Orthogonal Polynomials Associated with an Inverse Spectral Transform. The Cubic Case

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Abstract. The purpose of this work is to give some new algebraic properties of the orthogonality of a monic polynomial sequence \( \{Q_n\}_{n \geq 0} \) defined by
\[
Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) + r_n P_{n-3}(x), \quad n \geq 1,
\]
where \( r_n \neq 0, n \geq 3 \), and \( \{P_n\}_{n \geq 0} \) is a given sequence of monic orthogonal polynomials. Essentially, we consider some cases in which the parameters \( r_n, s_n, \) and \( t_n \) can be computed more easily. Also, as a consequence, a matrix interpretation using \( \text{LU} \) and \( \text{UL} \) factorization is done. Some applications for Laguerre, Bessel and Tchebychev orthogonal polynomials of second kind are obtained.

1. Introduction and Preliminaries

Let \( \{P_n\}_{n \geq 0} \) be a sequence of monic orthogonal polynomials with respect to a regular linear functional \( \mu \). We define a new sequence of monic polynomials \( \{Q_n\}_{n \geq 0} \) by the \( M-N \) type linear structure relation
\[
Q_n(x) + \sum_{i=1}^{M-1} a_{i,n} Q_{n-i}(x) = P_n(x) + \sum_{i=1}^{N-1} b_{i,n} P_{n-i}(x), \quad n \geq 1,
\]
where \( M \) and \( N \) are fixed positive integer numbers, and \( \{a_{i,n}\}_n \) and \( \{b_{i,n}\}_n \) are sequences of complex numbers with \( a_{M-1,n} b_{N-1,n} \neq 0 \). The study of the regularity of the sequence \( \{Q_n\}_{n \geq 0} \) is said to be an inverse problem. This problem has been studied in some particular cases. Indeed, the relations of types 1-2 and 2-1 have been studied in [9], the 1-3 type relation in [2], the 2-2 type relation in [4] and the 2-3 type relation in [1]. In addition, the 1-N type relation with constant coefficients has been analyzed in [3]. Recently, in [8] and for \( M = 1, N = 4 \), F. Marcellán and S. Varma determine necessary and sufficient conditions such that \( \{Q_n\}_{n \geq 0} \) becomes also orthogonal. This article is a continuation of [8]. It deals with some new results about the sequence \( \{Q_n\}_{n \geq 0} \) defined by
\[
Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) + r_n P_{n-3}(x), \quad r_n \neq 0, \quad n \geq 3.
\]
Firstly, we give some new results concerning the regularity conditions of the sequence $\{Q_n\}_{n \geq 0}$. In particular, we obtain a new characterization of the orthogonality of this sequence with respect to a linear functional $v^\ast$, in terms of the coefficients of a cubic polynomial $q$ such that $q(x)v^\ast = u$. Indeed, it is known [17] that up to some natural conditions the $M$ - $N$ type structure relation leads to a rational transformation $\Phi u = \Psi v$ where $\Phi$ and $\Psi$ are polynomials. Secondly, since the cases 1-2 and 1-3 type structure relation have been already considered in previous works (see [2, 9]), we obtain necessary and sufficient conditions so that the above 1-4 relation can be decomposed in three 1-2 relations or two relations of types 1-2 and 1-3 and then proceed by iteration. This study is based on the factorization of $q(x)$. We will study the case when $\{P_n\}_{n \geq 0}$ is symmetric and $\{Q_n\}_{n \geq 0}$ is quasi-antisymmetric. In any situation, the matrix interpretation of this problem in terms of monic Jacobi matrices is done carefully. Finally, we give a detailed study of three examples.

Now, we are going to introduce some basic definitions and results. The field of complex numbers is denoted by $\mathbb{C}$. The vector space of polynomials with coefficients in $\mathbb{C}$ is denoted by $\mathcal{P}$ and its dual space is presented as $\mathcal{P}^\ast$. We will simply call polynomial every element of $\mathcal{P}$ and linear functional to the elements in $\mathcal{P}^\ast$. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}^\ast$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the moments of $u$.

