Strong Convergence Result of Split Feasibility Problems in Banach Spaces

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Abstract. The purpose of this paper is to introduce and study an iterative scheme for solving the split feasibility problems in the setting of \( p \)-uniformly convex and uniformly smooth Banach spaces. Under suitable conditions a strong convergence theorem is established. The main result presented in this paper extends some recent results done by Jitsupa Deepho and Poom Kumam [Jitsupa Deepho and Poom Kumam, A Modified Halperns Iterative Scheme for Solving Split Feasibility Problems, Abstract and Applied Analysis, Volume 2012, Article ID 876069, 8 pages] and some others.

1. Introduction

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Let \( I \) denote the identity operator on \( H \). Let \( C \) and \( Q \) be nonempty, closed and convex subsets of real Hilbert spaces \( H_1 \) and \( H_2 \), respectively. The split feasibility problem (SFP) is to find a point \( x \in C \) such that \( Ax \in Q \),

\[
    x \in C \text{ such that } Ax \in Q, \tag{1}
\]

where \( A : H_1 \to H_2 \) is a bounded linear operator. We say that SFP is consistent if (1) has a solution. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [4] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, for example, [3, 13, 16, 22, 27–32] and references therein).

Recently, Deepho and Kumam [7] introduced and studied a modified Halperns iterative scheme for solving the split feasibility problem in the setting of infinite-dimensional Hilbert spaces. Under suitable conditions, they established the following strong convergence theorem.

**Theorem 1.1.** Suppose that the SFP is consistent and \( 0 < \xi < \frac{2}{\|A\|^2} \). Let \( \{x_n\} \) be a sequence defined by

\[
    x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \xi A^*(I - P_Q)A)x_n, \quad n \geq 1, \tag{2}
\]

where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \([0,1]\) and satisfy \( \alpha_n + \beta_n + \gamma_n = 1 \). If the following assumptions are satisfied:

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(C1) \( \lim_{n \to \infty} \alpha_n = 0 \) but \( \sum_{n=1}^{\infty} \alpha_n = \infty \)

(C2) \( \limsup_{n \to \infty} \beta_n < 1 \)

(C3) the sums \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| \) and \( \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| \) are finite.

Then \( \{x_n\} \) converges strongly to a solution of the SFP (1).

The following question naturally arises.

**Question:** Can we extend the results of problem (1) to higher Banach spaces (i.e., \( L_p \) spaces, \( 1 < p < \infty \)) other than Hilbert spaces considered by many authors?

Let \( E_1 \) and \( E_2 \) be two \( p \)-uniformly convex real Banach spaces which are also uniformly smooth. Let \( C \) and \( Q \) be nonempty, closed and convex subsets of \( E_1 \) and \( E_2 \) respectively, \( A : E_1 \to E_2 \) be a bounded linear operator and \( A^* : E_2^* \to E_1^* \) be the adjoint of \( A \). The split feasibility problem (SFP) in (1) is reformulated as find a point

\[
x \in C \text{ such that } Ax \in Q.
\]

We assume that SFP (3) has a nonempty solution set \( \Omega := \{y \in C : Ay \in Q\} = C \cap A^{-1}(Q) \). Then, we have that \( \Omega \) is a closed and convex subset of \( E_1 \).

In solving SFP (3) in \( p \)-uniformly convex real Banach spaces which are also uniformly smooth, Schöpfer et al. [21] proposed the following algorithm: For \( x_1 \in E_1 \) and \( n \geq 1 \), set

\[
x_{n+1} = \Pi_C \left[ f_{E_1}(x_n) - t_n A^* f_{E_2}(Ax_n - P_Q(Ax_n)) \right],
\]

where \( \Pi_C \) denotes the Bregman projection and \( f \) the duality mapping. Clearly the above algorithm covers the Byrne’s CQ algorithm [2]

\[
x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), n \geq 1,
\]

which is found to be a gradient-projection method (GPM) in convex minimization as a special case. They established the weak convergence of algorithm (4) under the condition that \( E_1 \) is \( p \)-uniformly convex, uniformly smooth and the duality mapping of \( E_1 \), is sequentially weak-to-weak-continuous.

We remark here that the condition that the duality mapping of \( E_1 \) is sequentially weak-to-weak-continuous assumed in [21] excludes some important Banach spaces, such as the classical \( L_p(2 < p < \infty) \) spaces.

Recently, Wang [25] modified the above algorithm (4) and proved strong convergence for the following multiple-sets split feasibility problem (MSSFP): find \( x \in E_1 \) satisfying

\[
x \in \bigcap_{i=1}^{r} C_i, Ax \in \bigcap_{j=1+r}^{r+s} Q_j,
\]

where \( r, s \) are two given integers, \( C_i, i = 1, \ldots, r \) is a closed convex subset in \( E_1 \), and \( Q_j, j = r + 1, \ldots, r + s, \) is a closed convex subset in \( E_2 \). He defined for each \( n \in \mathbb{N} \),

\[
T_n(x) = \begin{cases} 
\Pi_{C_{i(n)}}(x), & 1 \leq i(n) \leq r, \\
J_{E_1}(x) - t_n A^* J_{E_2}(Ax - P_{Q_{i(n)}}(Ax)), & r + 1 \leq i(n) \leq r + s,
\end{cases}
\]

where \( i : \mathbb{N} \to I \) is the cyclic control mapping

\[
i(n) = n \mod (r + s) + 1,
\]
and \( t_n \) satisfies
\[
0 < t \leq t_n \leq \left( \frac{q}{c_n \|A\|} \right)^{1/n},
\]
with \( c_n \) a constant defined as in Lemma 2.1 and proposed the following algorithm: For any initial guess \( x_1 = x \), define \( \{x_n\} \) recursively by
\[
\begin{aligned}
\{\begin{array}{ll}
y_n &= T_n x_n \\
D_n &= \{ w \in E_1 : \Delta_p(y_n, w) \leq \Delta_p(x_n, w) \} \\
E_n &= \{ w \in E_1 : \langle x_n - w, f_p(x) - f_p(x_n) \rangle \geq 0 \} \\
x_{n+1} &= \Pi_{D_n \cap E_n}(x).
\end{array}\}
\]
(7)

Using the idea in the work of Nakajo and Takahashi [14], he proved the following strong convergence theorem in \( p \)-uniformly convex Banach spaces which is also uniformly smooth.

