On \([m, C]\)-Isometric Operators

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Abstract. In this paper we introduce an \([m, C]\)-isometric operator \(T\) on a complex Hilbert space \(\mathcal{H}\) and study its spectral properties. We show that if \(T\) is an \([m, C]\)-isometric operator and \(N\) is an \(n\)-nilpotent operator, respectively, then \(T + N\) is an \([m + 2n - 2, C]\)-isometric operator. Finally we give a short proof of Duggal’s result for tensor product of \(m\)-isometries and give a similar result for \([m, C]\)-isometric operators.

1. Introduction

Let \(\mathcal{H}\) be a complex Hilbert space and \(\mathcal{L}(\mathcal{H})\) be the set of bounded linear operators on \(\mathcal{H}\). For an integer \(m \in \mathbb{N}\) and an operator \(T \in \mathcal{L}(\mathcal{H})\) is said to be an \(m\)-isometric operator if

\[
\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^m - j T^{m-j} = 0.
\]

In 1995, J. Agler and M. Stankus [1] introduced an \(m\)-isometric operator and showed nice results. An antilinear operator \(C\) on \(\mathcal{H}\) is said to be conjugation if \(C^2 = I\) and \(\langle Cx, Cy \rangle = \langle y, x \rangle\) for all \(x, y \in \mathcal{H}\). An operator \(T \in \mathcal{L}(\mathcal{H})\) is said to be complex symmetric if \(CTC = T^*\). In [11], S. Jung, E. Ko, M. Lee and J. Lee studied spectral properties of complex symmetric operators. In [4], M. Chō, E. Ko and J. Lee introduced \((m, C)\)-isometric operators with conjugation \(C\) as follows; For an operator \(T \in \mathcal{L}(\mathcal{H})\) and an integer \(m \geq 1\), \(T\) is said to be an \((m, C)\)-isometric operator if there exists some conjugation \(C\) such that

\[
\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{m-j} C T^{m-j} = 0.
\]

According to definitions of \(m\)-isometry, \((m, C)\)-isometry and complex symmetric, we define an \([m, C]\)-isometry \(T\) as follows; An operator \(T\) is said to be an \([m, C]\)-isometric operator if there exists some conjugation \(C\) such that

\[
\sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} T^{m-j} C = 0.
\]
It is easy to see that if $T$ is complex symmetric and an $[m, C]$-isometry, then $T$ is an $m$-isometry. Throughout the paper, let $I$ be the identity operator on $\mathcal{H}$.

2. Example

(i) Let $\mathcal{H} = \mathbb{C}^2$ and let $C$ be a conjugation on $\mathcal{H}$ given by $C \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$.

If $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{C}^2$, then $CTC = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = T^*$. Since $T^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, it follows that

$$\sum_{j=0}^{2} (-1)^j \binom{2}{j} T^{2j} C = \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + I = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \neq 0.$$

Therefore, $T$ is a $(2, C)$-isometric operator. On the other hand, $T$ is not a $[2, C]$-isometric operator due to the fact that

(ii) Under the same space $\mathcal{H}$ and the same conjugation $C$ to (i), let $S$ be an operator given by $S = \begin{pmatrix} i & \sqrt{2} \\ \sqrt{2} & -i \end{pmatrix}$.

Then CSC $= \begin{pmatrix} i & \sqrt{2} \\ \sqrt{2} & -i \end{pmatrix}$ and CSC $= S \neq S^*$. Moreover, it holds CSC $\cdot S - I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - I = 0$ and hence $S$ is a $[1, C]$-isometry. But $S^* \cdot CSC - I = \begin{pmatrix} 2 & 1 \\ 5 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \neq 0$ and hence $S$ is not a $(1, C)$-isometry.