For any linear functional $v^\ast$ and any polynomial $h$ let $hv^\ast, \delta_v^\ast$, and $(x - c)^{-1}v^\ast$ be the linear functionals defined by: $\langle hv^\ast, f \rangle := \langle v^\ast, hf \rangle$, $\langle \delta_v^\ast, f \rangle := \langle v^\ast, f \rangle$ and $\langle (x - c)^{-1}v^\ast, f \rangle := \langle v^\ast, \theta_c f \rangle$ where $\theta_c f(x) = \frac{f(x) - f(c)}{x - c}$, $c \in \mathbb{C}$, $f \in \mathcal{P}$. Then, it is straightforward to prove that for $c \in \mathbb{C}$, and $v^\ast \in \mathcal{P}^\ast$, we have [15]

\[ (x - c)^{-1}((x - c)v^\ast) = v^\ast - (v^\ast)_0 \delta_v, \] (1.1)

\[ (x - c)((x - c)^{-1}v^\ast) = v^\ast, \] (1.2)

A linear functional $u^\ast$ is called regular if there exists a sequence of polynomials $\{P_n\}_{n \geq 0}$ (deg $P_n \leq n$) such that $\langle u^\ast, P_n P_m \rangle = r_n \delta_{n,m}$, $r_n \neq 0$, $n \geq 0$.

Then deg $P_n = n$, $n \geq 0$ and we can always suppose each $P_n$ is monic. In such a case, the sequence $\{P_n\}_{n \geq 0}$ is unique. It is said to be the sequence of monic orthogonal polynomials with respect to $u^\ast$. In the sequel it will be denoted as SMOP. It is a very well known fact that the sequence $\{P_n\}_{n \geq 0}$ satisfies the recurrence relation (see, for instance, the monograph by Chihara [6])

\[ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \]
\[ P_1(x) = x - \beta_0, \quad P_0(x) = 1, \] (1.3)

with $(\beta_n, \gamma_{n+1}) \in \mathbb{C} \times \mathbb{C} - \{0\}$, $n \geq 0$. By convention we set $\gamma_0 = (u^\ast)_0$.

The linear functional $u^\ast$ is said to be normalized if $(u^\ast)_0 = 1$. In this paper, we suppose that any linear functional will be normalized.

2. Some Algebraic Properties

In the sequel $\{P_n\}_{n \geq 0}$ denotes a SMOP with respect to a regular linear functional $u^\ast$. By giving three sequences of complex numbers $\{s_n\}_{n \geq 1}$, $\{t_n\}_{n \geq 2}$, and $\{r_n\}_{n \geq 3}$, we define a new sequence of monic polynomials $\{Q_n\}_{n \geq 0}$ such that

\[ Q_1(x) = P_1(x) + s_1, \]
\[ Q_2(x) = P_2(x) + s_2P_1(x) + t_2, \]
\[ Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) + r_n P_{n-3}(x), \quad n \geq 3, \quad \text{with } r_n \neq 0, \ n \geq 3. \] (2.1)

Let us recall the following result:
Theorem 2.1. [8] \( \{Q_n\}_{n \geq 0} \) is an SMOP if and only if as well as \( \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \neq 0 \) with

\[
\begin{align*}
    s_{n-1} \tilde{\gamma}_n &= s_n \gamma_{n-1} + t_n (\beta_{n-2} - \tilde{\beta}_n) + r_n - r_{n+1}, \quad n \geq 2, \\
    t_{n-1} \tilde{\gamma}_n &= t_n \gamma_{n-2} + r_n (\beta_{n-3} - \tilde{\beta}_n), \quad n \geq 3, \\
    r_{n-1} \tilde{\gamma}_n &= r_n \gamma_{n-3}, \quad n \geq 4,
\end{align*}
\]  

where

\[
\begin{align*}
    \tilde{\beta}_n &= \beta_n + s_n - s_{n+1}, \quad n \geq 0, \\
    \tilde{\gamma}_n &= \gamma_n + t_n - t_{n+1} + s_n (\beta_{n-1} - \beta_n - s_n + s_{n+1}), \quad n \geq 1,
\end{align*}
\]

with \( s_0 = t_0 = l_1 = r_0 = t_1 = r_2 = 0. \)

Furthermore, \( \{Q_n\}_{n \geq 0} \) satisfies the three-term recurrence relation

\[
\begin{align*}
    Q_{n+2}(x) &= (x - \tilde{\beta}_{n+1}) Q_{n+1}(x) - \gamma_{n+1} Q_n(x), \quad n \geq 0, \\
    Q_1(x) &= x - \tilde{\beta}_0, \\
    Q_0(x) &= 1.
\end{align*}
\]  