**Theorem 1.2.** Let \( E_1 \) and \( E_2 \) be two \( p \)-uniformly convex real Banach spaces which are also uniformly smooth. Let \( C \) and \( Q \) be nonempty, closed and convex subsets of \( E_1 \) and \( E_2 \) respectively, \( A : E_1 \rightarrow E_2 \) be a bounded linear operator and \( A^* : E_2^* \rightarrow E_1^* \) be the adjoint of \( A \). Suppose that SFP (5) has a nonempty solution set \( \Omega \). Let the sequence \( \{x_n\}_{n=1}^{\infty} \) be generated by (7). Then \( \{x_n\}_{n=1}^{\infty} \) converges strongly to the Bregman projection of \( x \) onto the solution set \( \Omega \).

The main advantage of result of Wang [25] is that the weak-to-weak continuity of the duality mapping, assumed in [21] is dispensed with and strong convergence result was achieved. On the other hand, to implement the algorithm (7) of Wang [25], one has to calculate, at each iteration, the Bregman projection onto the intersection of two half spaces \( D_n \) and \( E_n \).

Our aim in this paper is to construct another iterative scheme for solving problem (3) for which its implementation does not involve calculation of Bregman projection onto the intersection of two half spaces at each step of the iteration for which strong convergence is achieved in \( p \)-uniformly convex real Banach spaces which are also uniformly smooth.

### 2. Preliminaries

Let \( E_1 \) and \( E_2 \) be real Banach spaces and let \( A : E_1 \rightarrow E_2 \) be a bounded linear operator. The dual (adjoint) operator of \( A \), denoted by \( A^* \), is a bounded linear operator defined by \( A^* : E_2^* \rightarrow E_1^* \)
\[
\langle A^* y, x \rangle := \langle y, Ax \rangle, \quad \forall x \in E_1, y \in E_2^*
\]
and the equalities \( \|A^*\| = \|A\| \) and \( N(A^*) = R(A)^\perp \) are valid, where \( R(A)^\perp := \{ x^* \in E_2^* : \langle x^*, u \rangle = 0, \forall u \in R(A) \} \).

For more details on bounded linear operators and their duals, please see [8, 23, 24].

Let \( 1 < q \leq 2 \leq p \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( E \) be a real Banach space. The modulus of convexity \( \delta_E : [0, 2] \rightarrow [0, 1] \) is defined as
\[
\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = 1 = \|y\|, \|x - y\| \geq \epsilon \right\}.
\]

\( E \) is called uniformly convex if \( \delta_E(\epsilon) > 0 \) for any \( \epsilon \in (0, 2) \); \( p \)-uniformly convex if there is a \( c_p > 0 \) so that \( \delta_E(\epsilon) \geq c_p \epsilon^p \) for any \( \epsilon \in (0, 2) \). The modulus of smoothness \( \rho_E(\tau) : [0, \infty) \rightarrow [0, \infty) \) is defined by
\[
\rho_E(\tau) = \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1 \right\}.
\]

\( E \) is called uniformly smooth if \( \lim_{\tau \rightarrow 0} \frac{\rho_E(\tau)}{\tau} = 0 \); \( q \)-uniformly smooth if there is a \( C_q > 0 \) so that \( \rho_E(\tau) \leq C_q \tau^q \) for any \( \tau > 0 \). The \( L_p \) space is \( 2 \)-uniformly convex for \( 1 < p < 2 \) and \( p \)-uniformly convex for \( p \geq 2 \). It is known that \( E \) is \( p \)-uniformly convex if and only if its dual \( E^* \) is \( q \)-uniformly smooth (see [12]).

The \( q \)-uniformly smooth spaces have the following estimate [26].
Lemma 2.1. (Xu, [26]) Let $x, y \in E$. If $E$ is $q$-uniformly smooth, then there is a $C_q > 0$ so that
\[ \|x - y\|^p \leq \|x\|^p - q\langle y, J^p_E(x) \rangle + C_q \|y\|^p. \]

Here and hereafter, we assume that $E$ is a $p$-uniformly convex and uniformly smooth, which implies that its dual space, $E^*$, is $q$-uniformly smooth and uniformly convex. In this situation, it is known that the duality mapping $J^p_E$ is one-to-one, single-valued and satisfies $J^p_E = (J^p_E)^{-1} = (J^p_E^*)^{-1}$, where $J^p_E = J^p_E^*$ is the duality mapping of $E^*$ (see [1, 6]). Here the duality mapping $J^p_E : E \to 2^{E^*}$ is defined by
\[ J^p_E(x) = \{ x \in E^* : \langle x, x \rangle = \|x\|^p, \|x\| = \|x\|^{-1} \}. \]

The duality mapping $J^p_E$ is said to be weak-to-weak continuous if
\[ x_n \to x \Rightarrow \langle J^p_E x_n, y \rangle \to \langle J^p_E x, y \rangle \]
holds true for any $y \in E$. It is worth noting that the $\ell_p(p > 1)$ space has such a property, but the $J^p_E(p > 2)$ space does not share this property.

