Positive Solutions for a Fractional $p$-Laplacian Boundary Value Problem

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Abstract. In this paper we study the existence of positive solutions for the fractional $p$-Laplacian boundary value problem

\[
\begin{aligned}
D_0^\beta (\phi_p(D_0^\alpha u(t))) &= f(t, u(t)), \quad t \in (0, 1), \\
u(0) &= u'(0) = 0, \quad u'(1) = au'(\xi), \quad D_0^\alpha u(0) = 0, \quad D_0^\alpha u(1) = bD_0^\alpha u(\eta),
\end{aligned}
\]

where \(2 < \alpha \leq 3, 1 < \beta \leq 2, D_0^\alpha, D_0^\beta\) are the standard Riemann-Liouville fractional derivatives, \(\phi_p(s) = |sp^{-2}s, p > 1, \phi_p^{-1} = \phi_p, 1/p + 1/q = 1, 0 < \xi, \eta < 1, 0 \leq a < \xi^{2-\alpha}, 0 \leq b < \eta^{2-\alpha}\) and \(f \in C([0, 1] \times [0, +\infty), [0, +\infty))\).

Using the monotone iterative method and the fixed point index theory in cones, we establish two new existence results when the nonlinearity \(f\) is allowed to grow \((p - 1)\)-sublinearly and \((p - 1)\)-superlinearly at infinity.

1. Introduction

In this paper we discuss the existence of positive solutions for the fractional $p$-Laplacian boundary value problem

\[
\begin{aligned}
D_0^\beta (\phi_p(D_0^\alpha u(t))) &= f(t, u(t)), \quad t \in (0, 1), \\
u(0) &= u'(0) = 0, \quad u'(1) = au'(\xi), \quad D_0^\alpha u(0) = 0, \quad D_0^\alpha u(1) = bD_0^\alpha u(\eta),
\end{aligned}
\]

where \(2 < \alpha \leq 3, 1 < \beta \leq 2, D_0^\alpha, D_0^\beta\) are the standard Riemann-Liouville fractional derivatives, \(\phi_p(s) = |sp^{-2}s, p > 1, \phi_p^{-1} = \phi_p, 1/p + 1/q = 1, 0 < \xi, \eta < 1, 0 \leq a < \xi^{2-\alpha}, 0 \leq b < \eta^{2-\alpha}\) and \(f \in C([0, 1] \times [0, +\infty), [0, +\infty))\).
Fractional differential equations arise naturally for example in physics, chemistry, diffusion and transport theory, chaos and turbulence, viscoelastic mechanics and non-newtonian fluid mechanics; for more details on fractional applications, we refer the reader to [1–3]. There are many papers in the literature on the existence of solutions for fractional boundary value problems; see for example [4–12] and the references therein. In [4], the authors investigated the existence of positive solutions for the fractional differential equation with integral boundary conditions

\[
\begin{aligned}
D_0^\alpha u(t) + q(t) f(t, u(t)) &= 0, \\ u(0) = u'(0) = 0, \\ u(1) = \int_0^1 g(s) u(s) ds,
\end{aligned}
\]

and obtained an existence result if the following condition is satisfied:

\((H_f)\) there exist \(a, \Lambda > 0\) such that \(f(t, x) \leq f(t, y) \leq \Lambda a\), for \(0 \leq x \leq y \leq a, t \in [0, 1]\).

Note for multi-point boundary value problems the Green’s functions may be complicated. Bai [5] considered the fractional three point boundary value problem

\[
\begin{aligned}
D_0^\alpha u(t) + f(t, u(t)) &= 0, 0 < t < 1, \\
u(0) = 0, \beta u(\eta) = u(1),
\end{aligned}
\]

where \(\alpha \in (1, 2], \beta \eta^{\alpha-1}, \eta \in (0, 1).\) The Green’s function is