(iii) Let $F$ and $J$ be conjugations on a Hilbert space $\mathcal{H}$ such that $JF \neq I$. Define $T$ and $C$ by $T = \begin{pmatrix} 0 & F \\ J & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & J \\ I & 0 \end{pmatrix}$. Then it is easy to see that $C$ is a conjugation on $\mathcal{H} \oplus \mathcal{H}$, $CTC \cdot T = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and $T^* \cdot CTC = \begin{pmatrix} 1 & 0 \\ F & \overline{J} \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ I & 0 \end{pmatrix}$. Hence $T$ is a $[1, C]$-isometric operator and not a $(1, C)$-isometric operator.

3. $[m, C]$-Isometric Operators

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a conjugation $C$, we define the operator $\lambda_m(T; C)$ by

$$\lambda_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} C T^{m-j} C \cdot T^{m-j}.$$ 

Then $T$ is an $[m, C]$-isometry if and only if $\lambda_m(T; C) = 0$. Moreover, it holds that

$$CTC \cdot \lambda_m(T; C) \cdot T - \lambda_m(T; C) = \lambda_{m+1}(T; C).$$

Hence if $T$ is an $[m, C]$-isometry, then $T$ is an $[n, C]$-isometry for every $n \geq m$.

Let $C$ be a conjugation on $\mathcal{H}$. Then $C$ satisfies $\|Cx\| = \|x\|$ and $C(\alpha x) = \overline{\alpha}Cx$ for all $x \in \mathcal{H}$ and all $\alpha \in \mathbb{C}$. Moreover, since $C^2 = I$, it follows that $(CTC)^n = CT^nC$ and $(CTC)^* = CT^nC$ for every positive integer $n$ (see [10] for more details). For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\sigma_p(T)$ and $\sigma_d(T)$ be the point spectrum and the approximate point spectrum of $T$, respectively. We denote the range of $T$ by $R(T)$. Then we have
\textbf{Theorem 3.1.} Let $T \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$-isometric operator. Then the following statements hold:

(i) $T$ is bounded below.

(ii) $0 \notin \sigma_a(T)$.

(iii) $T$ is injective and $\text{R}(T)$ is closed.

\textbf{Proof.} If $0 \in \sigma_a(T)$, then there exists a sequence of unit vectors $\{x_n\}$ of $\mathcal{H}$ such that $\lim_{n \to \infty} T x_n = 0$. Since $T$ is an $[m, C]$-isometric operator, it follows that

$$
\sum_{j=0}^{m-1} (-1)^j \binom{m}{j} CT^{m-j} C \cdot T^{m-j} = (-1)^{m+1} I.
$$

Moreover, since $\lim_{n \to \infty} \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} CT^{m-j} C \cdot T^{m-j} x_n = 0$, it follows from (2) that $\lim_{n \to \infty} x_n = 0$, which is a contradiction. Hence $0 \notin \sigma_a(T)$. Since (i), (ii), and (iii) are equivalent, this completes the proof. \hfill \Box

\textbf{Theorem 3.2.} Let $T \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$-isometric operator. If $\alpha \in \sigma_a(T)$, then $\overline{\alpha}^{-1} \in \sigma_a(T)$. In particular, if $\alpha$ is an eigenvalue of $T$, then $\overline{\alpha}^{-1}$ is an eigenvalue of $T$.

\textbf{Proof.} Let $\{x_n\}$ be a sequence of unit vectors such that $\lim_{n \to \infty} (T - \alpha) x_n = 0$. Since $T$ is an $[m, C]$-isometric operator, $C$ is bounded, and $\lim_{n \to \infty} (T^\ell - \alpha^\ell) x_n = 0$ for all $\ell \in \mathbb{N}$, it holds that

$$
0 = \lim_{n \to \infty} \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} CT^{m-j} C \cdot T^{m-j} x_n
= C \lim_{n \to \infty} \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} T^{m-j} \overline{\alpha}^{m-j} Cx_n
= C \lim_{n \to \infty} \overline{\alpha} T - 1)^m Cx_n.
$$

Moreover, since $C^2 = I$, it holds $\lim_{n \to \infty} (\overline{\alpha} T - 1)^m Cx_n = 0$. Since $\|Cx_n\| = 1$ and $\alpha \neq 0$ by Theorem 3.1, it follows that $\lim_{n \to \infty} (T - \overline{\alpha}^{-1})^m Cx_n = 0$ and hence $\overline{\alpha}^{-1} \in \sigma_a(T)$. \hfill \Box

\textbf{Corollary 3.3.} Let $T \in \mathcal{L}(\mathcal{H})$ be an $[m, C]$-isometric operator. Then $\|T\| \geq 1$.