Given a Gâteaux differentiable convex function $f : E \to \mathbb{R}$, the Bregman distance with respect to $f$ is defined as:
\[ \Delta_f(x, y) = f(y) - f(x) - \langle f'(x), y - x \rangle, \quad x, y \in E. \]

It is worth noting that the duality mapping $J^p_E$ is in fact the derivative of the function $f_p(x) = (\frac{1}{p})\|x\|^p$. Then the Bregman distance with respect to $f_p$ is given by
\[ \Delta_p(x, y) = \frac{1}{q} \|x\|^q - \langle J^p_E x, y \rangle + \frac{1}{p} \|y\|^p \]
\[ = \frac{1}{p} (\|y\|^p - \|x\|^p) + \langle J^p_E x, x - y \rangle \]
\[ = \frac{1}{q} (\|x\|^q - \|y\|^q) - \langle J^p_E x - J^p_E y, x \rangle. \]

Given $x, y, z \in E$, one can easily get
\[ \Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J^p_E x - J^p_E z \rangle, \quad \text{(8)} \]
\[ \Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J^p_E x - J^p_E y \rangle. \quad \text{(9)} \]

Generally speaking, the Bregman distance is not a metric due to the absence of symmetry, but it has some distance-like properties. For the $p$-uniformly convex space, the metric and Bregman distance has the following relation (see [21]):
\[ \tau \|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J^p_E x - J^p_E y \rangle, \quad \text{(10)} \]
where $\tau > 0$ is some fixed number.

It is easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a $p$-uniformly convex and uniformly smooth $E$, then $x_n - y_n \to 0$, $n \to \infty$ implies that $\Delta_p(x_n, y_n) \to 0$, $n \to \infty$.

Let $C$ be a nonempty, closed and convex subset of $E$. The metric projection
\[ P_{C} x = \text{argmin}_{y \in C} \|x - y\|, \quad x \in E, \]
is the unique minimizer of the norm distance, which can be characterized by a variational inequality:
\[ \langle J^p_E(x - P_{C}x), z - P_{C}x \rangle \leq 0, \quad \forall z \in C. \quad \text{(11)} \]
Following [1, 5], we make use of the function \( V_p \) as the Bregman projection in Banach spaces.

Then \( V_p \) is nonnegative and
\[
V_p(x, x) = \Delta_p(f_E^p(x), x) = \Delta_p(f_E^p(x), x)
\]
for all \( x \in E \) and \( x \in E^* \). Moreover, by the subdifferential inequality,
\[
V_p(x, x) + \langle g, f_E^p(x) - x \rangle \leq V_p(x + g, x)
\]
for all \( x \in E \) and \( x, g \in E^* \) (see also [11], Lemmas 3.2 and 3.3; [15]). In addition, \( V_p \) is convex in the first variable. Thus, for all \( z \in E \),
\[
\Delta_p(\sum_{i=1}^{N} t_i f_E^p(x_i), z) = \Delta_p(\sum_{i=1}^{N} t_i f_E^p(x_i), z) \leq \sum_{i=1}^{N} t_i \Delta_p(x_i, z),
\]
where \( \{x_i\}_{i=1}^{N} \subset E \) and \( \{t_i\}_{i=1}^{N} \subset (0, 1) \) with \( \sum_{i=1}^{N} t_i = 1 \).

We next state the following lemmas which will be used in the sequel.

**Lemma 2.2.** (Xu [26]) Let \( \{a_n\} \) be a sequence of nonnegative real numbers satisfying the following relation:
\[
a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 1,
\]
where, (i) \( \{a_n\} \subset [0, 1] \), \( \sum a_n = \infty \); (ii) \( \limsup \sigma_n \leq 0 \); (iii) \( \gamma_n \geq 0 \); (\( n \geq 1 \), \( \sum \gamma_n < \infty \). Then, \( a_n \to 0 \) as \( n \to \infty \).

We shall adopt the following notations in this paper:
. \( x_n \to x \) means that \( x_n \to x \) strongly;
. \( x_n \rightharpoonup x \) means that \( x_n \rightharpoonup x \) weakly;
. \( \omega_n(\{x_n\}) := \{x : \exists x_n \rightharpoonup x \} \) is the weak \( \omega \)-limit set of the sequence \( \{x_n\}\).

In this paper, we assume that \( E_1 \) and \( E_2 \) are \( p \)-uniformly convex real Banach spaces which are also uniformly smooth, \( E_1^* \) is \( q \)-uniformly smooth real Banach space which is also uniformly convex where \( 1 < q \leq 2 < p < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). We further assume that \( f_{E_1}^* \) and \( f_{E_2}^* \) represent the duality mappings of \( E_1 \) and \( E_2 \) respectively, and \( f_{E_1}^* = (f_{E_1}^*)^{-1} = (f_{E_1}^*)^{-1}, \) where \( f_{E_1}^* = f_{E_1}^* \) is the duality mapping of \( E_1^* \).
3. Main Results

**Theorem 3.1.** Let $E_1$ and $E_2$ be two $p$-uniformly convex real Banach spaces which are also uniformly smooth. Let $C$ and $Q$ be nonempty, closed and convex subsets of $E_1$ and $E_2$ respectively, $A : E_1 \rightarrow E_2$ be a bounded linear operator and $A^* : E_2^* \rightarrow E_1^*$ be the adjoint of $A$. Suppose that SFP (3) has a nonempty solution set $\Omega$. Let $\{\alpha_n\} \subset (0, 1 - \epsilon)$ for some $\epsilon > 0$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. For a fixed $u \in C$, let sequences $\{y_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in C$,

\[
\begin{align*}
\{ & y_n = \int_{E_1}^p [f^p_{E_1}(x_n) - t_n A^* f^p_{E_2}(A x_n - P_Q(A x_n))] \\
& x_{n+1} = \Pi_{C} \int_{E_1}^p (\alpha_n f^p_{E_1}(u) + \beta_n f^p_{E_1}(x_n) + \gamma_n f^p_{E_1}(y_n)), \ n \geq 1. \tag{17}
\end{align*}
\]

Suppose the following conditions are satisfied:

(a) $\lim_{n \to \infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < t_n \leq k < \left(\frac{q}{\sqrt{p} \alpha_n}\right)^{\frac{1}{q}}$ and

(d) $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element $x^* \in \Omega$, where $x^* = \Pi_{\Omega} u$.