\[
G(t, s) = \begin{cases}
\frac{\Gamma(\alpha+1)}{\Gamma(1)} \left[ \frac{t(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{1-\beta s^{\alpha-1}} \right], & 0 \leq s \leq t \leq 1, s \leq \eta, \\
\frac{\Gamma(\alpha+1)}{\Gamma(1)} \left[ \frac{(1-t)^{\alpha-1} - (1-t-s)^{\alpha-1}}{1-\beta s^{\alpha-1}} \right], & 0 < \eta \leq s \leq t \leq 1, \\
\frac{\Gamma(\alpha+1)}{\Gamma(1)} \left[ \frac{(1-t)^{\alpha-1} - (1-t-s)^{\alpha-1}}{1-\beta s^{\alpha-1}} \right], & 0 \leq t \leq s \leq \eta \leq 1, \\
\frac{\Gamma(\alpha+1)}{\Gamma(1)} \left[ \frac{(1-t)^{\alpha-1}}{1-\beta s^{\alpha-1}} \right], & 0 \leq t \leq s \leq 1, \eta \leq s.
\end{cases}
\]

Note if \(\beta = 0\), then (2) reduces to the problem

\[
\begin{aligned}
D_0^\alpha u(t) + f(t, u(t)) &= 0, 0 < t < 1, \\
u(0) = u(1) = 0.
\end{aligned}
\]

The Green’s function is

\[
g(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases}
\left[ t(1-s) \right]^{\alpha-1} - \left( t-s \right)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\
\left[ t(1-s) \right]^{\alpha-1}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

Now if the three point problem (2) is considered as a perturbation of the two point problem (4), we can use (5) to obtain (3), i.e.,

\[
G(t, s) = g(t, s) + \frac{\beta \eta^{\alpha-1}}{1-\beta \eta^{\alpha-1}} g(\eta, s).
\]

This simple idea motivates our study in Section 2.

In this paper we first obtain an existence result with \(f\) growing \((p-1)\)-sublinearly at infinity. Moreover, we establish an iterative sequence for approximating the solution. Next, using the fixed point index theory, we obtain an existence result with \(f\) growing \((p-1)\)-superlinearly at infinity.
2. Preliminaries

For convenience, in this section we present some basic definitions and notations from fractional calculus.

Definition 2.1 Let \( f : (0, +\infty) \to (-\infty, +\infty) \) be a continuous function. Then the Riemann-Liouville fractional derivative of order \( \alpha > 0 \) is given by

\[
D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t \frac{f(s)}{(t-s)^{n+1}} ds,
\]

where \( n = [\alpha] + 1 \), \([\alpha]\) denotes the integer part of the number \( \alpha \), provided that the right side is pointwise defined on \((0, +\infty)\).

Definition 2.2 Let \( f : (0, +\infty) \to (-\infty, +\infty) \) be a function. Then the Riemann-Liouville fractional integral of order \( \alpha > 0 \) is given by

\[
I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,
\]

provided that the right side is pointwise defined on \((0, +\infty)\).

From the definition of the Riemann-Liouville derivative one obtains the following result.

Lemma 2.1 Let \( u \in C(0,1) \cap L(0,1) \). Then the fractional differential equation \( D_0^\alpha u(t) = 0 \) has a unique solution

\[
u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_N t^{\alpha-N}, \quad c_i \in \mathbb{R}, \quad i = 1, 2, \ldots, N,
\]

where \( N \) is the smallest integer greater than or equal to \( \alpha \).

Lemma 2.2 Assume that \( u \in C(0,1) \cap L(0,1) \) with a fractional derivative of order \( \alpha > 0 \) that belongs to \( C(0,1) \cap L(0,1) \). Then

\[
I_0^\alpha D_0^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_N t^{\alpha-N}, \quad c_i \in \mathbb{R}, \quad i = 1, 2, \ldots, N,
\]

where \( N \) is the smallest integer greater than or equal to \( \alpha \).

Lemma 2.3 Let \( \alpha, \xi, a \) be as in (1) and \( y \in C[0,1] \). Then solving

\[
\begin{cases}
D_0^\alpha u(t) + y(t) = 0, t \in (0,1), \\
u(0) = u'(0) = 0, u'(1) = a u(\xi),
\end{cases}
\] (6)

is equivalent to solving

\[
u(t) = \int_0^t G(t,s) y(s) ds,
\]

where

\[
G(t,s) = \begin{cases}
g_1(t,s) + \frac{at^{\alpha-1}}{1-a\xi^{\alpha-1}} g_2(\xi,s),
\end{cases}
\]

\[
g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases}
t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\
t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \leq t \leq s \leq 1,
\end{cases}
\]

\[
g_2(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases}
t^{\alpha-2}(1-s)^{\alpha-2} - (t-s)^{\alpha-2}, & 0 \leq s \leq t \leq 1, \\
t^{\alpha-2}(1-s)^{\alpha-2}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

