}}{q_{ml}l_n} \right) \right|^p < \infty \text{ and } \left( Q_mT_{k,l} \frac{l_m}{q_{ml}l_n} \right) \in M_u \right\}.$$ 

Then, $(R'_{\mathfrak{s}}(L_s))^{(\beta(bp))} = d_1 \cap d_2 \cap d_3$. 
Proof. Let \( x = (x_{mn}) \in R^d(\mathcal{L}_s) \). Then, there exists a double sequence \( y = (y_{mn}) \in \mathcal{L}_s \) by Part (ii) of Theorem 2.1. Also, we have \( s = (s_{mn}) \) from (4) such that

\[
\begin{align*}
    s_{mn} &= \sum_{k,l=0}^{m,n} x_{kl} - \frac{1}{q_k l_l} (Q_k T_l y_{kl} - Q_k T_{l-1} y_{k,l-1} - Q_k T_{l-1} y_{k-1,l-1} + Q_k T_{l-1} y_{k-1,l-1})
\end{align*}
\]

for all \( m, n \in \mathbb{N} \). Now, by the generalized Abel transformation for double sequences we obtain that

\[
\begin{align*}
    z_{mn} &= \sum_{k,l=0}^{m,n} a_{kl} X_{kl} = \sum_{k,l=0}^{m,n} s_{kl} A_{11} a_{kl} + \sum_{k=0}^{m-1} s_{kn} A_{10} a_{kn} + \sum_{l=0}^{n-1} s_{ml} A_{01} a_{ml} + s_{mn} a_{mn}
\end{align*}
\]

for all \( m, n \in \mathbb{N} \). With some straightforward calculation, we can rewrite the relation (10) as follows

\[
\begin{align*}
    z_{mn} &= \sum_{k,l=0}^{m,n} q_{kl} X_{kl} = \sum_{k,l=0}^{m,n} Q_k T_l A_{11} \left( \frac{a_{kl}}{q_k l_l} \right) y_{kl} + \sum_{k=0}^{m-1} Q_k T_n A_{10} \left( \frac{a_{kn}}{q_k l_n} \right) y_{kn} \\
    &\quad + \sum_{l=0}^{n-1} Q_m T_l A_{01} \left( \frac{a_{ml}}{q_m l_m} \right) y_{ml} + Q_m T_n \left( \frac{a_{mn}}{q_m l_n} \right) y_{mn} = (By)_{mn}
\end{align*}
\]

for all \( m, n \in \mathbb{N} \), where the four-dimensional matrix \( B = (b_{mnkl}) \) is defined by

\[
\begin{align*}
    b_{mnkl} = \begin{cases} 
        Q_k T_l A_{11} \left( \frac{a_{kl}}{q_k l_l} \right), & 0 \leq k \leq m - 1 \text{ and } 0 \leq l \leq n - 1, \\
        Q_k T_n A_{10} \left( \frac{a_{kn}}{q_k l_n} \right), & 0 \leq k \leq m - 1 \text{ and } l = n, \\
        Q_m T_l A_{01} \left( \frac{a_{ml}}{q_m l_m} \right), & k = m \text{ and } 0 \leq l \leq n - 1, \\
        Q_m T_n \left( \frac{a_{mn}}{q_m l_n} \right), & k = m \text{ and } l = n, \\
        0, & \text{otherwise}
    \end{cases}
\end{align*}
\]

for all \( m, n, k, l \in \mathbb{N} \). Thus, we see that \( ax = (a_{mn} x_{mn}) \in C_{\mathcal{L}_p} \) whenever \( x = (x_{mn}) \in R^d(\mathcal{L}_s) \) if and only if \( z = (z_{mn}) \in C_{\mathcal{L}_p} \) whenever \( y = (y_{mn}) \in \mathcal{L}_s \). This leads us to the fact that \( B \in (\mathcal{L}_s : C_{\mathcal{L}_p}) \). Hence, from Part (ii) of Theorem 4.3, the following statement

