A Trace Formula for Discontinuous Eigenvalue Problem

Fatma Hıra, Nihat Altınısık

Hitit University, Faculty of Arts and Sciences, Department of Mathematics, 19030 Çorum, TURKEY
Ondokuz Mayıs University, Faculty of Arts and Sciences, Department of Mathematics, 55139 Samsun, TURKEY

Abstract. In this paper, we deal with a Sturm-Liouville problem which has discontinuity at one point and contains an eigenparameter in a boundary condition. We obtain a regularized trace formula for the problem.

1. Introduction

Consider the boundary value problem

$$\tau(u) := -y'' + q(x)y = \lambda y, \quad x \in I,$$  \hspace{1cm} (1.1)

with boundary conditions

$$y(0) = 0,$$  \hspace{1cm} (1.2)

$$y'(\pi) - \lambda y(\pi) = 0,$$  \hspace{1cm} (1.3)

and transmission conditions

$$y\left(\frac{\pi}{2} + 0\right) = a_1 y\left(\frac{\pi}{2} - 0\right),$$  \hspace{1cm} (1.4)

$$y'\left(\frac{\pi}{2} + 0\right) = a_1^{-1} y'\left(\frac{\pi}{2} - 0\right) + a_2 y\left(\frac{\pi}{2} - 0\right) = 0,$$  \hspace{1cm} (1.5)

where $I := \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, $\lambda$ is an eigenparameter, $q(x)$ is a real valued function which is continuous in $\left[0, \frac{\pi}{2}\right]$ and $\left(\frac{\pi}{2}, \pi\right]$ and has finite limits $q\left(\frac{\pi}{2} \pm 0\right) := \lim_{x \rightarrow \frac{\pi}{2} \pm 0} q(x)$, $a_1, a_2$ are real numbers and $a_1 > 0$.

Gelfand and Levitan [9] first calculated the regularized trace for the classical Sturm-Liouville problem. After this work, developing trace formulas for continuous problems were investigated by many authors (see [1, 2, 4-8, 10-13, 17, 21]). The history and the current state of the theory of the regularized traces of the linear operators were presented in the survey paper [16]. As far as we know, there are a few works about the regularized trace of discontinuous eigenvalue problems (see [19, 20]). In [20], the author obtained some formulas for the regularized traces of similar problem that none of the boundary conditions contains an eigenparameter.

This paper is organized as follows: Firstly, the asymptotic formulas of the eigenvalues and eigenfunctions are derived. Then the regularized trace formula for the problem (1.1)-(1.5) is obtained similar to the techniques of [14,15].

2010 Mathematics Subject Classification. Primary 34B24; Secondary 34L20, 47E05

Keywords. Regularized trace formula, Gelfand-Levitan trace formula, discontinuous eigenvalue problem.

Received: 04 February 2016; Accepted: 18 October 2016

Communicated by Dragan S. Djordjević, Jelena Manojlović

Email addresses: fatmahira@yahoo.com.tr or fatmahira@hitit.edu.tr (Fatma Hıra), anihat@omu.edu.tr (Nihat Altınısık)
2. Preliminaries

The asymptotic formulas of the eigenvalues and eigenfunctions can be derived similar to the classical techniques of [3, 18].

We will define the solution of (1.1) by

\[
\phi(x, \lambda) = \begin{cases} 
\phi_1(x, \lambda), & x \in \left[0, \frac{\pi}{2}\right), \\
\phi_2(x, \lambda), & x \in \left[\frac{\pi}{2}, \pi\right],
\end{cases}
\]

(2.1)
as follows: Let \( \phi_1(x, \lambda) \) be the solution of (1.1) on \( \left[0, \frac{\pi}{2}\right) \) which satisfies the initial conditions

\[
y(0, \lambda) = 0, \quad y'(0, \lambda) = 1.
\]

(2.2)