Proof. It is enough to consider the case when \( u \) is a solution of (2.1). From Definition 2.2 and Lemma 2.2 we have

\[
u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds,
\]
for some constants \( c_i \in \mathbb{R}, i = 1, 2, 3 \).
From \( u(0) = u'(0) = 0 \) we have \( c_2 = c_3 = 0 \). Hence

\[
 u(t) = c_1 t^{\alpha - 1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds,
\]

and

\[
 u'(t) = c_1 (\alpha - 1) t^{\alpha - 2} - (\alpha - 1) \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds.
\]

Consequently, we obtain

\[
 u'(1) = c_1 (\alpha - 1) - (\alpha - 1) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds,
\]

and

\[
 u' (\xi) = c_1 (\alpha - 1) \xi^{\alpha - 2} - (\alpha - 1) \int_0^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds.
\]

Then \( u'(1) = au'(\xi) \) implies that

\[
 c_1 = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds = c_1 a \xi^{\alpha - 2} - a \int_0^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds,
\]

and

\[
 c_1 = \frac{1}{1 - a \xi^{\alpha - 2}} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds - \frac{a}{1 - a \xi^{\alpha - 2}} \int_0^\xi \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds.
\]

As a result,

\[
 u(t) = \frac{1}{1 - a \xi^{\alpha - 2}} \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds - \frac{at^{\alpha-1}}{1 - a \xi^{\alpha - 2}} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{at^{\alpha-1}}{1 - a \xi^{\alpha - 2}} \int_0^1 \frac{\xi^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds
\]

\[
 = \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{at^{\alpha-1}}{1 - a \xi^{\alpha - 2}} \int_0^1 \frac{\xi^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds
\]

\[
 = \int_0^1 G(t,s)y(s)ds.
\]

This completes the proof. \( \square \)

**Lemma 2.4** Let \( \alpha, \beta, \xi, \eta, a, b \) be as in (1) and \( y \in C[0,1] \). Then solving

\[
\begin{align*}
    \begin{cases}
        D_0^\beta (\phi_\beta (D_0^\alpha u(t))) = y(t), t \in (0, 1), \\
        u(0) = u'(0) = 0, u'(1) = au'(\xi), D_0^\alpha u(0) = 0, D_0^\alpha u(1) = b D_0^\alpha u(\eta),
    \end{cases}
\end{align*}
\]

(8)

is equivalent to solving

\[
 u(t) = \int_0^1 G(t,s) \phi_\beta \left( \int_0^1 H(s, \tau) y(\tau) d\tau \right) ds,
\]
Therefore, 

$$H(t, s) = h_1(t, s) + \frac{b^{\beta-1} t^{\beta-1}}{1 - b^{\beta-1} \eta^{\beta-1}} h_1(\eta, s),$$

$$h_1(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta-1}(1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$ (9)

**Proof.** It is enough to consider the case when \( u \) is a solution of (2.3). From Lemma 2.2 we have

$$I_{0^+}^\beta D_{0^+}^\alpha (\phi_p(D_{0+}^\beta u(t))) = \phi_p(D_{0+}^\beta u(t)) + c_1 b^{\beta-1} + c_2 b^{\beta-2},$$

for some constants \( c_i \in \mathbb{R}, i = 1, 2 \). In view of (8), we obtain

$$I_{0^+}^\beta D_{0^+}^\alpha (\phi_p(D_{0+}^\beta u(t))) = I_{0^+}^\beta y(t).$$

Also we find

$$\phi_p(D_{0+}^\alpha u(t)) = I_{0^+}^\alpha y(t) + c_1 b^{\beta-1} + c_2 b^{\beta-2}$$

$$= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds + c_1 b^{\beta-1} + c_2 b^{\beta-2}.$$

Then \( D_{0+}^\alpha u(0) = 0 \) implies that \( c_2 = 0 \). Hence,

$$\phi_p(D_{0+}^\alpha u(1)) = \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds + c_1,$$

and

$$\phi_p(D_{0+}^\alpha u(\eta)) = \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds + c_1 \eta^{\beta-1}.$$