Remark 1. When \( \{Q_n\}_{n \geq 0} \) is an SMOP, then (2.2)-(2.4) can be written as

\[
\begin{align*}
    r_3 &= t_2 (\beta_1 - \beta_2 - s_2 + s_3) + s_2 \gamma_4 - s_1 [\gamma_2 + t_2 - l_3 + s_2 (\beta_1 - \beta_2 - s_2 + s_3)], \\
    r_4 &= r_3 + t_3 (\beta_2 - \beta_3 - s_3 + s_4) + s_3 [\gamma_2 - s_2 [\gamma_3 + t_3 - l_4 + s_3 (\beta_2 - \beta_3 - s_3 + s_4)], \\
    s_{n+5} &= s_{n+4} + \beta_{n+4} - \beta_{n+1} + \frac{t_{n+3}}{t_{n+3}} \gamma_{n+1} - \frac{t_{n+4}}{t_{n+4}} \gamma_{n+2}, \quad n \geq 0, \\
    t_{n+5} &= t_{n+4} + s_{n+4} (\beta_{n+3} - \beta_{n+4} - s_{n+4} + s_{n+5}) - \frac{r_{n+4}}{r_{n+3}} \gamma_{n+1} + \gamma_{n+2}, \quad n \geq 0, \\
    r_{n+5} &= r_{n+4} \left(1 - \frac{\gamma_{n+1}}{t_{n+3} \gamma_{n+3}}\right) + s_{n+4} \gamma_{n+4} + t_{n+4} (\beta_{n+2} - \beta_{n+4} - s_{n+4} + s_{n+5}), \quad n \geq 0,
\end{align*}
\]

where the initial conditions are

\[
\begin{align*}
    (a) \quad &t_2, \ t_3, \ t_4, \ s_1, \ s_2, \ s_3, \text{ and } \left\{ \begin{array}{ll}
        s_4 &= s_3 + \beta_3 - \beta_0 - \frac{s_2 \gamma_1}{t_{2s_3 - r_2}}, \quad \text{if } t_2 = 0, \\
        s_4 &= \frac{(s_3 + \beta_3 - \beta_0 - \frac{s_2 \gamma_1}{t_{2s_3 - r_2}}) t_{2s_3 - r_2}}{t_{2s_3 - r_2}}, \quad \text{if } t_2 \neq 0, \ t_2 s_3 - r_2 \neq 0.
    \end{array} \right. \\
    (b) \quad &t_2, \ t_3, \ s_1, \ s_2, \ s_3, \ s_4, \text{ and } t_4 = s_3 (\beta_2 - \beta_3 - s_3) + \gamma_5 + t_3 - \frac{t_2 \gamma_1 + r_3 (\beta_0 - \beta_3 - s_3)}{t_2}, \quad \text{if } t_2 \neq 0, \ t_5 = t_2 s_3.
\end{align*}
\]

Furthermore, \( t_2, \ t_3, \ t_4, \ s_1, \ s_2, \ s_3, \ s_4 \) verify

\[
\begin{align*}
    \gamma_1 + t_2 - l_3 + s_2 (\beta_1 - \beta_2 - s_2 + s_3) \neq 0, \\
    \gamma_2 + t_2 - l_3 + s_2 (\beta_1 - \beta_2 - s_2 + s_3) \neq 0, \\
    \gamma_3 + t_3 - l_4 + s_3 (\beta_2 - \beta_3 - s_3 + s_4) \neq 0.
\end{align*}
\]

Theorem 2.2. The following statements are equivalent:

(i) \( \{Q_n\}_{n \geq 0} \) is an SMOP with \( (\tilde{\beta}_n)_{n} \) and \( (\tilde{\gamma}_n)_{n} \) given by (2.5) and (2.6) the corresponding series of recurrence coefficients.