\[
\sup_{m,n \in \mathbb{N}} \sum_{k,l} |b_{mnkl}|^{r'} < \infty,
\]

holds. Therefore, we derive that

\[
\begin{align*}
    \sup_{m,n \in \mathbb{N}} \sum_{k,l} \left| Q_k T_l A_{11} \left( \frac{a_{kl}}{q_k l_l} \right) \right|^{r'} < \infty,
    \\
    \sup_{k \in \mathbb{N}} \sum_{l} \left| Q_k T_n A_{10} \left( \frac{a_{kn}}{q_k l_n} \right) \right|^{r'} < \infty,
    \\
    \sup_{l \in \mathbb{N}} \sum_{k} \left| Q_m T_l A_{01} \left( \frac{a_{ml}}{q_m l_m} \right) \right|^{r'} < \infty,
    \\
    \left| Q_m T_n \left( \frac{a_{mn}}{q_m l_n} \right) \right|^{r'} \in \mathcal{M}_r,
\end{align*}
\]

Hence, \( [R^d(\mathcal{L}_s)]^{(p,p')} = d_1 \cap d_2 \cap d_3 \). \( \Box \)
4. Characterization of Some Classes of Matrix Mappings

In this section, we characterize the classes \((L_s : M_u), (L_s : C_{tp}), (R_{0t}(L_s) : M_u)\) and \((R_{0t}(L_s) : C_{tp})\) of four dimensional matrices, in the cases both \(0 < s \leq 1\) and \(1 < s < \infty\). We also characterize the class \((L_s : L_n)\) of four dimensional matrices in the cases \(0 < s \leq 1\) and \(1 \leq s_1 < \infty\).

**Theorem 4.1.** Let \(A = (a_{mnlk})\) be any four dimensional matrix. Then, the following statements are satisfied:

(i) Let \(0 < s \leq 1\). Then, \(A \in (L_s : M_u)\) if and only if

\[
N = \sup_{m,n,k,l \in \mathbb{N}} |a_{mnlk}| < \infty. \tag{12}
\]

(ii) Let \(1 < s < \infty\). Then, \(A \in (L_s : M_u)\) if and only if

\[
M_1 = \sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnlk}|^\gamma < \infty. \tag{13}
\]

**Proof.** (i) Let \(0 < s \leq 1\) and \(A = (a_{mnlk}) \in (L_s : M_u)\). Then, \(Ax\) exists and belongs to \(M_u\) for all \(x \in L_s\), and \(A_{mn} \in M_u\) by Corollary 3.2 for each \(m, n \in \mathbb{N}\). Therefore, we obtain for \(e^{kl} \in L_s\) that

\[
\|Ax^{kl}\|_\infty = \sup_{m,n \in \mathbb{N}} |a_{mnlk}| < \infty
\]

for each fixed \(k, l \in \mathbb{N}\). That is to say that the condition (12) is necessary.

Conversely, suppose that (12) holds and take any \(x = (x_{kl}) \in L_s\). Then, \(A_{mn} \in M_u\) by Corollary 3.2 for each \(m, n \in \mathbb{N}\) which implies the existence of \(Ax\). Let \(m, n \in \mathbb{N}\) be fixed. Then, since

\[
\left| \sum_{k,l} a_{mnlk} x_{kl} \right|^s \leq \left( \sum_{k,l} |a_{mnlk}| |x_{kl}| \right)^s \leq \left( \sup_{k,l \in \mathbb{N}} |a_{mnlk}| \right)^s \left( \sum_{k,l} |x_{kl}| \right)^s \leq \left( \sup_{k,l \in \mathbb{N}} |a_{mnlk}| \right)^s \sum_{k,l} |x_{kl}|^s
\]

one can obtain by taking supremum over \(m, n \in \mathbb{N}\) that

\[
\|Ax\|_\infty = \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l} a_{mnlk} x_{kl} \right| \leq N (\|x\|_s)^1/s.
\]

This shows the sufficiency of the condition (12).