Consequently, \( D_{0+}^\alpha u(1) = bD_{0+}^\alpha u(\eta) \) implies that

$$\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds + c_1 = b b^{\beta-1} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds + c_1 b^{\beta-1} \eta^{\beta-1},$$

and

$$c_1 = \frac{b b^{\beta-1}}{1 - b^{\beta-1} \eta^{\beta-1}} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds - \frac{1}{1 - b^{\beta-1} \eta^{\beta-1}} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds.$$

Therefore,

$$\phi_p(D_{0+}^\alpha u(t)) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds + \frac{b b^{\beta-1} \eta^{\beta-1}}{1 - b^{\beta-1} \eta^{\beta-1}} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds$$

$$- \frac{1}{1 - b^{\beta-1} \eta^{\beta-1}} \int_0^1 \frac{b^{\beta-1} (1-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds$$

$$= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds - \int_0^1 \frac{b^{\beta-1} (1-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds + \frac{b b^{\beta-1} \eta^{\beta-1}}{1 - b^{\beta-1} \eta^{\beta-1}} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds$$

$$- \frac{b b^{\beta-1} \eta^{\beta-1}}{1 - b^{\beta-1} \eta^{\beta-1}} \int_0^1 \frac{b^{\beta-1} (1-s)^{\beta-1}}{\Gamma(\beta)} y(s) ds$$

$$= - \int_0^1 H(t, s) y(s) ds.$$
Also we have
\[ D_0^\alpha u(t) + \phi(t) \left( \int_0^1 H(t, s)y(s)ds \right) = 0. \]

Note Lemma 2.3 and the boundary conditions \( u(0) = u'(0) = 0, u'(1) = au'(\xi) \), so we have
\[ u(t) = \int_0^1 G(t, s)\phi(t) \left( \int_0^1 H(s, \tau)y(\tau)d\tau \right) ds. \]

This completes the proof. □

Lemma 2.5 The functions \( G, H \) have the following properties:
(i) \( G, H \in C([0, 1] \times [0, 1], [0, +\infty)) \) and \( G(t, s), H(t, s) > 0 \) for \( t, s \in (0, 1) \),
(ii) \( G(t, s) \leq \delta_1 t^{\alpha-1} \) for \( t, s \in [0, 1] \), where \( \delta_1 := \frac{1}{\Gamma(\alpha)} \left[ 1 + \frac{a\xi^{\alpha-3}}{1-a\xi^{\alpha-1}} \right] > 0. \)
(iii) \( \delta_2 t^{\alpha-1}s(1-s)^{\alpha-2} \leq G(t, s) \leq \delta_1 s(1-s)^{\alpha-2} \), for \( t, s \in [0, 1] \), where \( \delta_2 := \frac{a(\alpha-2)\xi^{\alpha-1}(1-\xi)}{\Gamma(\alpha)(1-a\xi^{\alpha-1})} \).

Proof. From [7-10] we have \( g_1, g_2, h_1 \in C([0, 1] \times [0, 1], [0, +\infty)) \) and \( g_1(t, s), h_1(t, s) > 0 \) for \( t, s \in (0, 1) \), so \( G, H \) have these properties.

From [10, Lemma 4] we have
\[ g_1(t, s) \leq \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-2}, \quad g_1(t, s) \leq \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, \quad \text{for } t, s \in [0, 1], \]
and
\[ \frac{(\alpha-2)t^{\alpha-3}(1-t)s(1-s)^{\alpha-2}}{\Gamma(\alpha)} \leq g_2(t, s) \leq \frac{1}{\Gamma(\alpha)} t^{\alpha-3}s(1-s)^{\alpha-2} \leq \frac{1}{\Gamma(\alpha)} t^{\alpha-3}, \quad \text{for } t, s \in [0, 1]. \]