(ii) It holds \( \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \neq 0 \) together with the initial conditions (2.8) and

\[
l_2 (\gamma_3 + t_3 - l_4 + s_3 (\beta_2 - \beta_3 - s_3 + s_4)) = t_3 \gamma_1 + r_3 (\beta_2 - \beta_3 - s_3 + s_4).
\]  

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and there exist three complex numbers \(a, b\) and \(c\) such that, for \(n \geq 1\)

\[
A_n := \frac{t_{n+2}}{r_{n+2}} \gamma_n - \beta_n - \beta_{n+1} - \beta_{n+2} + s_{n+3} = a,
\]

\[
B_n := \frac{1}{r_{n+2}} \gamma_n \left[ s_{n+2} \gamma_{n+1} + t_{n+2} (s_{n+3} - \beta_{n+2} - \beta_{n+3}) \right] - \gamma_{n+1} - \gamma_{n+2} - \gamma_{n+3} + t_{n+4}
- \left( s_{n+1} - \beta_{n+2} - \beta_{n+3} \right) (\beta_{n+1} + \beta_n) + s_{n+3} \left( s_{n+3} - s_{n+4} - \beta_{n+2} - \beta_{n+3} \right) - \beta_{n+1}^2 = b,
\]

\[
C_n := \frac{1}{r_{n+2}} \gamma_n \left[ y_{n+1} (\gamma_{n+2} + s_{n+2} (s_{n+3} - \beta_{n+2})) + t_{n+2} \left( s_{n+3} (\beta_{n+3} - \beta_{n+2} + s_{n+3} - s_{n+4}) + \beta_{n+1} (\beta_{n+3} - s_{n+3}) - \gamma_{n+2} - \gamma_{n+3} + t_{n+4} \right) + \gamma_{n+1} (\beta_{n+2} - s_{n+3}) + (s_{n+4} - \beta_{n+3}) (\beta_{n+4} - s_{n+5}) + \beta_{n+1} (\gamma_{n+3} - t_{n+4}) + r_{n+4} = c.
\]

Furthermore, if \(u\) and \(v\) are the linear functionals associated with the sequences \([P_n]_{n \geq 0}\) and \([Q_n]_{n \geq 0}\), respectively, then

\[
q(x)u = -ku,
\]

with \(q(x) = x^3 + ax^2 + bx + c, k \in \mathbb{C} - \{0\}\).

**Proof.** Notice that, by Remark 1, \([Q_n]_{n \geq 0}\) is an SMOP if and only if the condition \(\gamma_1 \gamma_2 \gamma_3 \neq 0\) and the initial conditions (2.8), (2.12) and the above Eqs (2.9)-(2.11) hold. To conclude the proof we need to show that Eqs. (2.13)-(2.15) are equivalent to (2.9)-(2.11).

- We first prove that (2.9)-(2.11) \(\Rightarrow\) (2.13)-(2.15). Using (2.9), we get

\[
A_{n+1} = A_{n+2}, \quad n \geq 0.
\]

Hence (2.13). Now, we will deduce (2.14).

Multiplying the expression (2.11) by \(\gamma_{n+2}/r_{n+4}\), we obtain

\[
\frac{s_{n+3} r_{n+4}}{r_{n+3}} \gamma_{n+1} \gamma_{n+2} + \frac{t_{n+4}}{r_{n+4}} \gamma_{n+2} (s_{n+4} - \beta_{n+2} - \beta_{n+3}) = \frac{s_{n+4}}{r_{n+4}} \gamma_{n+3} \gamma_{n+2} + \frac{t_{n+4}}{r_{n+4}} (s_{n+5} - \beta_{n+3} - \beta_{n+4}) + \left( 1 - \frac{r_{n+5}}{r_{n+4}} \right) \gamma_{n+2}.
\]

Besides, from (2.10) we have, for \(n \geq 0\)

\[
\frac{s_{n+3} r_{n+4}}{r_{n+3}} \gamma_{n+1} \gamma_{n+2} + \frac{t_{n+4}}{r_{n+4}} \gamma_{n+2} (s_{n+4} - \beta_{n+2} - \beta_{n+3}) = \frac{s_{n+4}}{r_{n+4}} \gamma_{n+3} \gamma_{n+2} + \frac{t_{n+4}}{r_{n+4}} (s_{n+5} - \beta_{n+3} - \beta_{n+4}) + \gamma_{n+2} - \gamma_{n+5} - t_{n+6} - s_{n+5} (\beta_{n+4} - \beta_{n+5} - s_{n+5} + s_{n+6}).
\]

Using (2.17) in the expression of \(\frac{s_{n+4} \gamma_{n+2}}{r_{n+4}}\) which appears in the right hand side of the above formula, we obtain

\[
B_{n+2} = B_{n+1}, \quad n \geq 0.
\]

Hence (2.14). Now, we will deduce (2.15).