\textbf{Proof.} If $0 < \|T\| < 1$, then there exists $\alpha \in \sigma(T)$ and a sequence $\{x_n\}$ of unit vectors such that $0 < |\alpha| < 1$ and $\|T - \alpha x_n\| \to 0$. By Theorem 3.2, it holds $\overline{\alpha}^{-1} \in \sigma(T)$. Since $|\overline{\alpha}^{-1}| > 1$, it is a contradiction. \hfill \Box

\textbf{Theorem 3.4.} Let $C$ be a conjugation on $\mathcal{H}$ and let $T \in \mathcal{L}(\mathcal{H})$. Then the following assertions hold.

(i) If $T$ is invertible, then $T$ is an $[m, C]$-isometric operator if and only if so is $T^{-1}$.

(ii) If $T$ is an $[m, C]$-isometric operator, then $T^n$ is also an $[m, C]$-isometric operator for any $n \in \mathbb{N}$.

\textbf{Proof.} (i) Suppose that $T$ is invertible and an $[m, C]$-isometry. Since $C^2 = I$, it follows that

$$
0 = (CT^{-m} C) \left[ \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (CT^{m-j} C) T^{m-j} \right] T^{-m}
= \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (C(T^{-1})^j) (T^{-1})^j.
$$

Since

$$
\sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (C(T^{-1})^j) (T^{-1})^j = 0
$$

then $T$ is invertible and an $[m, C]$-isometry. Since $C^2 = I$, it follows that

$$
0 = (CT^{-m} C) \left[ \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (CT^{m-j} C) T^{m-j} \right] T^{-m}
= \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (C(T^{-1})^j) (T^{-1})^j.
$$

Since

$$
\sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (C(T^{-1})^j) (T^{-1})^j = 0
$$
is equivalent to
\[\sum_{j=0}^{m} (-1)^j \binom{m}{j} (C(T^{-1})^j C)(T^{-1})^j = 0,\]

\(T^{-1}\) is an \([m, C]\)-isometry. Hence the statement (i) holds.

(ii) Since
\[
a^n - 1)^m = (a - 1)^m (a^{n-1} + a^{n-2} + \cdots + a + 1)^m
\]
\[= (a - 1)^m (\xi_0 a^{m(n-1)} + \xi_1 a^{m(n-1)-1} + \xi_2 a^{m(n-1)-2} + \cdots + \xi_{m(n-1)})\]
where \(\xi_i\) are coefficients \((i = 0, \ldots, m(n-1))\), it follows that
\[
\lambda_m(T^n; C) = \sum_{i=0}^{m(n-1)} \xi_i C T^{m(n-1)-i} C \cdot \lambda_m(T; C) \cdot T^{m(n-1)-i}.
\]

From (3), if \(\lambda_m(T; C) = 0\), then \(\lambda_m(T^n; C) = 0\). Hence \(T^n\) is an \([m, C]\)-isometric operator for any \(n \in \mathbb{N}\). So this completes the proof. \(\square\)

An operator \(N \in L(H)\) is said to be \(n\)-nilpotent if \(N^n = 0\) \((n \in \mathbb{N})\). In [2] T. Bermúdez, A. Martinón, V. Müller and A.J. Noda proved the following.

**Proposition 3.5.** (Theorem 3.1, [2]) Let \(T\) be an \(m\)-isometry on \(H\) and \(N\) be an \(n\)-nilpotent operator such that \(TN = NT\). Then \(T + N\) is an \((m + 2n - 2)\)-isometry.

We have following similar result.

