Existence and Structure of the Common Fixed Points Based on TVS

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Abstract. In this paper, we investigate the common fixed point property for commutative nonexpansive mappings on $\tau$-compact convex sets in normed and Banach spaces, where $\tau$ is a Hausdorff topological vector space topology that is weaker than the norm topology. As a consequence of our main results, we obtain that the set of common fixed points of any commutative family of nonexpansive self-mappings of a nonempty $\text{clm}$-compact (resp. weak* compact) convex subset $C$ of $L_1(\mu)$ with a $\sigma$-finite $\mu$ (resp. the James space $J_0$) is a nonempty nonexpansive retract of $C$.

1. Introduction

Let $E$ be a normed space and $\tau$ be a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology. $E$ is said to have the fixed point property with respect to $\tau$ ($\tau$-fpp) if the following holds: For each nonempty, norm bounded, $\tau$-compact, convex subset $C$ of $E$, every nonexpansive mapping $T: C \rightarrow C$ (i.e., $\|Tx - Ty\| \leq \|x - y\|, x, y \in C$) has a fixed point. We say that a nonempty closed and convex subset $C$ of $E$ has the fpp if every nonexpansive mapping $T: C \rightarrow C$ has a fixed point, and also $C$ is said to have the $\tau$-fpp if each nonempty, norm bounded, $\tau$-compact, convex subset of $C$ has the fpp. It is not clear that a set having the fpp must be bounded. It is known that the norm boundedness assumption for $\tau$-compact, convex sets is redundant (see, for example, [12]). An standard example of such a $\tau$ is where $\tau$ is the weak topology on $E$. Another example is where $E$ is a dual Banach space and $\tau$ is the weak* topology. Yet another example is when $E = L_1(\mu)$ and $\tau$ is the topology $\text{clm}$ of convergence locally in measure (see, e.g., [18]).

Determining conditions on a Banach space $E$ so that it has the fixed point property has been of considerable interest for many years. Kirk [13] proved that a weakly compact convex subset of a Banach space with weak normal structure has the fpp. It is known that every compact convex subset of a Banach space has normal structure. Kirk's proof of his result also yields that a weak* compact convex subset of a Banach space with weak* normal structure has the fixed point property (see [19]). The condition above that $C$ has normal structure can not be dropped. In fact, Alspach [2] showed that $L_1[0,1]$ fails the weak fpp.

We say that $E$ has the (common) $\tau$-fpp for commutative semigroups if whenever $S = \{T_s : s \in S\}$ is a commutative semigroup of nonexpansive self-mappings on a nonempty, $\tau$-compact, convex subset of $E$, then the common fixed point set of $S$, $\text{Fix}(S)$, is nonempty. Bruck [4] showed that a Banach space $E$ having the weak fpp has the weak fpp for commutative semigroups. We refer to [23] for a simple proof.
to a more general version of Bruck’s result. For a dual Banach space $E$ satisfying the weak* fpp, it is still unknown whether $E$ has the weak* fpp for commutative semigroups. Very recently, Borzdynski and Wisnicki [3] proved that if $S$ is a commuting family of weak* continuous nonexpansive mappings acting on a weak* compact convex subset $C$ of the dual Banach space $E$, then the set of common fixed points of $S$ is a nonempty nonexpansive retract of $C$. This partially answers a long-standing open problem posed by Lau in [15] (see also [17]). Examples of Banach spaces with the weak* fpp for commutative semigroups include $ℓ₁$, trace class operators on a Hilbert space, Hardy space $H^1$ and the Fourier algebra of a compact group (see [16, 19–22]).

In this paper, by using the retraction tool, we study the common fixed point property for commutative nonexpansive mappings on $τ$-compact convex sets in $E$, where $τ$ is a Hausdorff topological vector space topology that is weaker than the norm topology. In Section 2, we shall prove the following: Let $E$ be a Banach space, $τ$ a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology and the norm of $E$ is lsc with respect to $τ$, and $C$ be a nonempty, $τ$-compact, separable, convex subset of $E$ which has the $τ$-fpp. Then any commutative family of nonexpansive self-mappings of $C$ has a common fixed point and the set of common fixed points is a nonexpansive retract of $C$. In Section 4, we obtain the same result by replacing the separability with the $τ$-Opial condition. As a consequence, we shall show that the set of common fixed points of any commutative family of nonexpansive self-mappings of a nonempty $clm$-compact (resp. weak* compact) convex subset $C$ of $L₁(µ)$ with a $σ$-finite $µ$ (resp. the James space $ℓ₀$) is a nonempty nonexpansive retract of $C$.