Multiplying (2.10) by \(\gamma_{n+2} \gamma_{n+3}/r_{n+4}\), we get

\[
\frac{\gamma_{n+1} \gamma_{n+2} \gamma_{n+3}}{r_{n+3}} \gamma_{n+2} + (s_{n+4} - \beta_{n+3}) \frac{s_{n+4}}{r_{n+4}} \gamma_{n+2} \gamma_{n+3} = \frac{\gamma_{n+2} \gamma_{n+3}}{r_{n+4}} \gamma_{n+2} \gamma_{n+3} + (s_{n+5} - \beta_{n+4}) \frac{s_{n+4}}{r_{n+4}} \gamma_{n+2} \gamma_{n+3} + \left[ \frac{t_{n+4} - t_{n+5}}{r_{n+4}} \right] \gamma_{n+2} \gamma_{n+3}.
\]
Using (2.9) and (2.10), we have, for $n \geq 0$
\[
\frac{t_{n+5}}{r_{n+4}} \gamma_{n+2} = \left[ \gamma_{n+5} + t_{n+5} - t_{n+6} + S_{n+5}(B_{n+4} - B_{n+5} - S_{n+5} + S_{n+6}) \right]
\]
and, therefore, for $n \geq 0$
\[
\frac{t_{n+6}}{r_{n+4}} \gamma_{n+2} = \frac{t_{n+5}}{r_{n+4}} \gamma_{n+2} - \beta_{n+5} - S_{n+5} + S_{n+6},
\]
\[
\gamma_{n+1} + \beta_{n+5} - S_{n+5} + S_{n+6}
\]

Hence (2.15).

Then, we deduce (2.10).

Taking into account the new expression of $\gamma_{n+1}$ obtained from (2.18) and $\frac{t_{n+3}}{r_{n+5}} \gamma_{n+1}$ obtained from (2.17) written for $n+1$ instead of $n$, we can reformulate (2.20)
\[
\gamma_{n+2} \frac{t_{n+5}}{r_{n+4}} \left( \gamma_{n+1} + \frac{r_{n+3}}{r_{n+4}} \gamma_{n+1} - S_{n+4}(S_{n+5} + B_{n+5} - B_{n+4} + S_{n+4} - B_{n+5} - S_{n+5}) - t_{n+4} + t_{n+5} - \gamma_{n+4} \right) = 0.
\]

Then, we deduce (2.10).

Taking into account the new expression of $\gamma_{n+1}$ obtained from (2.17), the (2.18) reads as
\[
\gamma_{n+2} \frac{r_{n+4}}{r_{n+3}} \left[ S_{n+3}(S_{n+4} - S_{n+4} - S_{n+4} - B_{n+4} + B_{n+2} - S_{n+4}) - r_{n+4} \right]
\]
\[
= - \gamma_{n+5} - t_{n+5} + t_{n+4} - S_{n+5}(B_{n+5} - B_{n+5} - S_{n+5} + S_{n+6}).
\]

From (2.4) and (2.6), we have
\[
\gamma_{n+2} \frac{r_{n+4}}{r_{n+3}} \left[ S_{n+3}(S_{n+4} - S_{n+4} - S_{n+4} - B_{n+4} + B_{n+2} - S_{n+4}) - r_{n+4} + r_{n+5} \right] = 0,
\]
therefore (2.11) holds.
Let $q$ be an SMOP with respect to a linear functional $w$.

Suppose also that $n_0 \geq 0$, and $n_1 \geq 0$.

Furthermore, $\epsilon_n - \mu_{n+1} - \frac{\rho_n}{\mu_n} = x_1$, $n \geq 1$.

Remark 2. The converse problem, i.e., the analysis of the regularity of a linear functional $v$ such that there exists a polynomial $q(x)$ such that $q(x)v = -ku$, $k \in \mathbb{C} \setminus \{0\}$, has been studied by many authors. In particular, in [10], [11] and [12] the cases $q(x) = x^4$ and $q(x) = x^3$ have been deeply analyzed.

3. Reducible Cases

The next Theorem will play an important role in the sequel.