Consequently,
\[ G(t, s) = g_1(t, s) + \frac{at^{\alpha-1}}{1-a\xi^{\alpha-1}} g_2(\xi, s) \leq \frac{1}{\Gamma(\alpha)} \left[ 1 + \frac{a\xi^{\alpha-3}}{1-a\xi^{\alpha-1}} \right] t^{\alpha-1}, \]
\[ G(t, s) = g_1(t, s) + \frac{at^{\alpha-1}}{1-a\xi^{\alpha-1}} g_2(\xi, s) \leq \frac{1}{\Gamma(\alpha)} \left[ 1 + \frac{a\xi^{\alpha-3}}{1-a\xi^{\alpha-1}} \right] s(1-s)^{\alpha-2}, \]
\[ G(t, s) = g_1(t, s) + \frac{at^{\alpha-1}}{1-a\xi^{\alpha-1}} g_2(\xi, s) \geq \frac{a(\alpha-2)\xi^{\alpha-1}(1-\xi)}{\Gamma(\alpha)(1-a\xi^{\alpha-1})} t^{\alpha-1}s(1-s)^{\alpha-2}. \]

This completes the proof. □

Let \( E := C([0, 1], \mathbb{R}) \), \( ||u|| := \max_{t \in [0, 1]} ||u(t)|| \), \( P := \{ u \in E : u(t) \geq 0, \forall t \in [0, 1] \} \). Then \( (E, || \cdot ||) \) is a real Banach space and \( P \) is a cone on \( E \). We let \( \rho_p := \{ u \in E : ||u|| < p \} \) for \( p > 0 \) in the sequel. Define \( A : P \rightarrow P \) by
\[ (Au)(t) = \int_0^1 G(t, s) \left( \int_0^1 H(s, \tau)f(\tau, u(\tau))d\tau \right)^\frac{1}{p} ds. \]

Then, by Lemma 2.4 the existence of solutions for (1) is equivalent to the existence of fixed points for the operator \( A \). Furthermore, in view of the continuity \( G, H \) and \( f \), we can use the Ascoli-Arzela theorem to show that \( A \) is a completely continuous operator.

Lemma 2.6 Let \( P_0 := \{ u \in P : \min_{t \in [0, 1]} u(t) \geq \delta_1 \theta_1 ||u||, \text{ where } 0 < \theta_1 < \theta_2 \leq 1 \}. \) Then \( A(P) \subset P_0 \).

Proof. For any \( u \in P \), from (iii) of Lemma 2.5 we have
\[ (Au)(t) \leq \delta_1 \int_0^1 s(1-s)^{\alpha-2} \left( \int_0^1 H(s, \tau)f(\tau, u(\tau))d\tau \right)^\frac{1}{p} ds. \]
Also for \( t \in [\theta_1, \theta_2] \), we obtain
\[
(Au)(t) = \int_0^t G(t, s) \left( \int_0^1 H(s, \tau)f(\tau, u(\tau))d\tau \right) \frac{1}{\delta_1} ds \\
\geq \int_0^1 \delta_2 \theta_1^{-1} s(1-s)^{a-2} \left( \int_0^1 H(s, \tau)f(\tau, u(\tau))d\tau \right) \frac{1}{\delta_1} ds \\
\geq \delta_2 \theta_1^{-1} \int_0^1 s(1-s)^{a-2} \left( \int_0^1 H(s, \tau)f(\tau, u(\tau))d\tau \right) \frac{1}{\delta_1} ds.
\]
Consequently,
\[
(Au)(t) \geq \frac{\delta_2 \theta_1^{-1}}{\delta_1} \delta_1 \int_0^1 s(1-s)^{a-2} \left( \int_0^1 H(s, \tau)f(\tau, u(\tau))d\tau \right) \frac{1}{\delta_1} ds \geq \frac{\delta_2 \theta_1^{-1}}{\delta_1} \|Au\|.
\]
This completes the proof. \( \square \)

**Lemma 2.7** (see [13, Lemma 2.6]) Let \( \theta > 0 \) and \( \varphi \in P \). Then
\[
\left( \int_0^\theta \varphi(t) dt \right)^\theta \leq \int_0^\theta \left( \varphi(t) \right)^\theta dt, \ \forall \theta \geq 1, \quad \left( \int_0^\theta \varphi(t) dt \right)^\theta \geq \int_0^\theta \left( \varphi(t) \right)^\theta dt, \ \forall 0 < \theta \leq 1.
\]

**Lemma 2.8** (see [14]) Let \( R > 0 \) and \( A : \overline{B}_R \cap P \to P \) a continuous compact operator. If there exists \( u_0 \in P \setminus \{0\} \) such that \( u - Au \neq \mu u_0 \) for all \( \mu \geq 0 \) and \( u \in \partial B_R \cap P \), then \( i(A, B_R \cap P, P) = 0 \), where \( i \) denotes the fixed point index on \( P \).