After defining this solution, we define the solution \( \phi_2(x, \lambda) \) of (1.1) on \( \left[\frac{\pi}{2}, \pi\right] \) by means of the solution \( \phi_1(x, \lambda) \) by the initial conditions

\[
y\left(\frac{\pi}{2}, \lambda\right) = a_1 \phi_1\left(\frac{\pi}{2}, \lambda\right), \quad y'(\frac{\pi}{2}, \lambda) = a_1^{-1} \phi_1'(\frac{\pi}{2}, \lambda) + a_2 \phi_1\left(\frac{\pi}{2}, \lambda\right).
\]

(2.3)

Consequently, \( \phi(x, \lambda) \) satisfies of (1.1) on \( I \), the boundary condition (1.2) and the transmission conditions (1.4) and (1.5).

Let \( \lambda = s^2 \). Then the following integral equations hold for \( j = 0 \) and \( j = 1 \):

\[
\phi_j(x, \lambda) = \frac{1}{s} (\sin sx)^{(j)} + \frac{1}{s} \int_0^x (\sin s(x-t))^{(j)} q(t) \phi_1(t, \lambda) \, dt,
\]

(2.4)

and

\[
\phi_j(x, \lambda) = \phi_2\left(\frac{\pi}{2}, \lambda\right)\cos s\left(x - \frac{\pi}{2}\right)^{(j)} + \frac{1}{s} \phi_j'\left(\frac{\pi}{2}, \lambda\right)\left(\sin s\left(x - \frac{\pi}{2}\right)\right)^{(j)}
\]

\[
+ \frac{1}{s} \int_{\frac{\pi}{2}}^x (\sin s(x-t))^{(j)} q(t) \phi_2(t, \lambda) \, dt.
\]

(2.5)

Solving the equations (2.4) and (2.5) by the method of successive approximations, we obtain the following asymptotic representation for \( |\lambda| \to \infty \):

\[
\phi_1(x, \lambda) = \frac{1}{s} \sin sx - \frac{1}{s^2} Q_1(x) \cos sx + \frac{1}{s^3} \frac{q(x) + q(0)}{4} \sin sx + O\left(\frac{e^{ln|x|}}{s^4}\right),
\]

(2.6)

\[
\phi_1'(x, \lambda) = \cos sx - \frac{1}{s^2} Q_1(x) \sin sx - \frac{1}{s^3} \frac{q(x) - q(0)}{4} \cos sx + \frac{1}{s^3} \frac{q'(x) + q'(0)}{8} \sin sx
\]

\[
+ O\left(\frac{e^{ln|x|}}{s^4}\right).
\]

(2.7)

and

\[
\phi_2(x, \lambda) = \frac{1}{s} \left[ A_1 \sin sx - A_2 \sin s(x - \pi) \right] - \frac{1}{s^2} \left[ A_1 \left(Q_1\left(\frac{x}{2}\right) + Q_2(x)\right) + \frac{a_2}{2} \right] \cos sx
\]

\[
+ A_2 \left(Q_1\left(\frac{x}{2}\right) - Q_2(x)\right) - \frac{a_2}{2} \cos s(x - \pi)
\]

\[
+ \frac{1}{s^3} \left[ A_1 \left(\frac{q(x) + q(0)}{4}\right) - \frac{Q_1\left(\frac{x}{2}\right) - Q_2(x)}{2} \left(Q_1\left(\frac{x}{2}\right) - Q_2(x)\right) \right] \sin sx
\]

\[
- \frac{1}{s^3} \left[ A_2 \left(\frac{q(x) + q(0)}{4}\right) - \frac{Q_1\left(\frac{x}{2}\right) + Q_2(x)}{2} \left(Q_1\left(\frac{x}{2}\right) + Q_2(x)\right) \right] \sin s(x - \pi)
\]

\[
+ O\left(\frac{e^{ln|x|}}{s^4}\right),
\]