Theorem 3.1. [9] Let $\{S_n\}_{n \geq 0}$, $\{\mu_n\}_{n \geq 0}$ be a SMOP with respect to a linear functional $w$, $\{\mu_n\}_{n \geq 0}$ a sequence of complex parameters and $\{Z_n\}_{n \geq 0}$ a simple set of monic polynomials, such that

$$Z_n = S_n + \mu_n S_{n-1}, \\ n \geq 1, \quad \text{with } \mu_n \neq 0.$$

Suppose also that $(\varepsilon_n, \rho_n)_{n \geq 0}$ is the set of parameters of the recurrence relation of the sequence $\{S_n\}_{n \geq 0}$. Then, $\{Z_n\}_{n \geq 0}$ is an SMOP with respect to a linear functional $\delta$ if and only there exist complex numbers $x_1 \neq \varepsilon_0 - \mu_1$ such that, for $n \geq 1$,

$$\varepsilon_n - \mu_{n+1} - \frac{\rho_n}{\mu_n} = x_1, \quad n \geq 1.$$

Furthermore,

$$(x - x_1)\delta = (\varepsilon_0 - x_1 - \mu_1)w.$$
Notice that if \(|Q_n|_{n \geq 0}\) satisfies (2.1), then the polynomials \(Q_n\) cannot be represented as a linear combination of the at most three consecutive polynomials \(P_n, P_{n-1}\) and \(P_{n-2}\). A natural question arises: Can the SMOP \(|Q_n|_{n \geq 0}\) be generated from \(|P_n|_{n \geq 0}\) in two or three steps with the help of some intermediate SMOPs? The interest of this question is to simplify the computation of the parameters \(s_n, t_n\) and \(r_n\), in each case via one of the other parameters noted \(a_n, b_n\) and \(c_n\).

From now on, let \(|P_n|_{n \geq 0}\) and \(|Q_n|_{n \geq 0}\) be two SMOPs with respect to the regular linear functionals \(u\) and \(v\), respectively which are related by (2.1).

3.1. The split of a 1-4 relation in three 1-2 relations.

**Proposition 3.2.** Let \(|a_n|_{n \geq 0}\), \(|b_n|_{n \geq 0}\) and \(|c_n|_{n \geq 0}\) be three sequences of nonzero complex numbers. The representation (2.1) can be written as, for \(n \geq 1\)

\[
\begin{align*}
Q_n &= R_n + c_n R_{n-1}, \\
R_n &= R_n + b_n R_{n-1}, \\
P_n &= P_n + a_n P_{n-1},
\end{align*}
\]

(3.4)

where, \(|R_n|_{n \geq 0}\) and \(|\tilde{R}_n|_{n \geq 0}\) are two SMOPs, if and only if there exist two complex numbers \(\alpha, \beta\), such that

\[
\begin{align*}
D_n &= \beta_n - a_n + \frac{\gamma_n}{a_n} = \alpha, \quad n \geq 1, \\
E_n &= \beta_n + a_n - a_n + b_n - b_n - b_n = \beta, \quad n \geq 1,
\end{align*}
\]

(3.5)

and

\[
\begin{align*}
a_{n+1} &= s_{n+1} - t_{n+1} + \frac{s_n + a_n + b_n}{a_n} - b_n - b_0, \quad n \geq 1, \\
b_{n+2} &= s_{n+2} - a_n - a_n - r_{n+2} - a_n, \quad n \geq 1, \\
c_n &= s_n - a_n - b_n, \quad n \geq 1,
\end{align*}
\]

(3.6)

with \(a_1 \neq s_1 - b_1, a_2 \neq s_2 - b_2\).

Under such conditions, \(\alpha\) and \(\beta\) are two of the zeros of \(q(x) := x^3 + ax^2 + bx + c\).

Furthermore \(\frac{a(x)}{x-\lambda} u\) and \(\frac{a(x)}{x-\lambda(x-\beta)} v\) are the linear functionals respect to which \(|R_n|_{n \geq 0}\) and \(|\tilde{R}_n|_{n \geq 0}\) are orthogonal, respectively.

**Proof.** From Theorem 3.1 and the two first equations of (3.4), we have (3.5) and

\[
(x - \alpha)w_1 = -\lambda u, \quad (x - \beta)w_2 = -\varepsilon w_1,
\]