2. The $τ$-fpp for Commutative Mappings on Separable Subsets

Recall some general concepts and definitions. Let $E$ be a normed space and $C$ be a nonempty subset of $E$. A mapping $T$ on $C$ is said to be a retraction if $T² = T$. A subset $F$ of $C$ is called a nonexpansive retract of $C$ if either $F = ∅$ or there exists a retraction of $C$ onto $F$ which is a nonexpansive mapping. Nonexpansive retract plays an important role in the study of the structure of fixed point sets of nonexpansive mappings. We refer the reader to [4–7] for more information concerning nonexpansive retracts.

Let $E$ be a normed space, and $τ$ be a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology. The purpose of this section is to study the $τ$-fpp (which implies the weak* fpp) for commutative semigroups of nonexpansive mappings. In fact, we give some partial answers to the following question:

If a dual Banach space $E$ has the $τ$-fpp, does $E$ have the $τ$-fpp for commuting semigroups?

The following theorem is essential to get the main results.

Theorem 2.1. Let $E$ be a normed space and $τ$ be a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology. Suppose that the norm of $E$ is lsc with respect to $τ$. Let $C$ be a nonempty, $τ$-compact, convex subset of $E$ with the $τ$-fpp. If $\{T₁, \ldots, Tₙ\}$ is a commutative family of nonexpansive mappings on $C$, then $∩_{i=1}^{n} Fix(T_i)$ is a nonempty nonexpansive retract of $C$.

Proof. First, we prove that for each nonexpansive mapping $T : C → C$, $Fix(T)$ is a nonempty nonexpansive retract of $C$. To this purpose, consider $C^C$ with the product topology induced by the topology $τ$ on $C$. Then by Tychonoff’s theorem $C^C$ is compact. Now, consider a nonexpansive mapping $T : C → C$ and define

$$R := \{S ∈ C^C : S is nonexpansive, Fix(T) ⊂ Fix(S)\}.$$ 

We show that $R$ is closed in $C^C$. Suppose that $\{Uₐ : a ∈ A\}$ is a net in $R$, which converges to $U$ in $C^C$. Then for $z ∈ Fix(T)$, $Uₐ(z) = z$ so $U(z) = \tau − \limₐ Uₐ(z) = z$. By the lower semi-continuity of the norm with respect to $τ$, for any $x, y$ in $C$, $||Ux − Uy|| ≤ \lim infₐ ||Uₐx − Uₐy|| ≤ ||x − y||$. So we have shown that $U ∈ R$, hence that $R$ is closed in $C^C$. Since $C^C$ is compact, therefore $R$ is compact (the topology on $R$ is that of $τ$-pointwise convergence). Define a preorder $≤$ in $R$ by $S ≤ U$ if $||Sx − Sy|| ≤ ||Ux − Uy||$ for all $x, y$ in $C$ and using the Bruck’s method [7] we obtain a minimal element $R ∈ R$. Indeed, by considering Zorn’s lemma it suffices to show every linearly ordered subset of $R$ has a lower bound in $R$. If $\{Uₐ\}$ is a linearly ordered subset of $R$ by $≤$, the family of sets $\{S ∈ R : S ≤ Uₐ\}$ is linearly ordered by inclusion. The proof that $R$ is
closed in \( C^c \) can be repeated to show that these sets are closed in \( K \), and hence compact. So there exists \( U \in \bigcap_{i=1}^{n} \{ S \in K : S \leq U \} \) with \( U \leq U_i \) for each \( \lambda \). Now, we have shown the existence of a minimal element \( P \in K \) in the following sense:

\[
\text{if } S \in K \text{ and } \| S(x) - S(y) \| \leq \| P(x) - P(y) \|, \forall x, y \in C,
\]

\[
\text{then } \| S(x) - S(y) \| = \| P(x) - P(y) \|. \quad (*)
\]

We shall prove that \( P(x) \in \text{Fix}(T) \) for all \( x \in C \). For a given \( x \in C \), consider the set \( K = \{ S(P(x)) : S \in K \} \). Then \( K \) is a nonempty \( \tau \)-compact convex subset of \( C \), because \( K \) is convex and compact. On the other hand, \( TS \in K, \forall S \in K \). Therefore, we have \( T(K) \subseteq K \), and then, by the \( \tau \)-fpp, there exists \( h \in K \) with \( h(P(x)) \in \text{Fix}(T) \).