**Theorem 3.6.** Let \(T\) be an \([m, C]\)-isometric operator on \(H\) and \(N\) be an \(n\)-nilpotent operator such that \(TN = NT\). Then \(T + N\) is an \([m + 2n - 2, C]\)-isometry.

**Proof.** In the proof, we denote \(\lambda_m(T; C)\) by \(\lambda_m(T)\) simply. First we show
\[
\lambda_m(T + N) = \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^j C \cdot \lambda_k(T) \cdot T^j \cdot N^k,
\]
where \(\binom{m}{i, j, k} = \frac{m!}{i! \cdot j! \cdot k!}\) and \(\lambda_0(\cdot) = I\). It is easy to see that (4) holds for \(m = 1\). Assume that (4) holds for \(m\). Then by (1) we have
\[
\lambda_{m+1}(T + N) = (C(T + N)C) \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^j C \cdot \lambda_k(T) \cdot T^j \cdot N^k (T + N)
\]
\[= \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^j C \cdot \lambda_k(T) \cdot T^j \cdot N^k (T + N)
\]
\[+ \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^{j+1} C \cdot \lambda_k(T) \cdot T^j \cdot N^{k+1}
\]
\[+ \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^j C \cdot \lambda_k(T) \cdot T^j \cdot N^{k+1}
\]
\[+ \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^j C \cdot \lambda_k(T) \cdot T^j \cdot N^{k+1}
\]
\[+ \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^{j+1} C \cdot \lambda_k(T) \cdot T^j \cdot N^{k+1}
\]
\[+ \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^j C \cdot \lambda_k(T) \cdot T^j \cdot N^{k+1}
\]
\[+ \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^{j+1} C \cdot \lambda_k(T) \cdot T^j \cdot N^{k+1}
\]
\[+ \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^j C \cdot \lambda_k(T) \cdot T^j \cdot N^{k+1}
\]
\[+ \sum_{i+j+k=m} \binom{m}{i, j, k} C(T + N)^i C \cdot C N^{j+1} C \cdot \lambda_k(T) \cdot T^j \cdot N^{k+1}
\]
Let \( T \) be given by (4) on \( H \). Then we have \( \lambda_{m+2n-2}(T + N) = 0 \). Therefore, \( T + N \) is an \([m + 2n - 2, C]\)-isometric operator. \( \square \)

**Remark 3.7.** Let \( T \in L(H) \). If \( \beta_m(T) \) is defined by

\[
\beta_m(T) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{m-j} T^m j,
\]

then \( T \) is an \( m \)-isometric operator if and only if \( \beta_m(T) = 0 \). Since, for any commuting pair \((T, S)\), it follows that

\[
\beta_m(T + S) = \sum_{i+j+k=m} \binom{m}{i, j, k} (T + S)^i \cdot S^j \cdot \beta_k(T) \cdot T^i \cdot S^j.
\]

So we have other proof of Proposition 3.5.

From Theorem 3.6, we get the following corollary.

**Corollary 3.8.** If \( T \) is a \([1, C]\)-isometric operator on \( H \) and \( N \) is an \( n \)-nilpotent operator such that \( TN = NT \), then \( T + N \) is an \([2n - 1, C]\)-isometry.

**Example 3.9.** Let \( C \) be a conjugation given by \( C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1}) \) on \( \mathbb{C}^3 \). If \( T = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) on \( \mathbb{C}^3 \), then

\[
T = I + N \text{ where } N = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Thus we have

\[
T^2 = \begin{pmatrix} 1 & 0 & 2a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 & 3a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad CTC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

\[
CT^2C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and } CT^3C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3a & 0 & 1 \end{pmatrix}.
\]

Then we have

\[
\lambda_3(T, C) = CT^3CT^3 - 3CT^2CT^2 + 3CTCT - I = 0.
\]

On the other hand, since \( N^2 = 0 \), it follows from Theorem 3.6 that \( T \) is a \([3, C]\)-isometric operator.
For an operator \( T \in \mathcal{L}(\mathcal{H}) \), the numerical range \( W(T) \) of \( T \) is \( W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \} \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be convexoid if \( W(T) = \text{co} \sigma(T) \), that is, the closure of \( W(T) \) is equal to the convex hull of \( \sigma(T) \). An operator \( T \) is called power bounded if there exists a positive number \( M \) such that \( \|T^n\| \leq M \) for all \( n \in \mathbb{N} \).