**Lemma 2.9** (see [14]) Let \( r > 0 \) and \( A : \overline{B}_r \cap P \to P \) a continuous compact operator. If \( \|Au\| \leq \|u\| \) and \( Au \neq u \) for \( u \in \partial B_r \cap P \), then \( i(A, B_r \cap P, P) = 1 \).

Let \( p_* = \min(p - 1, 1) \), \( p' = \max(p - 1, 1) \), \( \gamma(t) = t^{a-1} \) for \( t \in [0, 1] \), and \( t_0 \in (0, 1) \) is a given point. For convenience, we put
\[
\kappa_1 := 2^{p'-1} \int_0^r \int_0^1 H^{p'}(s, \tau)\gamma p'(\tau)d\tau ds, \quad \kappa_2 := 2^{p'-1} \int_0^1 \int_0^1 H^{p'}(s, \tau)d\tau ds.
\]
\[
\lambda_1 := \frac{1}{\delta_1 \sqrt{\kappa_1}}, \quad \lambda_2 = \sqrt{\frac{2}{\int_0^1 G_p'(t_0, s) \int_0^1 H^p(s, \tau)d\tau ds}} \delta_1, \quad \lambda_3 := \frac{1}{\left( \delta_1 \int_0^1 s(1-s)^{a-2} \left( \int_0^1 H(s, \tau)d\tau \right) \frac{1}{\delta_1} ds \right)^{p'-1}}.
\]

We now list our hypotheses:
1. \( f(t, u) \in C([0, 1] \times [0, +\infty), [0, +\infty)) \).
2. \( f(t, u) \) is nondecreasing with respect to \( u \) and \( f(t, 0) \neq 0 \) for \( t \in [0, 1] \).
3. \( \limsup_{u \to +\infty} \frac{f(t, u)}{u} < \lambda_1^{p'-1} \) uniformly on \( t \in [0, 1] \).
4. \( \liminf_{u \to +\infty} \frac{f(t, u)}{u} > \lambda_1^{p'-1} \) uniformly on \( t \in [\theta_1, \theta_2] \).
5. \( \exists \zeta > 0 \) such that \( f(t, u) \leq \phi_p(\zeta) \lambda_3, \ \forall 0 \leq u \leq \zeta, \ t \in [0, 1] \).

**Example 2.10** (1) Let
\[
f(t, u) = e^t + \sum_{i=1}^n m_i u^{\alpha_i - 1} \text{ for } t \in [0, 1] \text{ and } u \in \mathbb{R}^+,
\]
where \( m_1 \in (0, \lambda_1^{p-1}) \), \( m_i \geq 0 \) for \( i = 2, 3, \ldots, n \).

Let \( p = 2 \), \( a = 2.5 \), \( \beta = 1.5 \), \( \xi = 0.5 \), \( a = 1 \) and \( b = 0 \). Note,

\[
\delta_1 = \frac{4 \sqrt{2} + 3}{3 \sqrt{\pi} \sqrt{2} - 1}, \quad \kappa_1 = \int_0^1 \int_0^1 H(s, \tau) f(\tau) \, d\tau \, ds = \frac{5 \sqrt{\pi}}{96},
\]

and \( \lambda_1 \approx 4.5 \). Let \( m_1 \in (0, 4.5) \). Note (H1)-(H3) hold.

(2) Let \( \zeta = 1 \). Then \( \phi_p(\zeta) = 1 \). Let

\[
f(t, u) = \sum_{i=1}^n m_i u^{(p-1)} \quad \text{for } t \in [0, 1] \text{ and } u \in \mathbb{R}^+,
\]

where \( m_i \) are nonnegative numbers such that \( \sum_{i=1}^n m_i \leq \lambda_3 \).