Let \( y = h(P(x)) \). Then \( P(y) = h(y) = y \), and by using the minimality of \( P \), we have \( \| P(x) - y \| = \| P(x) - P(y) \| = \| h(P(x)) - h(P(y)) \| = \| h(P(x)) - y \| = 0 \). So \( P(x) = y \in \text{Fix}(T) \). Since this is so for each \( x \in C \) and \( P \) belongs to \( K \), it follows that \( P^2 = P \). So, we have shown that \( \text{Fix}(T) \) is a nonexpansive retract of \( C \). Now, let \( \{ T_1, \ldots, T_n \} \) be a commuting family of nonexpansive mappings on \( C \). We prove that \( \cap_{n=1}^{\infty} \text{Fix}(T_i) \) is a nonempty nonexpansive retract of \( C \). The proof is by induction. If \( n = 1 \), then \( \text{Fix}(T_1) \) is a nonempty nonexpansive retract of \( C \) by the above discussion. Now suppose that \( \cap_{i=1}^{n-1} \text{Fix}(T_i) \) is a nonempty nonexpansive retract of \( C \) and \( R : C \to \cap_{i=1}^{n-1} \text{Fix}(T_i) \) a nonexpansive retraction. Then, it is easy to check that \( \text{Fix}(T_{n+1}R) = \cap_{i=1}^{n} \text{Fix}(T_i) \) (see, e.g., [4] for the details). Another application of the first part of the proof implies that \( \text{Fix}(T_{n+1}R) = \cap_{i=1}^{n} \text{Fix}(T_j) \) is a nonempty nonexpansive retract of \( C \), which completes the induction. \( \square \)

We will also need the following lemma, due to Bruck [4], as an intermediary step.

**Lemma 2.2.** If \( C \) is a bounded closed convex subset of a Banach space \( E \) and \( \{ F_n \} \) is a descending sequence of nonempty nonexpansive retracts of \( C \), then \( \cap_{n=1}^{\infty} F_n \) is the fixed point set of some nonexpansive \( r : C \to C \).

**Theorem 2.3.** Let \( E \) be a Banach space and \( \tau \) be a Hausdorff topological vector space topology on \( E \) that is weaker than the norm topology. Suppose the norm of \( E \) is lsc with respect to \( \tau \), and \( C \) be a nonempty, \( \tau \)-compact, separable, convex subset of \( E \) which has the \( \tau \)-fpp. Then any commutative family of nonexpansive self-mappings of \( C \) has a common fixed point and the set of common fixed points is a nonexpansive retract of \( C \).

**Proof.** Let \( S = \{ T_i \}_{i=1}^{n} \) be a commutative family of nonexpansive mappings on \( C \), and let \( \mathcal{F} \) be the family of the finite intersections of fixed point sets of mappings in the commutative family \( S \). We have shown, in Theorem 2.1, that \( \mathcal{F} \) is a family of nonempty nonexpansive retracts of \( C \), and \( \mathcal{F} \) is obviously directed by \( \supset \). Now, since \( C \) is separable, there is a countable subfamily \( \mathcal{F}' \) of \( \mathcal{F} \) such that

\[
\text{Fix}(S) = \bigcap \{ F : F \in \mathcal{F} \} = \bigcap \{ F : F \in \mathcal{F}' \}.
\]

Using the fact that \( \mathcal{F} \) is directed by \( \supset \) we can therefore find a descending sequence \( \{ F_n \} \) in \( \mathcal{F} \) with \( \text{Fix}(S) = \bigcap_{n=1}^{\infty} F_n \). But, by Lemma 2.2, \( \text{Fix}(S) = \bigcap_{n=1}^{\infty} F_n = \text{Fix}(r) \), for some nonexpansive \( r : C \to C \). Since \( C \) has the \( \tau \)-fpp, Theorem 2.1 implies \( \text{Fix}(S) \) is a nonempty nonexpansive retract of \( C \). This completes the proof. \( \square \)

An standard example of such a pair \((E, \tau)\) is where \( E \) is a dual Banach space and \( \tau \) is the weak* topology. Thus, Theorem 2.3 yields the following result:

**Theorem 2.4.** Let \( E \) be a dual Banach space and suppose \( C \) is a nonempty, weak* compact, separable, convex subset of \( E \) which has the weak*-fpp. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of \( C \) is a nonempty nonexpansive retract of \( C \).
For commutative sequences of nonexpansive self-mappings it is possible to say even more (cf. [24]).