**Theorem 3.10.** Let \( T \) be a \([2, C]\)-isometric operator. If \( T \) is power bounded and \( \text{CTC} \cdot T - I \) is convexoid, then \( T \) is a \([1, C]\)-isometric operator.

**Proof.** For the proof, we will show that \( W(\text{CTC} \cdot T - I) = \{0\} \). Assume that \( W(\text{CTC} \cdot T - I) \neq \{0\} \). Since \( \text{CTC} \cdot T - I \) is convexoid, it holds \( W(\text{CTC} \cdot T - I) = \text{co} \sigma(\text{CTC} \cdot T - I) \). Then there exist a non-zero \( a \in \mathbb{C} \) and a sequence \( \{x_n\} \) of unit vectors in \( \mathcal{H} \) such that \( \lim_{n\to\infty} (\text{CTC} \cdot T - I)a x_n = 0 \). Since \( T \) is a \([2, C]\)-isometric operator, it holds \( \lim_{n\to\infty} (\text{CTC} \cdot T^n - (1 + na)) x_n = 0 \). Inductively, we have

\[
\lim_{n\to\infty} (\text{CTC} \cdot T^n - (1 + na)) x_n = 0.
\]

Therefore, it holds that \( \|\text{CTC} \cdot T^n\| \geq |1 + na| \). Since \( a \neq 0 \), it follows that \( \lim_{n\to\infty} |1 + na| = \infty \). Since \( T \) is power bounded, so is \( \text{CTC} \cdot T^n \) and hence it is a contradiction. \( \square \)

4. **Tensor Products of \([m, C]\)-Isometric Operators**

For a complex Hilbert space \( \mathcal{H} \), let \( \mathcal{H} \otimes \mathcal{H} \) denote the completion of the algebraic tensor product of \( \mathcal{H} \) and \( \mathcal{H} \) endowed a reasonable uniform cross-norm. For operators \( T \in \mathcal{L}(\mathcal{H}) \) and \( S \in \mathcal{L}(\mathcal{H}) \), \( T \otimes S \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \) denote the tensor product operator defined by \( T \) and \( S \). Note that \( T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I) \). Then B. Duggal in [9] proved the following result.

**Proposition 4.1.** (Theorem 2.10, [9]) Let \( T \) and \( S \) be an \( m \)-isometry and an \( n \)-isometry on \( \mathcal{H} \), respectively. Then \( T \otimes S \) is an \((m + n - 1)\)-isometry on \( \mathcal{H} \otimes \mathcal{H} \).

Since Duggal’s proof is long and difficult, we firstly give a short proof. A pair of operators \((T, S)\) is said to be a doubly commuting pair if \((T, S)\) satisfies \( TS = ST \) and \( T^* S = S T^* \). Then, for a doubly commuting pair \((T, S)\), it holds

\[
\beta_m(TS) = \sum_{k=0}^{m} \binom{m}{k} T^{*k} \cdot \beta_{m-k}(T) \cdot T^k \cdot \beta_k(S). \tag{5}
\]

Equation (5) is a result of Lemma 3.1 of [3]. It comes from the following equation;

\[
(ab - 1)^n = (a - 1 + a(b - 1))^n = \sum_{k=0}^{m} \binom{m}{k} (a - 1)^{m-k} a^k (b - 1)^k.
\]

**Proposition 4.2.** Let \( T \) and \( S \) be an \( m \)-isometry and an \( n \)-isometry on \( \mathcal{H} \), respectively. If \((T, S)\) is a doubly commuting pair, then \( TS \) is an \((m + n - 1)\)-isometry on \( \mathcal{H} \).

