An Application of Power Increasing Sequences to Infinite Series and Fourier Series

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Abstract. In this paper, we proved a known theorem under more weaker conditions dealing with absolute Riesz summability of infinite series involving a quasi-$\sigma$-power increasing sequence. And we applied it to the trigonometric Fourier series.

1. Introduction

A positive sequence $(b_n)$ is said to be an almost increasing sequence if there exists a positive increasing sequence $c_n$ and two positive constants $M$ and $N$ such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). A positive sequence $(X_n)$ is said to be quasi-$\sigma$-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^\sigma X_n \geq m^\sigma X_m$ for all $n \geq m \geq 1$. Every almost increasing sequence is a quasi-$\sigma$-power increasing sequence for any non-negative $\sigma$, but the converse is not true for $\sigma > 0$ (see [10]). For any sequence $(\lambda_n)$ we write that $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$. The sequence $(\lambda_n)$ is said to be of bounded variation, denoted by $(\lambda_n) \in BV$, if $\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty$. Let $\sum a_n$ be a given infinite series with the partial sums $(s_n)$. By $u^\alpha_n$ and $t^\alpha_n$ we denote the $n$th Cesàro means of order $\alpha$, with $\alpha > -1$, of the sequences $(s_n)$ and $(na_n)$, respectively, that is (see [5])

$$u^\alpha_n = \frac{1}{A^\alpha_n} \sum_{v=0}^{n} A^\alpha_{n-v}s_v \quad \text{and} \quad t^\alpha_n = \frac{1}{A^\alpha_n} \sum_{v=0}^{n} A^\alpha_{n-v}va_v, \quad (t^1_n = t_n)$$

(1)

where

$$A^\alpha_n = \frac{(\alpha + 1)(\alpha + 2)\ldots(\alpha + n)}{n!} = O(n^\alpha), \quad A^\alpha_0 = 0 \quad \text{for} \quad n > 0.$$ 

(2)

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [7], [9])

$$\sum_{n=1}^{\infty} n^{k-1} |u^\alpha_n - u^\alpha_{n-1}|^k = \sum_{n=1}^{\infty} \frac{1}{n} |u^\alpha_n|^k < \infty.$$ 

(3)

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If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability. Let $(p_n)$ be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \quad (4)$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (5)$$

defines the sequence $(w_n)$ of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence $(s_n)$, generated by the sequence of coefficients $(p_n)$ (see [8]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of $n$ (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$, (resp. $|\bar{N}, p_n|$) summability.

2. Known Result

The following theorem is known dealing with the $|\bar{N}, p_n|_k$ summability factors of infinite series.

**Theorem 2.1 ([11]).** Let $(X_n)$ be an almost increasing sequence. If the sequences $(X_n)$, $(\lambda_n)$, and $(p_n)$ satisfy the conditions

$$\lambda_m X_m = O(1) \text{ as } m \to \infty, \quad (6)$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) \text{ as } m \to \infty, \quad (7)$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \text{ as } m \to \infty, \quad (8)$$

$$\sum_{n=1}^{m} \frac{p_n |s_n|^k}{P_n} = O(X_m) \text{ as } m \to \infty, \quad (9)$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m) \text{ as } m \to \infty, \quad (10)$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$. It should be remarked that Theorem A also implies the known result of Bor dealing with the absolute $|\bar{N}, p_n|_k$ summability factors of infinite series (see [3]).

3. Main Result

The aim of this paper is to prove Theorem 2.1 under more weaker conditions. Now we shall prove the following theorem.
**Theorem 3.1** Let \((X_n)\) be a quasi-\(\sigma\)-power increasing sequence. If the sequences \((X_n), (\lambda_n),\) and \((p_n)\) satisfy the conditions (6), (7), (8), and

\[
\sum_{n=1}^{m} \frac{p_n}{p_n X_n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty,
\]

(11)

\[
\sum_{n=1}^{m} \frac{|t_n|^k}{n X_n} = O(X_m) \quad \text{as} \quad m \to \infty,
\]

(12)

then the series \(\sum a_n \lambda_n\) is summable \(|\tilde{N}, p_n|, k \geq 1\).