(ii) Let \(1 < s < \infty\) and \(A = (a_{mnlk}) \in (L_s : M_u)\). Then, \(Ax\) exists and is in \(M_u\) for all \(x \in L_s\). We assume that \(M_1 = \infty\). Then, we may choose the sequences \((m_i), (k_i), (n_i)\) and \((l_i)\) in \(\mathbb{N}\) with \(k_i < k_{i+1}\) and \(l_i < l_{i+1}\) for all \(i, j \in \mathbb{N}\) such that

\[
|a_{m,n,k,l}|^\gamma > (ij)^\gamma. \tag{14}
\]

Let us define the double sequence \(x = (x_{kl}) \in L_s\) by

\[
x_{kl} := \begin{cases} 
sgn (a_{m,n,k,l}) , & k = k_i \text{ and } l = l_i, \\
0 , & \text{otherwise}
\end{cases}
\]

for all \(k, l \in \mathbb{N}\). Since \(s' > 1\), using the inequality (14) we see that

\[
|(Ax)_{m,n}| = \left| \sum_{k,l} a_{m,n,k,l} x_{kl} \right| = |a_{m,n,k,l} x_{k,l}| > ij
\]
and so,
\[
\sup_{i,j \in \mathbb{N}} \|(Ax)_{m,n}\| > \infty,
\]
a contradiction. Therefore, the condition (13) is necessary.

Conversely, suppose that (13) holds and take any \( x = (x_{kl}) \in \mathcal{L}_s \). Then \( Ax \) exists, since \( A_{mn} \in \mathcal{L}_{s'} \) for each \( m, n \in \mathbb{N} \) by Theorem 2.7 in [2]. Therefore, we obtain by Hölder’s inequality that
\[
\|Ax\|_{s'} = \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l} a_{mnkl} x_{kl} \right| \\
\leq \sup_{m,n \in \mathbb{N}} \left( \sum_{k,l} |a_{mnkl}|^{s'} \right)^{1/s'} \left( \sum_{k,l} |x_{kl}|^{s} \right)^{1/s} \\
< M_1 ||x||_s,
\]
as desired.

This completes the proof. \( \square \)

**Theorem 4.2.** Let \( 0 < s \leq 1 \) and \( 1 \leq s_1 < \infty \). Then, \( A = (a_{mnkl}) \in (\mathcal{L}_s : \mathcal{L}_{s_1}) \) if and only if
\[
\sup_{k,l \in \mathbb{N}} \sum_{m,n} |a_{mnkl}|^{s_1} < \infty. \quad (15)
\]

**Proof.** Let \( 0 < s \leq 1 \), \( 1 \leq s_1 < \infty \) and \( A \in (\mathcal{L}_s : \mathcal{L}_{s_1}) \). Then, \( Ax \) exists and belongs to \( \mathcal{L}_{s_1} \) for all \( x \in \mathcal{L}_s \), and \( A_{mn} \in \mathcal{M}_s \) by Corollary 3.2 for each \( m, n \in \mathbb{N} \). Therefore, we obtain for \( e^{kl} \in \mathcal{L}_s \) that
\[
||Ae^{kl}||_{s_1} = \left( \sum_{m,n} |a_{mnkl}|^{s_1} \right)^{1/s_1} < \infty
\]
for each fixed \( k, l \in \mathbb{N} \). That is to say that the condition (15) is necessary.

Conversely, suppose that the condition (15) is satisfied and take any \( x = (x_{kl}) \in \mathcal{L}_s \). Then, \( A_{mn} \in \mathcal{M}_s \) by Corollary 3.2 for each \( m, n \in \mathbb{N} \) which implies the existence of \( Ax \). Then,
\[
\left( \sum_{m,n=0} a_{mn} x_{kl} \right)^{1/s_1} = \left( \sum_{m,n=0} \left| a_{mn} x_{kl} \right|^{s_1} \right)^{1/s_1} \\
\leq \sum_{k,l} \left( \sum_{m,n=0} |a_{mnkl}|^{s_1} \right)^{1/s_1} \\
= \sum_{k,l} |x_{kl}| \left( \sum_{m,n=0} |a_{mnkl}|^{s_1} \right)^{1/s_1} \\
\leq \sup_{k,l \in \mathbb{N}} \left( \sum_{m,n=0} |a_{mnkl}|^{s_1} \right)^{1/s_1} \sum_{k,l} |x_{kl}| < \infty.
\]
Since \( i, j \in \mathbb{N}'s \) are arbitrary, we obtain that \( ||Ax||_{s_1} < \infty \), as desired. \( \square \)