(2.8)
Using the Rouche theorem in (2.12), we obtain
\[ \phi^2(x, \lambda) = A_1 \cos sx - A_2 \cos (x - \pi) + \frac{1}{s} \left( \left[ A_1 \left( Q_1 \left( \frac{x}{\pi} \right) + Q_2(x) \right) + \frac{a_2}{2} \right] \sin sx + \right. \]
\[ \left. + A_2 \left( Q_1 \left( \frac{x}{\pi} \right) - Q_2(x) \right) - \frac{a_2}{2} \right] \sin s (x - \pi) \right) + \frac{1}{s^2} \left[ -A_1 \left( \frac{q(x) - q(\pi)}{4} + \right. \right. \]
\[ \left. + Q_1 \left( \frac{x}{\pi} \right) Q_2(x) - \frac{a_2}{2} \left( Q_1 \left( \frac{x}{\pi} \right) + Q_2(x) \right) \right] \cos sx + A_2 \left( \frac{q(x) - q(\pi)}{4} \right) \]
\[ - Q_1 \left( \frac{x}{\pi} \right) Q_2(x) - \frac{a_2}{2} \left( Q_1 \left( \frac{x}{\pi} \right) - Q_2(x) \right) \right] \cos s (x - \pi) + O \left( \frac{e^{i\mu x}}{s^3} \right), \]
where
\[ Q_1(x) = \frac{1}{2} \int_0^x q(t) dt, \quad Q_2(x) = \frac{1}{2} \int_0^x q(t) dt, \quad A_1 = \frac{a_1 + a_1^{-1}}{2}, \quad A_2 = \frac{a_1 - a_1^{-1}}{2}. \]

Since the function \( \phi(x, \lambda) \) satisfies the boundary condition (1.2) and the transmission conditions (1.4) and (1.5) to find the eigenvalues of the problem (1.1)-(1.5), we have to insert the function \( \phi(x, \lambda) \) in the boundary condition (1.3) and find the roots of this equation. It is obvious that the characteristic function \( \omega(\lambda) \) of the problem (1.1)-(1.5) is as follows
\[ \omega(\lambda) = \phi^2(\pi, \lambda) - s^2 \phi_2(\pi, \lambda), \]
and the eigenvalues of the problem (1.1)-(1.5) coincide with the roots of \( \omega(\lambda) \). Using equations (2.8) and (2.9), we obtain
\[ \omega(\lambda) = -s A_1 \sin \pi s + \left[ A_1 \left( 1 + Q_1 \left( \frac{x}{\pi} \right) + Q_2(x) \right) + \frac{a_2}{2} \right] \cos \pi s \]
\[ + A_2 \left( Q_1 \left( \frac{x}{\pi} \right) - Q_2(x) \right) - \frac{a_2}{2} \right] + \frac{1}{s} \left[ A_1 \left[ Q_1 \left( \frac{x}{\pi} \right) + Q_2(x) \right] + \right. \]
\[ + Q_1 \left( \frac{x}{\pi} \right) Q_2(x) - \frac{a_2}{2} \left( Q_1 \left( \frac{x}{\pi} \right) + Q_2(x) \right) \right] \cos s (x - \pi) + O \left( \frac{e^{i\mu x}}{s^3} \right). \]

Using the Rouche theorem in (2.12), we obtain
\[ s_n = n + \frac{C}{n A_1} + O \left( \frac{1}{n^3} \right), \]
where
\[ C = A_1 \left( 1 + Q_1 \left( \frac{\pi}{2} \right) + Q_2(x) \right) + \frac{a_2}{2} + (-1)^n \left[ A_2 \left( Q_1 \left( \frac{\pi}{2} \right) - Q_2(x) \right) - \right. \]
\[ \left. + Q_1 \left( \frac{\pi}{2} \right) Q_2(x) - \frac{a_2}{2} \left( Q_1 \left( \frac{\pi}{2} \right) + Q_2(x) \right) \right] \sin \pi s \]
\[ + O \left( \frac{e^{i\mu x}}{s^3} \right). \]