**Proof.** By Equation (5), we have

\[
\beta_{m+n-1}(TS) = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} T^{*k} \cdot \beta_{m+n-1-k}(T) \cdot T^k \cdot \beta_k(S).
\]

(i) If \( 0 \leq k \leq n - 1 \), then \( m + n - 1 - k \geq m \) and hence \( \beta_{m+n-1-k}(T) = 0 \).

(ii) If \( k \geq n \), then \( \beta_k(S) = 0 \).

Therefore, \( \beta_{m+n-1}(TS) = 0 \) and so \( TS \) is an \((m + n - 1)\)-isometry. \( \square \)
Proof of Proposition 4.1. It is clear that \( T \otimes I \) and \( I \otimes S \) are an \( m \)-isometry and an \( n \)-isometry on \( \mathcal{H} \otimes \mathcal{H} \), respectively. Since \( (T \otimes I, I \otimes S) \) is a doubly commuting pair, by Proposition 4.2, \( (T \otimes I)(I \otimes S) = T \otimes S \) is an \((m + n - 1)\)-isometry on \( \mathcal{H} \otimes \mathcal{H} \).

Next we show following similar result of Proposition 4.1. For \([m, C]\)-operators, let \((T, S)\) be a commuting pair and satisfies \( S \cdot CTC = CTC \cdot S \), where \( C \) is a conjugation. Then it holds

\[
\lambda_m(TS; C) = \sum_{k=0}^{m} \binom{m}{k} CT^k C \cdot \lambda_{m-k}(T; C) \cdot T^k \cdot \lambda_k(S; C).
\]

(6)

Then, by a similar proof of Proposition 4.2, we have

**Theorem 4.3.** Let \( T \) and \( S \) be an \([m, C]\)-isometry and an \([n, C]\)-isometry on \( \mathcal{H} \), respectively. If \((T, S)\) is a commuting pair and satisfies \( S \cdot CTC = CTC \cdot S \), then \( TS \) is an \([m + n - 1, C]\)-isometry on \( \mathcal{H} \).

**Proof.** By Equation (6), it holds

\[
\lambda_{m+n-1}(TS; C) = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} CT^k C \cdot \lambda_{m+n-1-k}(T; C) \cdot T^k \cdot \lambda_k(S; C).
\]

Hence \( TS \) is an \([m + n - 1, C]\)-isometry on \( \mathcal{H} \). \( \square \)

**Theorem 4.4.** Let \( T \) and \( S \) be an \([m, C]\)-isometry and an \([n, D]\)-isometry on \( \mathcal{H} \), respectively. Then \( T \otimes S \) is an \([m + n - 1, C \otimes D]\)-isometry on \( \mathcal{H} \otimes \mathcal{H} \).

For conjugations \( C \) and \( D \) on \( \mathcal{H} \), we define \( C \otimes D \) on \( \mathcal{H} \otimes \mathcal{H} \) by

\[
(C \otimes D)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \overline{\alpha_j} Cx_j \otimes Dy_j.
\]

First we prepare the following lemma.

**Lemma 4.5.** Let \( C \) and \( D \) be conjugations on \( \mathcal{H} \). Then \( C \otimes D \) is a conjugation on \( \mathcal{H} \otimes \mathcal{H} \).

**Proof.** Let \( x = \sum_{i=1}^{n} \alpha_i x_i^1 \otimes x_i^2 \) and \( y = \sum_{j=1}^{m} \beta_j y_j^1 \otimes y_j^2 \in \mathcal{H} \otimes \mathcal{H} \) where \( \alpha_i, \beta_j \in \mathbb{C} \). Since \( C \) and \( D \) are isometric, it follows that

\[
\langle (C \otimes D)x, (C \otimes D)y \rangle = \langle (C \otimes D)(\sum_{i=1}^{n} \alpha_i x_i^1 \otimes x_i^2), (C \otimes D)(\sum_{j=1}^{m} \beta_j y_j^1 \otimes y_j^2) \rangle
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{\alpha_i} \langle Cx_i^1, Cx_i^2 \rangle \cdot \overline{\beta_j} \langle Dy_j^1, Dy_j^2 \rangle
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{\alpha_i} \langle x_i^1, x_i^2 \rangle \cdot \overline{\beta_j} \langle y_j^1, y_j^2 \rangle
\]

\[
= \langle \sum_{j=1}^{m} \overline{\beta_j} y_j^1 \otimes y_j^2, \sum_{i=1}^{n} \overline{\alpha_i} x_i^1 \otimes x_i^2 \rangle = \langle y, x \rangle.
\]