**Theorem 4.3.** Let \( A = (a_{mnkl}) \) be any four dimensional matrix. Then, the following statements hold:

(i) Let \( 0 < s \leq 1 \). Then, \( A \in (\mathcal{L}_s : C_{bp}) \) if and only if (12) holds and there exists \( (\alpha_{kl}) \in \Omega \) such that
\[
bp - \lim_{m,n \to \infty} a_{mnkl} = \alpha_{kl}. \quad (16)
\]
(ii) Let $1 < s < \infty$. Then, $A \in (\mathcal{L}_s : C_{b\psi})$ if and only if (13) and (16) hold.

Proof. (i) Let $0 < s \leq 1$ and suppose that $A = (a_{mkl}) \in (\mathcal{L}_s : C_{b\psi})$. Then, since the inclusion $C_{b\psi} \subset \mathcal{M}_u$ holds, the necessity of the condition (12) is obtained from Part (i) of Theorem 4.1. Besides, since $Ax$ exists and belongs to $C_{b\psi}$ for every $x \in \mathcal{L}_s$ by hypothesis, this also holds for $e_{kl} \in \mathcal{L}_s$ which gives that $Ae_{kl} = (a_{mkl})_{m,n \in \mathbb{N}} \in C_{b\psi}$ for each fixed $k, l \in \mathbb{N}$. Hence, the condition (16) is necessary.

Conversely, suppose that (12) and (16) hold, and $x = (x_{kl})$ be any sequence in the space $\mathcal{L}_s$. Then, $A_{mn} \in \mathcal{L}_s^{(k)}$ for each $m, n \in \mathbb{N}$, $Ax$ exists. Therefore, we get by (16) for each fixed $k, l \in \mathbb{N}$ with (12) that

$$|\alpha_{kl}| = bp - \lim_{m,n \to \infty} |a_{mkl}| \leq \sup_{m,n \in \mathbb{N}} |a_{mkl}|$$

which gives that $(\alpha_{kl}) \in M_{b\psi}$. Hence, the series $\sum_{k,l} \alpha_{kl} x_{kl}$ converges for every $x \in \mathcal{L}_s$.

Additionally, for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|a_{mkl} - \alpha_{kl}| < \varepsilon$ for all $m, n > n_0$ by (16).

Then, we obtain that

$$\left| \sum_{k,l} a_{mkl} x_{kl} - \sum_{k,l} \alpha_{kl} x_{kl} \right|^s = \left| \sum_{k,l} (a_{mkl} - \alpha_{kl}) x_{kl} \right|^s \leq \sum_{k,l} |(a_{mkl} - \alpha_{kl}) x_{kl}|^s \leq \varepsilon^s \left( \sum_{k,l} |x_{kl}|^s \right) \leq \varepsilon^s \sum_{k,l} |x_{kl}|^s.$$ 

This shows that $bp - \lim_{m,n \to \infty} (Ax)_{mn} = \sum_{k,l} \alpha_{kl} x_{kl}$ as desired.

(ii) Let $s > 1$. Since the necessity of the conditions can be easily seen in the similar way used in Part (i), we omit the details.

It is obtained with (13) for all $i, j \in \mathbb{N}$ that

$$\sum_{k,l=0}^{i,j} |\alpha_{kl}|^r = bp - \lim_{m,n \to \infty} \sum_{k,l=0}^{i,j} |a_{mkl}|^r \leq \sup_{m,n \in \mathbb{N}} \sum_{k,l=0}^{i,j} |a_{mkl}|^r < \infty.$$ 

(17)

This means that $(\alpha_{kl}) \in \mathcal{L}_{r\psi}$. Hence, the double series $\sum_{k,l} \alpha_{kl} x_{kl}$ converges for every $x \in \mathcal{L}_s$.