**Proof.** Let $x^* \in \Omega$. Suppose $w_n := A x_n - P_Q(A x_n), \ \forall n \geq 1$. Then we have $y_n = \int_{E_1}^p [f^p_{E_1}(x_n) - t_n A^* f^p_{E_2}(w_n)], \ \forall n \geq 1$.

It follows from (11) that

\[
\langle f^p_{E_2}(w_n), A x_n - A x^* \rangle = \|A x_n - P_Q(A x_n)\|^p + \langle f^p_{E_2}(w_n), P_Q(A x_n) - A x^* \rangle \\
\geq \|A x_n - P_Q(A x_n)\|^p = ||w_n||^p,
\]

which, with Lemma 2.1, yields

\[
\begin{align*}
\Delta_p(y_n, x^*) &= \Delta_p(\int_{E_1}^p [f^p_{E_1}(x_n) - t_n A^* f^p_{E_2}(w_n)]_{E_1}, x^*) \\
&= \frac{1}{q} ||f^p_{E_1}(x_n) - t_n A^* f^p_{E_2}(w_n)||^p - \langle f^p_{E_2}(w_n), A x^* \rangle + \frac{1}{p} ||x^*||^p \\
&\leq \frac{1}{q} ||f^p_{E_1}(x_n)||^p - t_n \langle A x_n, f^p_{E_2}(w_n) \rangle + \frac{C_q(t_n \|A\|^q)}{q} ||f^p_{E_2}(w_n)||^p \\
&\quad - \langle f^p_{E_2}(w_n), A x^* \rangle + \frac{1}{p} ||x^*||^p \\
&= \frac{1}{q} ||x_n||^p - \langle f^p_{E_1}(x_n), x^* \rangle + \frac{1}{p} ||x^*||^p + t_n \langle f^p_{E_2}(w_n), A x^* - A x_n \rangle \\
&\quad + \frac{C_q(t_n \|A\|^q)}{q} ||f^p_{E_2}(w_n)||^p \\
&= \Delta_p(x_n, x^*) + t_n \langle f^p_{E_2}(w_n), A x^* - A x_n \rangle + \frac{C_q(t_n \|A\|^q)}{q} ||f^p_{E_2}(w_n)||^p \\
&\leq \Delta_p(x_n, x^*) + \left( t_n - \frac{C_q(t_n \|A\|^q)}{q} \right) ||w_n||^p. \tag{19}
\end{align*}
\]

Using the condition (c), we have

\[
\Delta_p(y_n, x^*) \leq \Delta_p(x_n, x^*), \ \forall n \geq 1.
\]
Now, using (17), we have
\[
\Delta_p(x_{n+1}, x') \leq \alpha_n \Delta_p(u, x') + \beta_n \Delta_p(x_n, x') + \gamma_n \Delta_p(y_n, x')
\]
\[
\leq \alpha_n \Delta_p(u, x') + \beta_n \Delta_p(x_n, x') + \gamma_n \Delta_p(x_1, x')
\]
\[
= \alpha_n \Delta_p(u, x') + (1 - \alpha_n) \Delta_p(x_n, x')
\]
\[
\leq \max(\Delta_p(u, x'), \Delta_p(x_n, x'))
\]
\[
\leq \max(\Delta_p(u, x'), \Delta_p(x_1, x')).
\]
Hence, \(\{x_n\}_{n=1}^{\infty}\) is bounded.
The rest of the proof will be divided into two parts.

**Case 1.** Suppose that there exists \(n_0 \in \mathbb{N}\) such that \(\{\Delta_p(x_{n_0}, x')\}_{n=n_0}^{\infty}\) is non-increasing. Then \(\{\Delta_p(x_{n_0}, x')\}_{n=n_0}^{\infty}\) converges and \(\Delta_p(x_{n_0}, x') - \Delta_p(x_{n_0+1}, x') \to 0, n \to \infty\). Then from (19), we obtain
\[
\left(t_n - \frac{C_p(t_n||A||)^q}{q}\right)\|Ax_n - P_Q(Ax_n)\|^p \leq \Delta_p(x_{n_0}, x') - \Delta_p(y_{n_0}, x').
\]
(21)

Also, from (20), we have
\[
\Delta_p(x_{n_0}, x') - \Delta_p(y_{n_0}, x') \leq \frac{\alpha_n}{\gamma_n} \Delta_p(u, x') + \frac{1 - \alpha_n}{\gamma_n} \Delta_p(x_n, x')
- \frac{1}{\gamma_n} \Delta_p(x_{n_0+1}, x')
\]
\[
= \frac{\alpha_n}{\gamma_n} \Delta_p(u, x') - \frac{\alpha_n}{\gamma_n} \Delta_p(x_n, x')
+ \frac{1}{\gamma_n} (\Delta_p(x_{n_0}, x') - \Delta_p(x_{n_0+1}, x'))
\]
\[
\leq \frac{\alpha_n}{\gamma_n} \Delta_p(u, x') + \frac{1}{\gamma_n} (\Delta_p(x_{n_0}, x') - \Delta_p(x_{n_0+1}, x')).
\]
(22)

Putting (21) into (22), we have
\[
\left(t_n - \frac{C_p(t_n||A||)^q}{q}\right)\|Ax_n - P_Q(Ax_n)\|^p \leq \Delta_p(x_{n_0}, x') - \Delta_p(y_{n_0}, x')
\]
\[
\leq \frac{\alpha_n}{\gamma_n} \Delta_p(u, x') + \frac{1}{\gamma_n} (\Delta_p(x_{n_0}, x') - \Delta_p(x_{n_0+1}, x')).
\]
(23)