Using the above values for \( p, a, \beta, \xi, a, b \), we have

\[
\lambda_3 = \left( \delta_1 \int_0^1 (1-s)^{b-2} \int_0^1 H(s, \tau) f(\tau) \, d\tau \, ds \right)^{-1} = \frac{\beta \Gamma(\beta)}{\delta_1} \left[ \frac{\Gamma(\beta + 1) \Gamma(\alpha - 1) - \Gamma(\beta + 2) \Gamma(\alpha - 1)}{\Gamma(\alpha + \beta + 1)} \right]^{-1} \approx 7.5.
\]

Let \( \sum_{i=1}^n m_i \leq 7.5 \). Note (H1), (H4) and (H5) hold.

3. Main Results

**Theorem 3.1** Suppose that (H1)-(H3) are satisfied. Then (1) has at least a positive solution \( u^* \). Moreover, there exists a monotone non-decreasing sequence \( \{u_n\}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} u_n = u^* \), where \( u_0(t) = M\gamma(t), \quad t \in [0, 1], \quad (M \text{ is defined in the proof}), \) and \( u_{n+1} = Au_n \) for \( n = 0, 1, 2, \ldots \).

**Proof.** From (H3) there exist \( \varepsilon_1 \in (0, \lambda_1) \) and \( c_1 > 0 \) such that

\[
f(t, u) \leq (\lambda_1 - \varepsilon_1)^{p-1} u^{p-1} + c_1, \quad \forall u \in [0, +\infty), \quad t \in [0, 1]. \tag{10}
\]

Take \( M \geq c_1^{\frac{1}{p-1}} \left( \frac{\varepsilon_1}{\kappa_1} \right)^{\frac{1}{p-1}} \), where \( \varepsilon_1, c_1 \) are defined in (10) and let \( u_0 = M\gamma \). Hence,

\[
(AM\gamma(t))(t) \leq \left[ \int_0^t G(t, s) \left( \int_0^1 H(s, \tau) f(\tau, M\gamma(\tau)) \, d\tau \right) \frac{ds}{\gamma(t)} \right]^{\frac{1}{p-1}}
\]

Then we have

\[
(AM\gamma(t))(t) \leq \frac{\delta_1 (\lambda_1 - \varepsilon_1)^{p-1} M\gamma(t)}{\kappa_1 + c_1^{\frac{1}{p-1}} \delta_1 (\gamma(t))^{p-1} \kappa_2}.
\]

Finally, we obtain

\[
(AM\gamma(t))(t) \leq M\gamma(t).
\]
This implies that

\[ u_1 = Au_0 \leq u_0. \]

Also we have from (H2),

\[
u_2(t) = (Au_1)(t) = \int_0^t G(t, s) \left( \int_0^s H(s, \tau) f(\tau, u_1(\tau)) d\tau \right) ds
\leq \int_0^t G(t, s) \left( \int_0^s H(s, \tau) f(\tau, u_0(\tau)) d\tau \right) ds
= (Au_0)(t) = u_1(t).
\]

By induction, \( u_{n+1} \leq u_n, n = 0, 1, 2, \ldots \). Also \( 0 \leq u_n(t) \leq M_\gamma(t) \leq M \) for \( t \in [0, 1] \) and \( n = 0, 1, 2, \ldots \). From the monotone bounded theorem we can take the limit as \( n \to \infty \) in \( u_{n+1} = Au_n \) and we obtain \( u^* = Au^* \).

Furthermore, because the zero function is not a solution of the problem (1), \( u^* \) is a positive solution for (1).

This completes the proof. \( \square \)

**Theorem 3.2** Suppose that (H1), (H4) and (H5) are satisfied. Then (1) has at least a positive solution.

**Proof.** From (H4) there exist \( \varepsilon > 0 \) and \( \varepsilon > 0 \) such that

\[ f(t, u) \geq (\lambda_2 + \varepsilon_2)u^{p-1} - c_2, \forall u \in [0, +\infty), t \in [\theta_1, \theta_2]. \]  

(11)

From (11) we have

\[
(\lambda_2 + \varepsilon_2)u^p = (\lambda_2 + \varepsilon_2)u^{p-1} \leq f(t, u) + \varepsilon \leq f(t, u) + \varepsilon_2 \leq f(t, u) + \varepsilon_2. 
\]

Hence,

\[
f(t, u) \geq (\lambda_2 + \varepsilon_2)u^p - \varepsilon_2. \]  

(12)