For any given $\varepsilon > 0$, let us choose fixed $k_0, l_0 \in \mathbb{N}$ such that

$$\sum_{k,l=0}^{k_0, l_0} |x_{kl}|^r + \sum_{k,l=k_0+1,0}^{i,j} |x_{kl}|^r + \sum_{k,l=k_0+1,0}^{i,j} |x_{kl}|^r < \left( \frac{\varepsilon}{12M_1^{1/r}} \right)^r.$$ 

(18)

Then, there exist an $n_0 \in \mathbb{N}$ by (16) such that

$$\sum_{k,l=0}^{k_0} |(a_{mkl} - \alpha_{kl}) x_{kl}| < \varepsilon$$ 

(19)

for every $m, n > n_0$. Therefore, by applying Hölder’s inequality with using relations (17)-(19) we have that

$$\left| \sum_{k,l} a_{mkl} x_{kl} - \sum_{k,l} \alpha_{kl} x_{kl} \right| = \left| \sum_{k,l} (a_{mkl} - \alpha_{kl}) x_{kl} \right| < \varepsilon$$

for all sufficiently large $m, n$. Hence, $Ax \in C_{b\psi}$.

This step completes the proof. □
Theorem 4.4. Let $A = (a_{mnkj})$ be any four dimensional matrix. Then, the following statements hold:

(i) Let $0 < s \leq 1$. Then, $A \in (R^s(L_0) : M_n)$ if and only if

\[
\sup_{m,n,k,l \in \mathbb{N}} \left| \sum_{h=0}^{i} Q_h T_i \frac{a_{mnkl}}{q_{kl}} \right| < \infty, 
\]

\[
\sup_{m,n,k,l \in \mathbb{N}} \left| \sum_{h=0}^{i} Q_h T_i \Delta_{11}^{kl} \left( \frac{a_{mnkl}}{q_{kl}} \right) \right| < \infty, 
\]

\[
\lim_{k \to \infty} Q_k T_i \Delta_{11}^{kl} \left( \frac{a_{mnkl}}{q_{kl}} \right) = 0 \quad \text{for each } l \in \mathbb{N}, 
\]

\[
\lim_{l \to \infty} Q_k T_i \Delta_{10}^{kl} \left( \frac{a_{mnkl}}{q_{kl}} \right) = 0 \quad \text{for each } k \in \mathbb{N}. 
\]

(ii) Let $1 < s < \infty$. Then, $A \in (R^s(L_0) : M_n)$ if and only if the conditions (22)-(23) hold and

\[
\sup_{m,n,k,l \in \mathbb{N}} \left| \sum_{h=0}^{i} Q_h T_i \Delta_{11}^{kl} \left( \frac{a_{mnkl}}{q_{kl}} \right) \right|^{s'} < \infty. 
\]

Proof. (i) Let $0 < s \leq 1$ and $x = (x_{mn}) \in R^s(L_0)$. Then, there exists a sequence $y = (y_{mn}) \in L_0$. For the $(i,j)$th rectangular partial sum of the series $\sum_{k,l} a_{mnkl} x_{kl}$, we have

\[
(Ax)^{i,j}_{mn} = \sum_{k,l=0}^{i,j} a_{mnkl} x_{kl} = \sum_{k,l=0}^{i-1,j-1} s_{kl} \Delta_{11}^{kl} a_{mnkl} + \sum_{k=0}^{i-1} s_{k} \Delta_{10}^{kj} a_{mnkj}
\]

for all $m, n \in \mathbb{N}$, where $s_{mn} = \sum_{k,j=0}^{m,n} x_{ij}$. Now, using the relation (4) we derive that