By conditions (c) and (d) and (23), we have
\[
0 \leq \left(t(1 - \frac{C_p||A||^q}{q})\|Ax_n - P_Q(Ax_n)\|^p\right)
\]
\[
\leq \left(t_n - \frac{C_p(t_n||A||)^q}{q}\right)\|Ax_n - P_Q(Ax_n)\|^p
\]
\[
\leq \frac{1}{\gamma_n} [\alpha_n \Delta_p(u, x') + (\Delta_p(x_{n_0}, x') - \Delta_p(x_{n_0+1}, x'))] \to 0, n \to \infty.
\]

Hence, we obtain
\[
\lim_{n \to \infty} \|Ax_n - P_Q(Ax_n)\| = 0.
\]
(24)
Since \( y_n = f_{E_1}^p (y_n) - t_n A f_{E_1}^p (Ax_n - P_Q(Ax_n)) \), then we have

\[
0 \leq \| f_{E_1}^p (y_n) - f_{E_1}^p (x_n) \| \leq t_n \| A^\|f_{E_1}^p (Ax_n - P_Q(Ax_n)) \|
\]

\[
\leq \left( \frac{q}{C_p \| A \|} \right)^{\frac{1}{q}} \| A^\|f_{E_1}^p (Ax_n - P_Q(Ax_n)) \| \to 0, n \to \infty.
\]

(25)

Therefore, we obtain

\[
\lim_{n \to \infty} \| f_{E_1}^p (y_n) - f_{E_1}^p (x_n) \| = 0.
\]

Since \( f_{E_1}^p \) is also norm-to-norm uniformly continuous on bounded subsets of \( E_1 \), we have

\[
\lim_{n \to \infty} \| y_n - x_n \| = 0.
\]

Furthermore,

\[
\| f_{E_1}^p (y_n) - f_{E_1}^p (x_n) - x_n \| = \| y_n - x_n \| \to 0, n \to \infty.
\]

Since \( f_{E_1}^p \) is norm-to-norm uniformly continuous on bounded sets, then

\[
f\| A^\| f_{E_1}^p (Ax_n - P_Q(Ax_n)) \| \leq t_n (A^\| f_{E_1}^p (Ax_n - P_Q(Ax_n)) \|
\]

\[
= \| f_{E_1}^p (x_n) - t_n A f_{E_1}^p (Ax_n - P_Q(Ax_n)) - f_{E_1}^p (x_n) \| \to 0, n \to \infty.
\]

Thus,

\[
\lim_{n \to \infty} \| A^\| f_{E_1}^p (Ax_n - P_Q(Ax_n)) \| = 0.
\]

(26)

Since \( \{x_n\} \) is bounded, there exists \( \{x_{n_j}\} \) of \( \{x_n\} \) such that \( x_{n_j} \to z \in \omega_{w_n}(x_n) \). From (9), (12) and (10), we have that

\[
\Delta_p(z, \Pi_C z) \leq (f_{E_1}^p (z) - f_{E_1}^p (\Pi_C z), z - \Pi_C z)
\]

\[
= (f_{E_1}^p (z) - f_{E_1}^p (\Pi_C z), z - x_{n_j}) + (f_{E_1}^p (z) - f_{E_1}^p (\Pi_C z), x_{n_j} - \Pi_C x_{n_j})
\]

\[
+ (f_{E_1}^p (z) - f_{E_1}^p (\Pi_C z), \Pi_C x_{n_j} - \Pi_C z)
\]

\[
\leq (f_{E_1}^p (z) - f_{E_1}^p (\Pi_C z), z - x_{n_j}) + (f_{E_1}^p (z) - f_{E_1}^p (\Pi_C z), x_{n_j} - \Pi_C x_{n_j}).
\]

As \( j \to \infty \), we obtain that \( \Delta_p(z, \Pi_C z) = 0 \). Thus, \( z \in C \). Let us now fix \( x \in C \). Then, \( Ax \in Q \) and

\[
\| (I - P_Q)Ax_n \|^p = \langle f_{E_2}^p (Ax_n - P_Q(Ax_n)), Ax_n - P_Q(Ax_n) \rangle
\]

\[
= \langle f_{E_2}^p (Ax_n - P_Q(Ax_n)), Ax_n - Ax \rangle
\]

\[
+ \langle f_{E_2}^p (Ax_n - P_Q(Ax_n)), Ax - P_Q(Ax_n) \rangle
\]

\[
\leq \langle f_{E_2}^p (Ax_n - P_Q(Ax_n)), Ax_n - Ax \rangle
\]

\[
\leq M \| A^\| f_{E_2}^p (I - P_Q)Ax_n \| \to 0, n \to \infty,
\]

where \( M > 0 \) is sufficiently large number. It then follows from (11) that

\[
\| (I - P_Q)Az \|^p = \langle f_{E_2}^p (Az - P_Q(Az)), Az - P_Q(Az) \rangle
\]

\[
= \langle f_{E_2}^p (Az - P_Q(Az)), Az - Ax_n \rangle + \langle f_{E_2}^p (Az - P_Q(Az)), Ax_n - P_Q(Ax_n) \rangle
\]

\[
+ \langle f_{E_2}^p (Az - P_Q(Az)), P_Q(Ax_n) - P_Q(Az) \rangle
\]

\[
\leq \langle f_{E_2}^p (Az - P_Q(Az)), Az - Ax_n \rangle + \langle f_{E_2}^p (Az - P_Q(Az)), Ax_n - P_Q(Ax_n) \rangle.
\]