In what follows, we shall show that there exists a large positive number \( R > \zeta (\zeta \text{ is defined in (H5)}) \) such that

\[ u - Au \neq \mu u_0 \text{ for all } \mu \geq 0 \text{ and } u \in \partial B_R \cap P, \]

(13)

where \( u_0 \) is a fixed element in \( P_0 \). If not, there exist \( \mu \geq 0 \) and \( u \in \partial B_R \cap P \) such that \( u - Au = \mu u_0 \), i.e., \( u(t) = (Au)(t) + \mu u_0(t) \) for \( t \in [0, 1] \). Hence \( ||u|| = ||Au + \mu u_0|| \geq ||Au|| \). Moreover, note that if \( u \in P \), by Lemma 2.6 we have \( Au + \mu u_0 \in P_0 \) and also \( u \in P_0 \).

Consequently, from (12), for a fixed point \( t_0 \in (0, 1) \), we have

\[
[(Au)(t_0)]^p = \left[ \int_0^1 G(t_0, s) \left( \int_0^s H(s, \tau)f(\tau, u(t_0)) d\tau \right) ds \right]^p
\geq \int_0^1 G^p(t_0, s) \left( \int_0^s H^p(s, \tau)f^p(\tau, u(t_0)) d\tau \right) ds
\geq \int_0^1 G^p(t_0, s) \int_0^s H^p(s, \tau)f^p(\tau, u(t_0)) d\tau ds
\geq \int_0^1 G^p(t_0, s) \int_0^{\theta_1} H^p(s, \tau) \left[ (\lambda_2 + \varepsilon_2)u^p - \varepsilon_2 \right] d\tau ds
\geq \left[ (\lambda_2 + \varepsilon_2) \left( \frac{\theta_2 - \theta_1}{\theta_1} \right) \right]^p R^p \int_0^1 G^p(t_0, s) \int_0^{\theta_2} H^p(s, \tau) d\tau ds - c_3,
\]

where \( c_3 > 0 \) is a constant. This contradicts (13). Therefore, (1) has at least a positive solution. \( \square \)
where \( c_3 = c_2^{\frac{p}{2}} \int_0^1 \int_0^{\theta^3} \int_0^{\theta^2} H(t,s)d\tau ds. \) Therefore, if \( R \) is large enough we have

\[
\|Au\| \geq (\|Au\|(t_0))^{\frac{1}{p}} > \lambda_2^{\frac{p}{2}} \left( \frac{\delta_1\theta_1^{1-a}}{\delta_1} \right)^{\frac{1}{p}} \int_0^1 \int_0^{\theta^3} \int_0^{\theta^2} H(t,s)d\tau ds - c_3
\]

\[
= 2R^{\frac{p}{2}} - c_3 \geq R^{\frac{p}{2}} = \|u\|^{\frac{p}{2}},
\]

e., \( \|Au\| > \|u\|, \) and this contradicts \( \|u\| \geq \|Au\|. \) Thus (13) holds true and Lemma 2.8 yields

\[
i(A, B_R \cap P, P) = 0.
\]

From (HS) for \( u \in \partial B \cap P \) we have

\[
\|Au\| = \max_{t \in [0,1]} \|Au(t)\| = \max_{t \in [0,1]} \left( \int_0^1 G(t,s) \left( \int_0^1 H(s,\tau)f(\tau, u(\tau))d\tau \right) ds \right)^{\frac{1}{p}}
\]

\[
\leq \zeta \lambda_3^{\frac{1}{p}} \delta_1 \int_0^1 \int_0^1 (1-s)^{1-a} \left( \int_0^1 H(s,\tau)d\tau \right)^{\frac{1}{p}} ds
\]

\[
= \zeta.
\]

Hence, \( \|Au\| \leq \|u\|, \) for \( u \in \partial B \cap P, \) and Lemma 2.9 implies that

\[
i(A, B_\zeta \cap P, P) = 1.
\]

Combining (14) and (15) gives

\[
i(A, (B_R \setminus \overline{B}_\zeta) \cap P, P) = i(A, B_R \cap P, P) - i(A, B_\zeta \cap P, P) = -1.
\]

Consequently the operator \( A \) has at least one fixed point on \((B_R \setminus \overline{B}_\zeta) \cap P, \) and hence (1) has at least one positive solution. This completes the proof. \( \square \)

References