\[
(Ax)^{i,j}_{mn} = \sum_{k,l=0}^{i,j} a_{mnkl} x_{kl} = \sum_{k,l=0}^{i-1,j-1} Q_k T_i \Delta_{11}^{kl} \left( \frac{a_{mnkl}}{q_{kl}} \right) y_{kl} + \sum_{k=0}^{i-1} Q_k T_i \Delta_{10}^{kj} \left( \frac{a_{mnkj}}{q_{kl}} \right) y_{kj}
\]

\[
+ \sum_{l=0}^{j-1} Q_l T_j \Delta_{10}^{jl} \left( \frac{a_{mnlj}}{q_{kl}} \right) y_{lj} + Q_j T_i \Delta_{10}^{ij} \left( \frac{a_{mnij}}{q_{lj}} \right) y_{lj} 
\]

for all $m, n, i, j \in \mathbb{N}$. Define the matrix $B_{mn} = \left( b_{mn}^{i,j} \right)$ by

\[
b_{mn}^{i,j} = \left\{ \begin{array}{ll}
Q_k T_i \Delta_{11}^{kl} \left( \frac{a_{mnkl}}{q_{kl}} \right), & 0 \leq k \leq i - 1 \quad \text{and} \quad 0 \leq l \leq j - 1 \\
Q_k T_i \Delta_{10}^{kl} \left( \frac{a_{mnkl}}{q_{kl}} \right), & 0 \leq k \leq i - 1 \quad \text{and} \quad l = j \\
Q_l T_j \Delta_{10}^{ij} \left( \frac{a_{mnij}}{q_{lj}} \right), & k = i \quad \text{and} \quad 0 \leq l \leq j - 1 \\
Q_l T_j \Delta_{10}^{ij} \left( \frac{a_{mnij}}{q_{lj}} \right), & k = i \quad \text{and} \quad l = j \\
0, & \text{otherwise}.
\end{array} \right.
\]

Therefore, (25) can be written as $(Ax)^{i,j}_{mn} = (B_{mn} y)^{i,j}$. Then, the $bp$-convergence of the rectangular partial sums $(Ax)^{i,j}_{mn}$ for all $m, n \in \mathbb{N}$ and for all $x \in R^s(L_0)$ is equivalent to the statement that $B_{mn} \in (L_0 : C_{bp})$ and
hence the conditions
\[
\sup_{k,j \in \mathbb{N}} \left| Q_k T_1 \frac{a_{mnkl}}{q_k l_l} \right| < \infty, \tag{27}
\]
\[
\sum_{k,l} \left| Q_k T_1 \Delta^{kl}_{11} \frac{a_{mnkl}}{q_k l_l} \right| < \infty, \tag{28}
\]
\[
\lim_{k \to \infty} Q_k T_1 \Delta^{kl}_{11} \left( \frac{a_{mnkl}}{q_k l_l} \right) = 0 \text{ for each } l \in \mathbb{N}, \tag{29}
\]
\[
\lim_{l \to \infty} Q_k T_1 \Delta^{kl}_{11} \left( \frac{a_{mnkl}}{q_k l_l} \right) = 0 \text{ for each } k \in \mathbb{N} \tag{30}
\]
must be satisfied for every fixed \( m, n \in \mathbb{N} \).

If we take \( bp \)-limit in the terms of the matrix \( B_{mn} = \left( b_{mn}^{i,j,k,l} \right) \) as \( i, j \to \infty \), we have
\[
bp - \lim_{i,j \to \infty} b_{mn}^{i,j,k,l} = Q_k T_1 \Delta^{kl}_{11} \left( \frac{a_{mnkl}}{q_k l_l} \right). \tag{31}
\]
Using the relation (31) we can define a four dimensional matrix \( B = (b_{mnkl}) \) by
\[
b_{mnkl} = Q_k T_1 \Delta^{kl}_{11} \left( \frac{a_{mnkl}}{q_k l_l} \right) \tag{32}
\]
for all \( m, n, k, l \in \mathbb{N} \). So, by the relations (25), (29), (30) and (31) we have
\[
bp - \lim_{i,j \to \infty} (Ax)_{mn}^{i,j} = bp - \lim_{i,j \to \infty} (By)_{mn}. \tag{33}
\]