Let \( w_n = f_{E_1}^p (u_n - a_n f_{E_1}^p (u_n)) \), \( n \geq 1 \), where \( u_n = f_{E_1}^p (\frac{\frac{u_n}{1 - a_n}}{1 - a_n}, f_{E_1}^p (y_n)) \), \( n \geq 1 \). By (10), we have

\[
0 < \Delta_p(y_n, x_n) \leq \| y_n - x_n \| \| f_{E_1}^p (y_n) - f_{E_1}^p (x_n) \| \to 0, n \to \infty.
\]
Furthermore,
\[
\begin{align*}
\Delta_p(u_n, x_n) & \leq \frac{\beta_n}{1 - \alpha_n} \Delta_p(x_n, x_n) + \frac{\gamma_n}{1 - \alpha_n} \Delta_p(y_n, x_n) \\
& = \frac{\gamma_n}{1 - \alpha_n} \Delta_p(y_n, x_n) \to 0, n \to \infty.
\end{align*}
\]
and
\[
\begin{align*}
\Delta_p(w_n, u_n) & \leq \alpha_n \Delta_p(u_n, u_n) + (1 - \alpha_n) \Delta_p(u_n, w_n) \\
& = \alpha_n \Delta_p(u_n, u_n) \to 0, n \to \infty.
\end{align*}
\]

Hence, by (10) we have \(\lim_{n \to \infty} ||w_n - u_n|| = 0\) and \(\lim_{n \to \infty} ||x_n - u_n|| = 0\). Now,
\[
||x_n - w_n|| \leq ||w_n - u_n|| + ||x_n - u_n|| \to 0, n \to \infty.
\]
Since \(x_n \to z\) and \(||w_n - x_n|| \to 0\), we have that \(w_n \to z\). Since \(Ax_n \to Az\), we have that
\[
\lim_{n \to \infty} (I - P_Q)Az = 0.
\]

Thus, \(Az \in Q\). Furthermore, let \(x' = \Pi_Q u\). Then we observe that
\[
\limsup_{n \to \infty} (w_n - x', \beta_{E_1}^{f_{E_1}}(u) - f_{E_1}^{q}(x')) = \limsup_{n \to \infty} (w_n - x', \beta_{E_1}^{f_{E_1}}(u) - f_{E_1}^{q}(x'))
\]
\[
= (z - x', \beta_{E_1}^{f_{E_1}}(u) - f_{E_1}^{q}(x')) \leq 0. \tag{27}
\]

Furthermore, by (15) and (14) we have
\[
\begin{align*}
\Delta_p(x_{n+1}, x') & \leq \Delta_p(f_{E_1}^{E_1}(\alpha_n^{f_{E_1}}(u), \gamma_n^{f_{E_1}}(y_n)), x') \\
& = \Delta_p(f_{E_1}^{E_1}(\alpha_n^{f_{E_1}}(u), \gamma_n^{f_{E_1}}(y_n)), x') \\
& = V_p(\alpha_n^{f_{E_1}}(u) + (1 - \alpha_n^{f_{E_1}}(u)), x') \\
& \leq V_p(\alpha_n^{f_{E_1}}(u) + (1 - \alpha_n^{f_{E_1}}(u)), x') - \alpha_n^{f_{E_1}}(u) \Delta_p(u_n, x') \\
& \leq \Delta_p(\alpha_n^{f_{E_1}}(u) + (1 - \alpha_n^{f_{E_1}}(u)), x') + \alpha_n^{f_{E_1}}(u) \delta_{E_1}(u_n, x') \\
& = \alpha_n \Delta_p(x_n, x') + \alpha_n(\alpha_n^{f_{E_1}}(u) + (1 - \alpha_n^{f_{E_1}}(u)), x') \\
& \leq \alpha_n \Delta_p(x_n, x') + \alpha_n(\alpha_n^{f_{E_1}}(u) + (1 - \alpha_n^{f_{E_1}}(u)), x') \\
& \leq (1 - \alpha_n) \Delta_p(x_n, x') + \alpha_n(\alpha_n^{f_{E_1}}(u) + (1 - \alpha_n^{f_{E_1}}(u)), x'). \tag{28}
\end{align*}
\]

Using Lemma 2.2 and (27) in (28), we obtain
\[
\lim_{n \to \infty} \Delta_p(x_n, x') = 0.
\]

Thus, \(x_n \to x', n \to \infty\).

**Case 2**

Assume that \(\{\Delta_p(x_n, x')\}_{n=1}^{\infty}\) is not monotonically decreasing sequence. Set \(\Gamma_n = \Delta_p(x_n, x')\), \(\forall n \geq 1\) and let \(\tau: \mathbb{N} \to \mathbb{N}\) be a mapping for all \(n \geq n_0\) (for some \(n_0\) large enough) by
\[
\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.
\]
Clearly, \( \tau \) is a non decreasing sequence such that \( \tau(n) \to \infty \) as \( n \to \infty \) and
\[
0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0.
\]
After a similar conclusion from (24), it is easy to see that
\[
\|Ax_{\tau(n)} - P_Q x_{\tau(n)}\| \to 0, n \to \infty.
\]
By the similar argument as above in Case 1, we conclude immediately that
\[
\lim_{n \to \infty} \|A^* f_{E_2}^n (Ax_{\tau(n)} - P_Q(Ax_{\tau(n)}))\| = 0.
\]
and
\[
\limsup_{n \to \infty} (w_{\tau(n)} - x^*, f_{E_1}^n (u) - f_{E_1}^n (x^*)) \leq 0.
\]
Since \( \{x_{\tau(n)}\} \) is bounded, there exists a subsequence of \( \{x_{\tau(n)}\} \), still denoted by \( \{x_{\tau(n)}\} \) which converges weakly to \( z \in C \) and \( Az \in Q \). From (28) we have that
\[
\Delta_p(x_{\tau(n)+1}, x^*) \leq (1 - \alpha_{\tau(n)} A_p(x_{\tau(n)}, x^*) + \alpha_n (w_{\tau(n)} - x^*, f_{E_1}^n (u) - f_{E_1}^n (x^*))
\]
which implies by Lemma 2.2
\[
\lim_{n \to \infty} \Delta_p(x_{\tau(n)}, x^*) = 0 \tag{29}
\]
and \( \lim_{n \to \infty} \Delta_p(x_{\tau(n)+1}, x^*) = 0 \). Furthermore, for \( n \geq n_0 \), it is easy to see that \( \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \) if \( n < \tau(n) \) (that is, \( \tau(n) < n \)), because \( \Gamma_j \geq \Gamma_{j+1} \) for \( \tau(n)+1 \leq j \leq n \). As a consequence, we obtain for all \( n \geq n_0 \),
\[
0 \leq \Gamma_n \leq \max(\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}) = \Gamma_{\tau(n)+1}.
\]
Hence \( \lim \Gamma_n = 0 \), that is, \( \{x_n\} \) converges strongly to \( x^* \). This completes the proof. \( \square \)