**Theorem 2.6.** Let $E$ be a Banach space and $\tau$ be a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology. Suppose the norm of $E$ is lsc with respect to $\tau$, and $C$ be a nonempty, $\tau$-compact, convex subset of $E$ which has the $\tau$-fpp. Then the set of common fixed points of any commutative sequence of nonexpansive self-mappings of $C$ is a nonempty nonexpansive retract of $C$.

**Proof.** Let $[T_n]$ be a commutative sequence of nonexpansive mappings on $C$. Then, by Theorem 2.1, for each natural number $n$, $F_n = \cap_{j=n}^{\infty} \text{Fix}(T_j)$ is a nonempty nonexpansive retract of $C$. Thus, applying Lemma 2.2, we deduce that $\cap_{n=1}^{\infty} F_n = \cap_{n=1}^{\infty} \text{Fix}(T_n)$ is the fixed point set of some nonexpansive mapping $r : C \to C$. Therefore, by Theorem 2.1, $\text{Fix}(r) = \cap_{n=1}^{\infty} \text{Fix}(T_n)$ is a nonempty nonexpansive retract of $C$. \[\square\]

3. Fixed Point Property with Respect to $\text{clm}$-Topology in $L_1(\mu)$

In this section, we will use the topology of convergence in measure which we now recall for the convenience of the reader. Let $(\Omega, \Sigma, \mu)$ be a positive $\sigma$-finite measure space and $L_0(\mu)$ be the set of all scalar-valued $\Sigma$-measurable functions on $\Omega$. The topological vector space topology $\text{clm}$, of convergence locally in measure on $L_0(\mu)$, is generated by the following translation-invariant metric: Let $(A_n)_{n=1}^{\infty}$ be a $\Sigma$-partition of $\Omega$, where $\Sigma := \{ A \in \Sigma : \mu(A) \in (0, \infty) \}$. Define $d_0$ by

$$
d_0(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(A_n)} \int_{A_n} \frac{|f - g|}{1 + |f - g|} d\mu, \text{ for all } f, g \in L_0(\mu).$$

If $\mu(\Omega) < \infty$, then the simpler metric

$$
d_0(f, g) = \int_{\Omega} \frac{|f - g|}{1 + |f - g|} d\mu, \text{ for all } f, g \in L_0(\mu),$$

generates the $\text{clm}$ topology. In this case, we simply refer to $\text{clm}$ as the topology of convergence in measure, denoted by $\text{cm}$. $L_0(\mu)$ is complete with respect to the above metric. For sequences, the $\text{clm}$-topology reduces, in a sense, to almost everywhere convergence. Indeed, any sequence in $L_0(\mu)$ that converges almost everywhere to $f \in L_0(\mu)$ must converge to $f$ locally in measure. On the other hand, every $\text{clm}$-convergent sequence of scalar-valued measurable functions has a subsequence that converges almost everywhere to the same limit function. Note that when we discuss $L_1(\mu)$, $\text{clm}$ or $\text{cm}$ will denote the topologies introduced above, restricted to $L_1(\mu)$. Further, the $L_1(\mu)$-norm is $\text{clm}$-lower semicontinuous. This follows from Fatou’s lemma and the fact that $\text{clm}$ is a metric topology. Thus, an example of the pair $(E, \tau)$ in Section 2 is where $E$ is $L_1(\mu)$ and $\tau$ is the topology $\text{clm}$ of convergence locally in measure.

We remark that one can show that in every $L_1(\mu)$, $\mu$ $\sigma$-finite, $\text{clm}$-compact sets must be norm separable. Besides, Lami Dozo and Turpin [14] showed that $L_1(\mu)$ has the fpp with respect to the topology $\text{clm}$ of convergence locally in measure. Combining the above discussion with Theorem 2.3 yields the following result:

**Theorem 3.1.** Let $C$ be a nonempty, $\text{clm}$-compact, convex subset of $L_1(\mu)$, where $\mu$ is $\sigma$-finite. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of $C$ is a nonempty nonexpansive retract of $C$.