**Remark 3.2** It should be noted that condition (11) is reduced to the condition (9), when \(k=1\). When \(k > 1\), condition (11) is weaker than condition (9) but the converse is not true. As in [12] we can show that if (9) is satisfied, then we get that

\[
\sum_{n=1}^{m} \frac{p_n}{p_n X_n |t_n|^k} = O(\frac{1}{X_{m}}) \sum_{n=1}^{m} \frac{p_n}{p_n} |t_n|^k = O(X_m).
\]

If (11) is satisfied, then for \(k > 1\) we obtain that

\[
\sum_{n=1}^{m} \frac{p_n}{p_n} |t_n|^k = \sum_{n=1}^{m} X_n^{k-1} \frac{p_n}{p_n} |t_n|^k = O(X_m^{k-1}) \sum_{n=1}^{m} \frac{p_n}{p_n} |t_n|^k = O(X_m^{k}) \neq O(X_m).
\]

The similar argument is also valid for the conditions (12) and (10). Also it should be noted that if we take \((X_n)\) as an almost increasing sequence, then we get some new results.

We need the following lemma for the proof of our theorem.

**Lemma 3.3 ([4])** Under the conditions of Theorem 3.1, we have that

\[
\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty,
\]

(13)

\[nX_n|\Delta \lambda_n| = O(1) \quad \text{as} \quad n \to \infty.
\]

(14)

4. **Proof of Theorem 3.1** Let \((T_n)\) be the sequence of \((\tilde{N}, p_n)\) mean of the series \(\sum a_n \lambda_n\). Then, by definition, we have

\[
T_n = \frac{1}{p_n} \sum_{r=0}^{n} p_r \sum_{v=0}^{\infty} a_r \lambda_r = \frac{1}{p_n} \sum_{r=0}^{n} (P_n - P_{n-1}) a_r \lambda_r.
\]

(15)

Then, for \(n \geq 1\), we get

\[
T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n} \frac{P_{v-1} \lambda_v}{v} v a_v.
\]

(16)

Applying Abel’s transformation to the right-hand side of (16), we have

\[
T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{r=1}^{n-1} \Delta \left( \frac{P_{r-1} \lambda_{v}}{v} \right) \sum_{v=0}^{r} v a_v + \frac{p_n \lambda_n}{nP_n} \sum_{r=1}^{n} v a_v
\]

\[= \frac{(n + 1)p_n \lambda_n}{nP_n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v-1} \lambda_v}{v} v + 1 \left( \frac{1}{v} \right)
\]

\[+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_{v-1} \Delta \lambda_v \left( \frac{1}{v} \right) + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_{v-1} \lambda_{v+1} \left( \frac{1}{v} \right)
\]

\[= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\]
To complete the proof of Theorem 3.1, by Minkowski’s inequality, it is sufficient to show that
\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n^k} \right)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4.
\]