Thus, it is seen by combining the fact "\( A = (a_{mnkl}) \in (R^q(L) : M_a) \) if and only if \( B \in (L : M_b) \)" with Part (i) of Theorem 4.1 that
\[
\sup_{m,n,k \in \mathbb{N}} \left| Q_k T_1 \Delta^{kl}_{11} \left( \frac{a_{mnkl}}{q_k l_l} \right) \right| < \infty. \tag{34}
\]

Therefore, from the conditions (27)-(33), we see that \( A = (a_{mnkl}) \in (R^q(L) : M_a) \) if and only if the conditions (20)-(23) hold.

(ii) Let \( 1 < s < \infty \). With the similar way used in the proof of Part (i), we have the \( bp \)-convergence of the rectangular partial sums \( (Ax)_{mn}^{i,j} \) for all \( m, n \in \mathbb{N} \) and for all \( x \in R^q(L) \) is equivalent to the statement that \( B_{mn} \in (L : C_{bp}) \) and hence the conditions (21)-(23) and
\[
\sum_{k,l} \left| Q_k T_1 \Delta^{kl}_{11} \left( \frac{a_{mnkl}}{q_k l_l} \right) \right|^{\prime} < \infty \tag{35}
\]
must be satisfied for every fixed \( m, n \in \mathbb{N} \). Also, by the definition of the matrix \( B_{mn} = \left( b_{mn}^{i,j,k,l} \right) \) in (26) we have the relation (32).

Thus, it is seen by combining the fact "\( A = (a_{mnkl}) \in (R^q(L) : M_a) \) if and only if \( B \in (L : M_b) \)" with Part (ii) of Theorem 4.1 that
\[
\sup_{m,n,k \in \mathbb{N}} \sum_{i,j} \left| Q_k T_1 \Delta^{kl}_{11} \left( \frac{a_{mnkl}}{q_k l_l} \right) \right|^{\prime} < \infty. \tag{36}
\]

Also, the condition (35) contains the conditions (21) and (34). Therefore, we see that \( A = (a_{mnkl}) \in (R^q(L) : M_a) \) if and only if the conditions (22)-(24) hold. This completes the proof. \( \square \)
Since Theorems 4.5 and 4.6 can be proved in a similar way to that used in the proof of Theorem 4.4, we give them without proof.

**Theorem 4.5.** Let \( A = (a_{mnkl}) \) be any four dimensional matrix. Then, the following statements hold:

(i) Let \( 0 < s \leq 1 \). Then, \( A \in \left( R^s(L_s) : C_{bp} \right) \) if and only if the conditions (20)-(23) hold and

\[ \exists (\alpha_{kl}) \in \Omega \text{ such that } bp - \lim_{m,n \to \infty} Q_T \Delta_{kl}^{ij} \left( \frac{a_{mnkl}}{q_{kl}} \right) = \alpha_{kl}. \]  

(ii) Let \( 1 < s < \infty \). Then, \( A \in \left( R^s(L_s) : C_{bp} \right) \) if and only if the conditions (22)-(24) and (36) hold.

**Theorem 4.6.** Let \( 0 < s < 1 \) and \( 1 < s_1 < \infty \). Then, \( A = (a_{mnkl}) \in \left( R^s(L_s) : L_{s_1} \right) \) if and only if the conditions (21)-(23) hold and

\[ \sup_{i,j \in \mathbb{N}} \left| \frac{Q_T a_{mnkl}}{q_{kl}} \right| < \infty, \]  

\[ \sup_{i,j \in \mathbb{N}} \left| \frac{Q_T a_{mnkl}}{q_{kl}} \right|^s < \infty. \]

**Theorem 4.7.** Let \( \lambda, \mu \) be two double sequence spaces, \( A = (a_{ijkl}) \) be any four dimensional matrix and \( B = (b_{mnij}) \) also be a four dimensional triangle matrix such that \( b_{mnij} = 0 \) if \( i > m \) and \( j > n \) for all \( m,n,i,j \in \mathbb{N} \). Then, \( A \in (\lambda : \mu_B) \) if and only if \( BA \in (\lambda : \mu) \).