In the particular case that $\mu$ is finite, Theorem 3.1 is a consequence of a result in [22] (see also [21]).

4. The $\tau$-fpp for Commutative Mappings Under the $\tau$-Opial Condition

In this section, we study the common fpp for commutative semigroups by considering conditions under which the fixed point sets of nonexpansive mappings are $\tau$-closed. For our purposes it will be convenient to prove the following result:
Theorem 4.1. Let $E$ be a normed space and $\tau$ be a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology. Suppose that the norm of $E$ is lsc with respect to $\tau$. Let $C$ be a nonempty, $\tau$-compact, convex subset of $E$ with the $\tau$-fpp. Then any commutative family of nonexpansive self-mappings of $C$ such that their fixed point sets are $\tau$-closed has a common fixed point and the set of common fixed points is a nonexpansive retract of $C$.

Proof. Let $\mathcal{S} = \{T_i\}_{i \in J}$ be a commutative family of nonexpansive mappings on $C$, and let $\mathcal{F}$ be the family of the finite intersections of fixed point sets of mappings in the commutative family $\mathcal{S}$. Since, by assumption, the fixed point sets are $\tau$-closed, Theorem 2.1 implies that $\mathcal{F}$ is a family of nonempty $\tau$-compact subsets of $C$ that is directed by $\supseteq$. Hence,

$$\text{Fix}(\mathcal{S}) = \bigcap \{F : F \in \mathcal{F}\} \neq \emptyset.$$ 

Now, defining

$$\mathcal{R} := \{T \in C^C : T \text{ is nonexpansive}, \text{Fix}(\mathcal{S}) \subset \text{Fix}(T)\},$$

and using an argument quite similar to the one used in the proof of Theorem 2.1, it is easy to show that $\mathcal{R}$ contains a nonexpansive retraction $P$ from $C$ onto $\text{Fix}(\mathcal{S})$. $\square$

Let $E$ be a normed space and $\tau$ be a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology. We say that a nonempty set $\tau$-fpp. Then any commutative family of nonexpansive self-mappings of $C$ such that their fixed point sets are $\tau$-closed has a common fixed point and the set of common fixed points is a nonexpansive retract of $C$.

Spaces with the weak Opial condition have weak normal structure [10] and hence the weak fpp. Similarly dual spaces with the weak* Opial condition can contain no nontrivial separable weak* sequentially compact convex diametral sets; in particular, separable duals with the weak* Opial condition have weak* normal structure [18], and hence the weak* fpp. Using the following argument, it is possible to say more:

Suppose $T$ is a nonexpansive self mapping of a nonempty bounded closed convex subset $C$ of a Banach space $E$. It is well known that $T$ admits an approximate fixed point sequence; that is, a sequence $(x_n)$ in $C$ with $\|Tx_n - x_n\| \to 0$. Now suppose $C$ is, in addition, $\tau$-compact, where $\tau$ is a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology. If the norm of $E$ satisfies $\tau$-Opial condition for nets (for sequences), then the fixed point set of $T$ is nonempty and $\tau$-compact (provided $C$ is metrizable with respect to $\tau$). In fact, if $(x_i)$ is net (sequence) in $C$ such that converges to $x$, with respect to $\tau$, and $\|Tx_i - x_i\| \to 0$, then

$$\lim_{i} \|Tx_i - x_i\| = \lim_{i} \|Tx_i - Tx_i\| \leq \lim_{i} \|x - x_i\|,$$

contradicting the $\tau$-Opial condition unless $Tx = x$. Moreover, if $(z_i)$ is a net (sequence) in $\text{Fix}(T)$ converging in $\tau$ to some $z \in C$, then

$$\lim_{i} \|z_i - z\| = \lim_{i} \|Tz_i - Tz\| \leq \lim_{i} \|z_i - z\|,$$

contradicting the $\tau$-Opial condition for nets (for sequences) unless $Tz = z$. This implies that if either the norm of $E$ satisfies $\tau$-Opial condition for nets, or the norm of $E$ satisfies $\tau$-Opial condition for sequences and $C$ is metrizable with respect to $\tau$, then $\text{Fix}(T)$ is nonempty and $\tau$-closed. Thus, Banach spaces with the $\tau$-Opial condition for nets (for sequences) have the $\tau$-fpp (provided $\tau$ is metrizable), and in this case the fixed point sets of nonexpansive mappings are $\tau$-compact.