**Corollary 3.2.** Let \( E_1 \) and \( E_2 \) be two \( L_p \) spaces with \( 2 \leq p < \infty \). Let \( C \) and \( Q \) be nonempty, closed and convex subsets of \( E_1 \) and \( E_2 \) respectively, \( A : E_1 \to E_2 \) be a bounded linear operator and \( A^* : E_2 \to E_1 \) be the adjoint of \( A \). Suppose that SFP (3) has a nonempty solution set \( \Omega \). Let \( \{\alpha_n\} \subset (0, 1 - \epsilon) \) for some \( \epsilon > 0 \), \( \{\beta_n\} \) and \( \{\gamma_n\} \) be sequences in (0, 1) such that \( \alpha_n + \beta_n + \gamma_n = 1 \). For a fixed \( u \in C \), let sequences \( \{y_n\}_{n=1}^{\infty} \) and \( \{x_n\}_{n=1}^{\infty} \) be generated by \( x_1 \in C \),
\[
\begin{align*}
\begin{cases}
y_n = \beta_n f_{E_1}^n (x_n) + \gamma_n f_{E_1}^n (y_n), \\
x_{n+1} = \Pi_C \Pi_{E_1} \alpha_n f_{E_1}^n (u) + \beta_n f_{E_1}^n (x_n) + \gamma_n f_{E_1}^n (y_n),
\end{cases}
\end{align*}
\]
\( n \geq 1 \).
Suppose the following conditions are satisfied:
\begin{enumerate}
\item \( \limsup_{n \to \infty} \alpha_n = 0; \)
\item \( \sum_{n=1}^{\infty} \alpha_n = \infty; \)
\item \( 0 < t \leq t_n \leq k < \left( \frac{\epsilon}{\|A\|} \right)^{\frac{t}{t_n}} \) and
\item \( 0 < \inf_{n \to \infty} \gamma_n \leq \sup_{n \to \infty} \gamma_n < 1. \)
\end{enumerate}
Then the sequence \( \{x_n\}_{n=1}^{\infty} \) converges strongly to an element \( x^* \in \Omega \), where \( x^* = \Pi_CL u \).

**Corollary 3.3.** Let \( H_1 \) and \( H_2 \) be two real Hilbert spaces. Let \( C \) and \( Q \) be nonempty, closed and convex subsets of \( H_1 \) and \( H_2 \) respectively, \( A : H_1 \to H_2 \) be a bounded linear operator and \( A^* : H_2 \to H_1 \) be the adjoint of \( A \). Suppose that SFP (3) has a nonempty solution set \( \Omega \). Let \( \{\alpha_n\} \subset (0, 1 - \epsilon) \) for some \( \epsilon > 0 \), \( \{\beta_n\} \) and \( \{\gamma_n\} \) be sequences in (0, 1) such that \( \alpha_n + \beta_n + \gamma_n = 1 \). For a fixed \( u \in C \), let sequences \( \{y_n\}_{n=1}^{\infty} \) and \( \{x_n\}_{n=1}^{\infty} \) be generated by \( x_1 \in C \),
\[
\begin{align*}
\begin{cases}
y_n = x_n - t_n A^* (Ax_n - P_Q(Ax_n)) \\
x_{n+1} = P_C (\alpha_n u + \beta_n x_n + \gamma_n y_n),
\end{cases}
\end{align*}
\]
\( n \geq 1 \).
Suppose the following conditions are satisfied:

(a) \( \lim_{n \to \infty} a_n = 0; \)

(b) \( \sum_{n=1}^{\infty} a_n = \infty; \)

(c) \( 0 < t \leq t_n \leq k < \frac{2}{\|A\|}; \) and

(d) \( 0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1. \)

Then the sequence \( \{x_n\}_{n=1}^{\infty} \) converges strongly to an element \( x^* \in \Omega, \) where \( x^* = P_{\Omega}u. \)

4. An Application

In this section, we give an application of Theorem 3.1 to the convexly constrained linear inverse problem in \( p \)-uniformly convex real Banach spaces which are also uniformly smooth. Consider the convexly constrained linear inverse problem (cf [9])

\[
\begin{aligned}
Ax &= b, \\
x &\in C,
\end{aligned}
\]

(30)

where \( E_1 \) and \( E_2 \) are two \( p \)-uniformly convex real Banach spaces which are also uniformly smooth and \( A : E_1 \to E_2 \) is a bounded linear mapping and \( b \in E_2. \)

It is well known that the projected Landweber method (see, [10]) given by

\[
\begin{aligned}
\{ x_1 \in C, \\
x_{n+1} &= P_{C}[x_n - \lambda A^*(Ax_n - b)], n \geq 1,
\end{aligned}
\]

where \( A^* \) is the adjoint of \( A \) and \( 0 < \lambda < 2\alpha \) with \( \alpha = \frac{1}{2\|A\|}, \) converges weakly to a solution of (30). In what follows, we present an algorithm with strong convergence for solving (30).