It follows from (2.13) that
\[ \lambda_n = n^2 + \frac{2C}{\pi A_1} + O \left( \frac{1}{n^2} \right). \]

3. Traces of the Problem

The series
\[ \sum_{n=0}^\infty \left( \lambda_n - n^2 - \frac{2C}{\pi A_1} \right) < \infty, \]
converges and is called the regularized first trace for the problem (1.1)-(1.5). The goal of this paper is to find its sum.
Theorem 3.1. Suppose that \( q(x) \) has a second-order piecewise integrable derivatives on \([0, \pi]\), then the following regularized trace formula holds

\[
s_\lambda = \sum_{n=1}^{\infty} \left( \frac{\lambda_n - n^2 - 2C}{\pi A_1} \right)
= Q_1 \left( \frac{x}{2} \right) + Q_2 (\pi) - \frac{q(\pi) + q(0)}{4} + \frac{\alpha_2}{A_1} \left( 1 + Q_1 \left( \frac{x}{2} \right) \right) - \frac{C}{\pi A_1} - \frac{C^2}{2A_1^2},
\]

(3.1)

where \( A_i, Q_i(x) \) (i = 1, 2) and \( C \) satisfy the equations (2.10) and (2.14).

Proof. Since \( \omega(\lambda) \) is an entire function from Hadamard’s theorem (see [11]), using (2.11) we have

\[
\omega(\lambda) = A \Phi(\lambda),
\]

(3.2)

where \( \Phi(\lambda) = \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right) \) and \( A \) is a certain constant to be determined below.

Let \( \lambda = -\mu^2 \). We calculate the sum of the series (3.1) by comparing the asymptotic expressions obtained from (3.2) on the right and left.

Put

\[
\Phi(-\mu^2) = \prod_{n=0}^{\infty} \left( 1 + \frac{\mu^2}{\lambda_n} \right) = \left( \frac{\lambda_0 + \mu^2}{\mu \pi} \right) B \Psi(\mu^2) \sinh \mu \pi,
\]

(3.3)

where

\[
B = \frac{1}{\lambda_0} \prod_{n=1}^{\infty} \left( \frac{n^2}{\lambda_n} \right), \quad \Psi(\mu^2) = \prod_{n=1}^{\infty} \left( 1 - \frac{n^2 - \lambda_n}{\mu^2 + n^2} \right).
\]

(3.4)

To study the asymptotic behaviour of \( \Psi(\mu^2) \) as \( \mu \to \infty \), we consider

\[
\ln \Psi(\mu^2) = \sum_{n=1}^{\infty} \ln \left( 1 - \frac{n^2 - \lambda_n}{\mu^2 + n^2} \right)
= -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{n^2 - \lambda_n}{\mu^2 + n^2} \right)^k
= -\sum_{n=1}^{\infty} \frac{n^2 - \lambda_n}{\mu^2 + n^2} - \sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left( \frac{n^2 - \lambda_n}{\mu^2 + n^2} \right)^k
= \sum_{n=1}^{\infty} \frac{C_1}{\mu^2 + n^2} + \sum_{n=1}^{\infty} \frac{\sum_{k=2}^{\infty} \left( \frac{n^2 - \lambda_n}{\mu^2 + n^2} \right)^k}{\mu^2 + n^2}

\]

(3.5)

where \( C_1 = \frac{2C}{\pi A_1} \).

Asymptotic expressions can be obtained according to the following lemma (similar to [15, Ch5]).