Combining the above facts with Theorem 4.1, we obtain the following results:
Theorem 4.2. Let $E$ be a Banach space and $\tau$ be a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology. Suppose that the norm of $E$ is lsc with respect to $\tau$ and satisfies $\tau$-Opial condition. Let $C$ be a nonempty, $\tau$-compact, convex subset of $E$ and let $C$ be metrizable with respect to $\tau$. Then any commuting family of nonexpansive self-mappings of $C$ has a common fixed point and the set of common fixed points is a nonexpansive retract of $C$.

Let $(\Omega, \Sigma, \mu)$ be a finite measure space. It follows from Proposition 5.2 in [21] (see also Lemma 2.6 in [22]) that $L_1(\mu)$ satisfies $cm$-Opial condition. Further, $cm$-topology is metrizable and the norm of $L_1(\mu)$ is lsc with respect to the topology of convergence in measure. Thus, $L_1(\mu)$ is an example that satisfies the assumptions of Theorem 4.2.

Theorem 4.3. Let $E$ be a Banach space and $\tau$ be a Hausdorff topological vector space topology on $E$ that is weaker than the norm topology. Suppose that the norm of $E$ is lsc with respect to $\tau$ and satisfies $\tau$-Opial condition for nets. Let $C$ be a nonempty, $\tau$-compact, convex subset of $E$. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of $C$ is a nonempty nonexpansive retract of $C$.

Corollary 4.4. Let $C$ be a nonempty, weak* compact and convex subset of a dual Banach space $E$. Suppose that $C$ satisfies weak*-Opial condition for nets. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of $C$ is a nonempty nonexpansive retract of $C$.

Let $E$ be a Banach space and let $\Gamma$ be a nonempty subspace of its dual $E^*$. If

$$\sup\{x'(\alpha) : x' \in \Gamma, \|x'\| = 1\} = \|x\|,$$

for each $x \in E$, then we say that $\Gamma$ is a norming set for $E$. It is obvious that a norming set generates a Hausdorff linear topology $\sigma(E, \Gamma)$ which is weaker than the weak topology $\sigma(E, E^*)$. It is worth noting here that $n(E) \subseteq E^{**}$ is a norming set for $E^{**}$, where $n$ is a natural embedding of $E$ into $E^{**}$, and hence, for $\Gamma = n(E)$, $\sigma(E, \Gamma)$ is the weak* topology on $E^*$. Throughout, $\Gamma$ denotes a norming set for $E$. It is easy to observe that the norm of $E$ is lower semicontinuous with respect to the $\sigma(E, \Gamma)$-topology [8]. It is shown in [8] that if $E$ is a Banach space, $\Gamma$ is a norming set for $E$ and $C$ is a nonempty, bounded and $\Gamma$-sequentially compact subset of $E$, then in $C$ the $\Gamma$-Opial condition for nets is equivalent to the $\Gamma$-Opial condition. Thus, we obtain the following:

Corollary 4.5. Let $E$ be a Banach space, $\Gamma$ be a norming set for $E$, $C$ be a nonempty, $\Gamma$-compact and $\Gamma$-sequentially compact convex subset of $E$. Suppose that $C$ satisfies $\Gamma$-Opial condition. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of $C$ is a nonempty nonexpansive retract of $C$.

Corollary 4.6. Let $E$ be a dual Banach space with a separable predual space, $C$ be a nonempty, weak* compact and convex subset of $E$. Suppose that $C$ satisfies weak*-Opial condition. Then the set of common fixed points of any commutative family of nonexpansive self-mappings of $C$ is a nonempty nonexpansive retract of $C$.

Finally, it is worth mentioning that some of the well-known classical dual Banach spaces satisfy weak*-Opial condition.

Example 4.7. The following dual Banach spaces satisfy the weak*-Opial condition for nets:

(i) $\ell_1$;
(ii) the James space $J_0$;
(iii) $B(G)$, the Fourier-Stieltjes algebra of a compact group $G$;

see [1, 9, 16, 19], for details. Hence, by Corollary 4.4 (or 4.6), the common fixed point set of any commutative family of nonexpansive self-mappings of a nonempty weak* compact convex subset $C$ in the above spaces is a nonempty nonexpansive retract of $C$.

Problem: Can Theorems 4.2 and 4.3 be extended to left reversible or amenable semigroups?

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