Corollary 4.1. Let \( E_1 \) and \( E_2 \) be two \( p \)-uniformly convex real Banach spaces which are also uniformly smooth. Let \( C \) and \( Q \) be nonempty, closed and convex subsets of \( E_1 \) and \( E_2 \) respectively, \( A : E_1 \to E_2 \) be a bounded linear operator and \( A^* : E_2^* \to E_1^* \) be the adjoint of \( A. \) Suppose that the convexly constrained linear inverse problem (30) is consistent and let \( \Omega \) denote its solution set. Let \( \{a_n\} \) be a sequence in \( (0, 1) \). Let \( \{a_n\} \subset (0, 1) \) for some \( \epsilon > 0, \) \( \{\beta_n\} \) and \( \{\gamma_n\} \) be sequences in \( (0, 1) \) such that \( a_n + \beta_n + \gamma_n = 1. \) For a fixed \( u \in C, \) let sequences \( \{y_n\}_{n=1}^{\infty} \) and \( \{x_n\}_{n=1}^{\infty} \) be generated by

\[
\begin{aligned}
y_n &= \frac{1}{\|E_1\|} \left| E_1 \left( J_{E_1} (x_n) - t_n A^* (Ax_n - b) \right) \right| \\
x_{n+1} &= \Pi_{E_1} \left( a_n J_{E_1} (x_n) + \beta_n J_{E_1} (x_n) + \gamma_n J_{E_1} (y_n) \right), n \geq 1.
\end{aligned}
\]

(31)

Suppose the following conditions are satisfied:

Suppose the following conditions are satisfied:

(a) \( \lim_{n \to \infty} a_n = 0; \)

(b) \( \sum_{n=1}^{\infty} a_n = \infty; \)

(c) \( 0 < t \leq t_n \leq k < \frac{2}{\|A\|}; \) and

(d) \( 0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1. \)

Then the sequence \( \{x_n\}_{n=1}^{\infty} \) converges strongly to an element \( x^* \in \Omega, \) where \( x^* = \Pi_{\Omega}u. \)

Proof. For each \( n \geq 1, \) replacing \( b = P_{\Omega}(Ax_n), x_n \in C \) implies that (17) reduces to (31). Thus, by Theorem 3.1 we obtain the desired conclusion. \( \square \)

We note that an iterative method is not of any serious interest unless it can be implemented to solve a problem. In view of this, we demonstrate the applicability of our result obtained in Theorem 3.1 to this example.
Example 4.2. Let $E_1 = L^2([a, b]) = E_2$ and suppose that $A : L^2([a, b]) \to L^2([a, b])$ is defined by

$$A(x)(s) = \int_a^b V(s, t)x(t)dt, \quad \forall x \in L^2([a, b]),$$

where $V : [a, b] \times [a, b] \to \mathbb{R}$ is continuous. It can be easily shown that $A$ is a bounded linear operator with the adjoint $A^*$ of $A$ defined by

$$A^*(x)(s) = \int_a^b V(t, s)x(t)dt, \quad \forall x \in L^2([a, b]).$$

Let $C = \{x \in L^2([a, b]) : \langle a, x \rangle = b\}$, for some $a \in L^2 - \{0\}$ and $Q = \{x \in L^2([a, b]) : \langle a, x \rangle \geq b\}$, for some $a \in L^2 - \{0\}, b \in \mathbb{R}$. In this case,

$$\Pi_C(x) = P_C(x) = \frac{b - \langle a, x \rangle}{\|a\|^2}a + x,$$

and

$$P_Q(x) = \max \left\{0, \frac{b - \langle a, x \rangle}{\|a\|^2}\right\}a + x.$$

We consider the following problem

find $x^* \in C$ such that $Ax^* \in Q$. \hfill (32)

To be more specific, let us consider a split feasibility problem in $L^2([0, 1])$. Suppose $A : L^2([0, 1]) \to L^2([0, 1])$ is defined by

$$Ax(s) = \int_0^s (s^2 + t^2)x(t)dt, \quad \forall x \in L^2([0, 1]).$$

Let $C = \{x \in L^2([0, 1]) : \int_0^1 2tx(t)dt = 0\}$ and $Q = \{x \in L^2([0, 1]) : \int_0^1 t^2x(t)dt \geq -1\}$. We consider the following problem

find $x^* \in C$ such that $Ax^* \in Q$. \hfill (33)

It is clear that Problem (33) has a nonempty solution set $\Omega$ since $0 \in \Omega$. In this case, our iterative scheme (17) becomes

$$x_n \in C, \text{ (with } \alpha_n = \frac{1}{n^2}, \beta_n = \frac{n}{2(m+n+1)}, \gamma_n = \frac{n}{2(m+n+1)} \text{ and } t_n = \frac{1}{2n^2})$$

$$\left\{\begin{array}{l}
y_n = x_n - \frac{\beta_n}{\beta_n + \gamma_n}A^*(Ax_n - P_Q(Ax_n)) \\
x_{n+1} = P_C\left(\frac{\alpha_n}{\alpha_n + \gamma_n}x_n + \frac{\beta_n}{\beta_n + \gamma_n}A^*(Ax_n - P_Q(Ax_n))\right), \quad n \geq 1.
\end{array}\right.$$

We see here that our iterative scheme can be implemented to solve the problem (33) considered in this example.

Remark 4.3. We make the following remark concerning our contributions in this paper.

1. The weak-to-weak continuity of the duality mapping assumed in [21] is dispensed with in this paper and strong convergence is achieved.

2. In implementing the algorithm (7), one has to calculate, at each iteration, the Bregman projection onto the intersection of two half spaces but in this our iterative algorithm (17), one does not have to calculate, at each iteration, the Bregman projection onto the intersection of two half spaces. Hence, our algorithm (17) appears more efficient and implementable than the algorithm of Wang [25].

3. The conditions $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ imposed in the result of Deepho and Kumam [7] are dispensed with in our result even in higher Banach spaces than Hilbert where the result of Deepho and Kumam [7] was proved.
References