Lemma 3.2. If \( |n^2 - \lambda_n| \leq \rho \), then

\[
\sum_{n=1}^{\infty} \frac{|n^2 - \lambda_n|^k}{(\mu^2 + n^2)^k} \leq \frac{\pi}{2} \frac{\rho^k}{\mu^{2k-1}}.
\]

(3.6)
It follows from (3.6) that
\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{|n^2 - \lambda_n|^k}{(\mu^2 + n^2)^k} \leq \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{\rho^k}{\mu^{2k-1}} = \frac{\pi}{2} \frac{\rho^2}{\mu^2} \sum_{k=0}^{\infty} \left( \frac{\rho}{\mu^2} \right)^k = O \left( \frac{1}{\mu^3} \right),
\]
(3.7)
and since \( \sup_n |\lambda_n - n^2 - C_1| n^2 < \infty \), it follows from (3.6) that
\[
\frac{1}{\mu^2} \sum_{n=1}^{\infty} \frac{\lambda_n - n^2 - C_1}{\mu^2 + n^2} = O \left( \frac{1}{\mu^3} \right).
\]
(3.8)
As we know,
\[
\sum_{n=1}^{\infty} \frac{1}{\mu^2 + n^2} = \frac{\pi \coth \mu \pi}{2\mu} - \frac{\pi}{2\mu} - \frac{1}{2\mu^2} + O(e^{-2\mu \pi}).
\]
(3.9)
Therefore, substituting (3.7)-(3.9) into (3.5) then, we obtain
\[
\ln \Psi (\mu^2) = \frac{\pi C_1}{2\mu} + \frac{1}{\mu^2} \left( s_\lambda - \lambda_0 + \frac{C_1}{2} \right) + O \left( \frac{1}{\mu^3} \right),
\]
where
\[
s_\lambda = \sum_{n=0}^{\infty} \left( \lambda_n - n^2 - C_1 \right).
\]
(3.10)
Therefore, we get
\[
\Psi (\mu^2) = \exp \left\{ \frac{\pi C_1}{2\mu} + \frac{1}{\mu^2} \left( s_\lambda - \lambda_0 + \frac{C_1}{2} \right) + O \left( \frac{1}{\mu^3} \right) \right\}
= 1 + \frac{\pi C_1}{2\mu} + \frac{1}{\mu^2} \left( s_\lambda - \lambda_0 + \frac{C_1}{2} + \frac{\pi^2 C_1^2}{8} \right) + O \left( \frac{1}{\mu^3} \right).
\]
(3.11)
Relying on (3.11), then we derive from (3.3) that
\[
\Phi (-\mu^2) = \frac{1}{2\pi} Be^{i\pi \eta} \left\{ \mu + \frac{\pi C_1}{2} + \frac{1}{\mu} \left( s_\lambda + \frac{C_1}{2} + \frac{\pi^2 C_1^2}{8} \right) + O \left( \frac{1}{\mu^2} \right) \right\}.
\]
(3.12)
We now study the asymptotic behaviour of the function
\[
\omega (-\mu^2) = \phi_2 (\pi, -\mu^2) + \mu^2 \phi_2 (\pi, -\mu^2),
\]
using the Liouville equation. Then according to formula (2.8) and (2.9), we have
\[
\phi_2 (x, -\mu^2) = \frac{1}{\mu} \left\{ A_1 \sinh \mu x - A_2 \sinh \mu (x - \pi) \right\} + \frac{1}{\mu^2} \left\{ A_1 \left( Q_1 \left( \frac{x}{2} \right) + Q_2 (x) \right) + \phi_2 \left( \frac{x}{2}, -\mu^2 \right) \right\} \cosh \mu x + O \left( \frac{1}{\mu^3} \right),
\]
(3.14)
and
\[
\phi'_2(x, -\mu^2) = A_1 \cosh \mu x - A_2 \cosh \mu (x - \pi) + \frac{1}{\mu} \left\{ A_1 \left( Q_1 \left( \frac{\pi}{2} \right) + Q_2 (\pi) \right) + \frac{a_2}{2} \right\} \sinh \mu x \\
+ \left[ A_2 \left( Q_1 \left( \frac{\pi}{2} \right) - Q_2 (\pi) \right) - \frac{a_2}{2} \right] \sinh \mu (x - \pi) + \frac{1}{\mu^2} \left\{ A_1 \left[ q (x) + q (0) \right] - \frac{a_2}{2} \right\} \cosh \mu (x - \pi) \\
+ \frac{a_2}{2} Q_1 \left( \frac{\pi}{2} \right) \cosh \mu x + \left[ \frac{a_2}{2} Q_1 \left( \frac{\pi}{2} \right) - A_2 \frac{q (x) - q (0)}{4} \right] \cosh \mu (x - \pi) \\
+ O \left( \frac{1}{\mu^3} \right).
\]  \tag{3.15}