Firstly, we have that
\[
\sum_{n=1}^{m} \left( \frac{p_n}{p_n^k} \right)^{k-1} |T_{n,1}|^k = O(1) \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n p_n P_n \lambda_n|^k = O(1) \sum_{n=1}^{m} |\lambda_n| p_n P_n X_n^{k-1}
\]
\[
= O(1) \sum_{n=1}^{m} |\lambda_n| \sum_{n=1}^{m} \sum_{n=1}^{m} p_n \frac{|t_n|^k}{p_n X_n^{k-1}} + O(1) |\lambda_m| \sum_{n=1}^{m} |\lambda_n| p_n \frac{|t_n|^k}{p_n X_n^{k-1}}
\]
\[
= O(1) \sum_{n=1}^{m} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \to \infty,
\]
by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. Also, as in \( T_{n,1} \), we have that
\[
\sum_{n=1}^{m+1} \left( \frac{p_n}{p_n^k} \right)^{k-1} |T_{n,2}|^k = O(1) \sum_{n=1}^{m+1} p_n \frac{p_n}{P_n^k P_n^{k-1}} \left( \sum_{n=1}^{m+1} p_n \frac{|t_n|^k}{P_n X_n^{k-1}} \right) \times \left( \frac{1}{P_n^{k-1}} \sum_{n=1}^{m+1} p_n \right)^{k-1}
\]
\[
= O(1) \sum_{n=1}^{m} |\lambda_n| \sum_{n=1}^{m} \sum_{n=1}^{m} p_n \frac{|t_n|^k}{P_n X_n^{k-1}} + O(1) |\Delta \lambda_m| X_m = O(1) \quad \text{as } m \to \infty.
\]
Again, by using (8), we get that
\[
\sum_{n=1}^{m+1} \left( \frac{p_n}{p_n^k} \right)^{k-1} |T_{n,3}|^k = O(1) \sum_{n=1}^{m+1} p_n \frac{p_n}{P_n^{k-1}} \left( \sum_{n=1}^{m+1} p_n \frac{|t_n|^k}{P_n X_n^{k-1}} \right)^k
\]
\[
= O(1) \sum_{n=1}^{m+1} p_n \frac{p_n}{P_n^{k-1}} \left( \sum_{n=1}^{m+1} \frac{p_n}{P_n X_n^{k-1}} |\Delta \lambda_n| X_n \right)^k
\]
\[
= O(1) \sum_{n=1}^{m+1} p_n \frac{p_n}{P_n^{k-1}} \left( \sum_{n=1}^{m+1} \frac{p_n}{P_n X_n^{k-1}} \frac{(v|\Delta \lambda_n|)^{k-1} v|\Delta \lambda_n| p_n |t_n|^k}{P_n X_n^{k-1}} \sum_{n=1}^{m+1} p_n \frac{p_n}{P_n^{k-1}} \right)^k
\]
\[
= \left( \sum_{n=1}^{m+1} p_n \frac{p_n}{P_n X_n^{k-1}} \frac{(v|\Delta \lambda_n|)^{k-1} v|\Delta \lambda_n| p_n |t_n|^k}{P_n X_n^{k-1}} \sum_{n=1}^{m+1} p_n \frac{p_n}{P_n^{k-1}} \right)^k
\]
\[
= O(1) \sum_{n=1}^{m+1} \Delta (v|\Delta \lambda_n|)^{k-1} v|\Delta \lambda_n| \|t_n|^k \sum_{n=1}^{m+1} p_n \frac{p_n}{P_n X_n^{k-1}} + O(1) m|\Delta \lambda_m| X_m = O(1) \quad \text{as } m \to \infty.
\]
by virtue of the hypotheses of Theorem 3.1 and and Lemma 3.3. Finally, by using (8), we have that
\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_k} \right)^{k-1} |T_{n,k}|^k \leq \sum_{n=2}^{m+1} \frac{P_n}{P_k} \sum_{v=1}^{n-1} \frac{1}{P_k} |\lambda_{v+1}|^k |t_v|^k
\]
\[
= O(1) \sum_{n=2}^{m+1} \frac{P_n}{P_k} \sum_{v=1}^{n-1} |\lambda_{v+1}|^k |t_v|^k \times \left( \frac{1}{P_k} \sum_{n=1}^{n-1} \frac{P_n}{P_k} \right)^{k-1}
\]
\[
= O(1) \sum_{v=1}^{m} |\lambda_{v+1}|^{k-1} |t_v|^k \sum_{n=v+1}^{m+1} \frac{P_n}{P_k} P_{n-1}
\]
\[
= O(1) \sum_{v=1}^{m} |\lambda_{v+1}|^{k-1} |t_v|^k = O(1) \quad \text{as} \quad m \to \infty.
\]

This completes the proof of Theorem 3.1.

5. Let \( f(t) \) be a periodic function with period \( 2\pi \) and integrable \((L)\) over \((-\pi, \pi)\). Write
\[
f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} C_n(x),
\]
\[
\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t)\}, \\
\phi_n(t) = \frac{1}{2} \int_{t}^{t+\pi} (t-u)^{n-1} \phi(u) du, \quad (a > 0).
\]
It is well known that if \( \phi(t) \in B^{BV}(0, \pi) \), then \( t_n(x) = O(1) \), where \( t_n(x) \) is the \((C, 1)\) mean of the sequence \((nC_n(x))\) (see [6]). Using this fact, we get the following main result dealing with the trigonometric Fourier series.

**Theorem 5.1** Let \((X_n)\) be a quasi-\(\sigma\)-power increasing sequence. If \( \phi_1(t) \in B^{BV}(0, \pi) \), and the sequences \((p_n)\), \((\lambda_n)\), and \((X_n)\) satisfy the conditions of Theorem 3.1, then the series \( \sum C_n(x) \lambda_n \) is summable \([N, p_n]_k\), \( k \geq 1 \).

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**References**