(7)

Moreover, since \( C \) and \( D \) are involutive, it follows that

\[
(C \otimes D)^2 = (C^2 \otimes D^2) = I \otimes I
\]

(8)

on the algebraic tensor product of \( \mathcal{H} \otimes \mathcal{H} \). Since \( C \) and \( D \) are bounded, it follows from (7) and (8) that \( C \otimes D \) is a conjugation on \( \mathcal{H} \otimes \mathcal{H} \). \( \square \)
Proof of Theorem 4.4. By Lemma 4.5, $C \otimes D$ is a conjugation. It is clear that $T \otimes I$ and $I \otimes S$ are $[m, C \otimes D]$-isometry and $[n, C \otimes D]$-isometry on $\mathcal{H} \otimes \mathcal{H}$, respectively. Since $(T \otimes I, I \otimes S)$ is a commuting pair and satisfies

$$(I \otimes S) \cdot ((C \otimes D)(T \otimes I)(C \otimes D)) = ((C \otimes D)(T \otimes I)(C \otimes D)) \cdot (I \otimes S),$$

by Theorem 4.3, $(T \otimes I)(I \otimes S) = T \otimes S$ is an $[m + n - 1, C \otimes D]$-isometry.

For an $(m, C)$-isometric operator $T$, $\Lambda_m(T; C)$ is defined by

$$\Lambda_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{m-j} \cdot C T C \cdot S^\ast C \cdot \Lambda_k(S; C).$$

Let a commuting pair $(T, S)$ satisfy $S^\ast \cdot C T C = C T C \cdot S^\ast$, where $C$ is a conjugation. Then it holds

$$\Lambda_m(TS; C) = \sum_{k=0}^{m} \binom{m}{k} T^{m-k} \cdot \Lambda_{m-k}(T; C) \cdot C T C \cdot \Lambda_k(S; C).$$

By similar proofs of Proposition 4.2 and Theorems 4.3 and 4.4, we have following results.

Theorem 4.6. Let $T$ and $S$ be an $(m, C)$-isometry and an $(n, C)$-isometry on $\mathcal{H}$, respectively. If $(T, S)$ is a commuting pair and satisfies $S^\ast \cdot C T C = C T C \cdot S^\ast$, then $T \otimes S$ is an $(m + n - 1, C \otimes D)$-isometry on $\mathcal{H} \otimes \mathcal{H}$.

Proof. The proof follows from Equation (9).

Theorem 4.7. Let $T$ and $S$ be an $(m, C)$-isometry and an $(n, D)$-isometry on $\mathcal{H}$, respectively. Then $T \otimes S$ is an $(m + n - 1, C \otimes D)$-isometry on $\mathcal{H} \otimes \mathcal{H}$.

Proof. Operators $T \otimes I$ and $I \otimes S$ are $(m, C \otimes D)$-isometry and $(n, C \otimes D)$-isometry on $\mathcal{H} \otimes \mathcal{H}$, respectively. Since $(T \otimes I, I \otimes S)$ is a commuting pair and satisfies $(I \otimes S)^\ast \cdot ((C \otimes D)(T \otimes I)(C \otimes D)) = ((C \otimes D)(T \otimes I)(C \otimes D)) \cdot (I \otimes S)^\ast$, by Theorem 4.6 $(T \otimes I)(I \otimes S) = T \otimes S$ is an $(m + n - 1, C \otimes D)$-isometry on $\mathcal{H} \otimes \mathcal{H}$.

References