Putting \( \phi_2(x, -\mu^2) \) and \( \phi'_2(x, -\mu^2) \) at \( \pi \) and also using formulas \( \sinh \mu \pi = \frac{\mu^\pi}{2} + O \left( \frac{1}{e^{2/2}} \right) \), \( \cosh \mu \pi = \frac{\mu^\pi}{2} + O \left( \frac{1}{e^{2/2}} \right) \) into (3.13), we have
\[
\omega \left( -\mu^2 \right) = \frac{\mu A_1 + A_1 \left( 1 + Q_1 \left( \frac{\pi}{2} \right) + Q_2 (\pi) \right) + \frac{a_2}{2} + (-1)^n \left( A_2 \left( Q_1 \left( \frac{\pi}{2} \right) - Q_2 (\pi) - 1 \right) - \frac{a_2}{2} \right) + \frac{1}{\mu} \left\{ A_1 \left[ Q_1 \left( \frac{\pi}{2} \right) + Q_2 (\pi) \right] - \frac{q (\pi) + q (0)}{4} \right\} + \frac{a_2}{2} \left( 1 + Q_1 \left( \frac{\pi}{2} \right) \right) \right\} + O \left( \frac{1}{\mu^2} \right). \tag{3.16}
\]

It follows from the equalities (3.2), (3.12), (3.16) and comparing the coefficients of \( \mu \), we obtain
\[
AB = A_1, \\
\frac{AB}{\pi} + \frac{1}{2} = A_1 \left( 1 + Q_1 \left( \frac{\pi}{2} \right) + Q_2 (\pi) \right) + \frac{a_2}{2} + (-1)^n \left( A_2 \left( Q_1 \left( \frac{\pi}{2} \right) - Q_2 (\pi) - 1 \right) - \frac{a_2}{2} \right), \tag{3.17}
\]

Therefore, we obtain
\[
s_1 = \sum_{n=0}^{\infty} \left( \lambda_n - n^2 - \frac{2C}{nA_1} \right) = Q_1 \left( \frac{\pi}{2} \right) + Q_2 (\pi) - \frac{q (\pi) + q (0)}{4} + \frac{a_2}{2A_1} \left( 1 + Q_1 \left( \frac{\pi}{2} \right) \right) - \frac{C}{\pi A_1} - \frac{C}{2A_1^2}.
\]

where
\[
C = A_1 \left( 1 + Q_1 \left( \frac{\pi}{2} \right) + Q_2 (\pi) \right) + \frac{a_2}{2} + (-1)^n \left( A_2 \left( Q_1 \left( \frac{\pi}{2} \right) - Q_2 (\pi) - 1 \right) - \frac{a_2}{2} \right),
\]
\[
A_1 = \frac{a_1 + a_1^{-1}}{2}, \quad A_2 = \frac{a_1 - a_1^{-1}}{2}, \quad Q_1 \left( \frac{\pi}{2} \right) = \frac{1}{2} \int_{0}^{\pi} q (t) \ dt, \quad Q_2 (\pi) = \frac{1}{2} \int_{\pi/2}^{\pi} q (t) \ dt,
\]

completing the proof of Theorem 3.1. \( \square \)

References