**Proof.** Suppose that \( \lambda, \mu \) are two double sequence spaces, \( A = (a_{ijkl}) \) is any four dimensional matrix and \( B = (b_{mnij}) \) is also a four dimensional triangle matrix such that \( b_{mnij} = 0 \) if \( i > m \) and \( j > n \) for all \( m,n,i,j \in \mathbb{N} \). Let \( x = (x_{kl}) \in \lambda \). Then, since the equality

\[ \sum_{i=0}^{m} \sum_{j=0}^{n} b_{mnij} a_{ijkl} x_{kl} = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{mnij} a_{ijkl} x_{kl} \]  

holds for all \( m,n,r,t \in \mathbb{N} \) one can obtain by letting \( r,t \to \infty \) in (39) that \( B(Ax) = (BA)x \). Therefore, it is immediate that \( Ax \in \mu_B \) whenever \( x \in \lambda \) if and only if \( (BA)x \in \mu \) whenever \( x \in \lambda \).

This completes the proof. \( \Box \)

Now, we define the four dimensional matrices \( C = (c_{mnkl}), D = (d_{mnkl}) \) and \( E = (e_{mnkl}) \) by

\[ c_{mnkl} = \sum_{i,j=0}^{m,n} a_{ijkl}, \quad d_{mnkl} = \sum_{i,j=0}^{m,n} \frac{a_{ijkl}}{(m+1)(n+1)} \quad \text{and} \quad e_{mnkl} = \sum_{i,j=0}^{m,n} \frac{q_{kl} a_{ijkl}}{Q_m T_n} \]

for all \( m,n,k,l \in \mathbb{N} \).

One can derive several new results from Theorems 4.1-4.7.

**Corollary 4.8.** Let \( 0 < s \leq 1 \). Then, the following statements hold:

(i) \( A = (a_{mnkl}) \in (L_s : BS) \) if and only if (12) holds with \( c_{mnkl} \) instead of \( a_{mnkl} \).

(ii) \( A = (a_{mnkl}) \in (L_s : M_0) \) if and only if (12) holds with \( d_{mnkl} \) instead of \( a_{mnkl} \).

(iii) \( A = (a_{mnkl}) \in (L_s : R^s(M_0)) \) if and only if (12) holds with \( e_{mnkl} \) instead of \( a_{mnkl} \).

(iv) \( A = (a_{mnkl}) \in (L_s : CS_{bp}) \) if and only if (12) and (16) hold with \( c_{mnkl} \) instead of \( a_{mnkl} \).

(v) \( A = (a_{mnkl}) \in (L_s : C_{bp}) \) if and only if (12) and (16) hold with \( d_{mnkl} \) instead of \( a_{mnkl} \).

(vi) \( A = (a_{mnkl}) \in (L_s : R^s(C_{bp})) \) if and only if (12) and (16) hold with \( e_{mnkl} \) instead of \( a_{mnkl} \).

**Corollary 4.9.** Let \( 1 < s < \infty \). Then, the following statements hold:

(i) \( A = (a_{mnkl}) \in (L_s : BS) \) if and only if (13) holds with \( c_{mnkl} \) instead of \( a_{mnkl} \).
Corollary 4.10. Let \( A = (a_{mn}) \in (\mathcal{L}_c : \widetilde{M}_d) \) if and only if (13) holds with \( d_{mn} \) instead of \( a_{mn} \).

Corollary 4.11. Let \( A = (a_{mn}) \in (\mathcal{L}_c : \widetilde{R}_d) \) if and only if (13) holds with \( e_{mn} \) instead of \( a_{mn} \).

References

Acknowledgements

We have a lot benefited from the excellent review, so we are indebted to three anonymous referees for helpful advice and constructive criticism on the earlier version of this paper. Also, we would like to thank Hüsamettin Çapan for his careful reading and valuable suggestions on the revised version of this paper which improved the presentation and readability.

References