(3.7)

where \(\alpha, \beta, \lambda\) and \(\varepsilon\) are certain complex numbers, \(\lambda, \varepsilon \neq 0, w_1\) and \(w_2\) are the linear functionals with respect to which \(|R_n|_{n \geq 0}\) and \(|\tilde{R}_n|_{n \geq 0}\) are orthogonal, respectively. Substituting \(R_n\) and \(\tilde{R}_n\) in (2.1), we get

\[
\begin{align*}
Q_1 &= R_1 + s_1 - a_1 - b_1, \\
Q_2 &= R_2 + t_2 - a_2 - b_2)R_1 + [t_2 - a_1 (s_2 - a_2) - b_1 (s_2 - a_2 - b_2)] , \\
Q_n &= R_n + (s_n - a_n - b_n)R_{n-1} + [t_n - a_{n-1} (s_n - a_n) - b_{n-1} (s_n - a_n - b_n)]R_{n-2} \\
&\quad + [r_n - a_{n-2} (s_n - a_n - b_n)] P_{n-3}, \quad n \geq 3,
\end{align*}
\]

then we also have \(t_n = (s_n - a_n - b_n) b_{n-1} + a_{n-1} (s_n - a_n)\) for all \(n \geq 2\) and \(r_n = a_{n-2} (s_n - a_n - b_n) b_{n-1}\) for all \(n \geq 3\), thus, \(s_n \neq a_n + b_n\) for every \(n \geq 3\). Hence (3.6) follows and, furthermore \(s_1 \neq a_1 + b_1, s_2 \neq a_2 + b_2\) hold.
Moreover, since \(|\{R_n\}_{n\geq 0}\) is an SMOP with respect to \(w_2\), being \(|Q_n|_{n\geq 0}\) an SMOP with respect to \(v\), then using Theorem 3.1, we find
\[
(x - \gamma)v = -\mu w_2,
\]
where \(\gamma\) and \(\mu\) are complex numbers, \(\mu \neq 0\). Thus,
\[
q(x)v = -\mu \lambda \epsilon (x - \alpha)(x - \beta)(x - \gamma)v.
\]
Since \(v\) is regular this gives \(k = \mu \lambda \epsilon \) and \(\alpha, \beta\) and \(\gamma\) are the zeros of \(q(x)\).

Conversely, from Theorem 3.1, (3.5) implies that the sequence \(|\{R_n\}_{n\geq 0}\) defined by \(R_n = P_n + a_n P_{n-1}, n \geq 1\), and \(R_n = R_n + b_n R_{n-1}, n \geq 1\), are SMOPs with respect to the linear functionals \(w_1\) and \(w_2\) such that \((x - \alpha)w_1 = kw_1\) and \((x - \beta)w_2 = k'w_1\) where \(k, k' \in \mathbb{C} - \{0\}\), respectively.

We have \(s_n \neq a_n + b_n, n \geq 1\). Taking \(t_n = (s_n - a_n - b_n)b_{n-1} + a_{n-1}(s_n - a_n), n \geq 2,\) and \(r_n = a_{n-2}(s_n - a_n - b_n)b_{n-1}, n \geq 3,\) we obtain
\[
\begin{align*}
\tilde{R}_n &= P_n + (b_n + a_n)P_{n-1} + b_n a_{n-1}P_{n-2}, n \geq 2, \\
Q_n &= P_n + s_n P_{n-1} + ((s_n - a_n - b_n)b_{n-1} + a_{n-1}(s_n - a_n))P_{n-2} + a_{n-2}(s_n - a_n - b_n)b_{n-1}P_{n-3}, n \geq 3, \\
Q_n &= \tilde{R}_n + (s_n - a_n - b_n)\tilde{R}_{n-1}, n \geq 1.
\end{align*}
\]

\[\square\]

**A matrix interpretation.** If \(w_1, w_2\) and \(v\) denote the corresponding linear functionals for \(|\{R_n\}_{n\geq 0}\), \(|\{\tilde{R}_n\}_{n\geq 0}\) and \(|Q_n|_{n\geq 0}\), respectively, defined by (3.4), (3.7) and (3.8), then it is well known (see [7]) that
\[
\begin{align*}
(x - \alpha)P_n &= R_{n+1} + d_n R_n, \\
(x - \beta)R_n &= \tilde{R}_{n+1} + d'_n \tilde{R}_n, \\
(x - \gamma)\tilde{R}_n &= Q_{n+1} + d''_n Q_n, n \geq 0,
\end{align*}
\]
with \(d_n, d'_n, d''_n \neq 0\).

Let \(P = (P_0, P_1, ...)^T, R = (R_0, R_1, ...)^T, \tilde{R} = (\tilde{R}_0, \tilde{R}_1, ...)^T,\) and \(Q = (Q_0, Q_1, ...)^T\) and \(I_P, I_R, I_{\tilde{R}}\) and \(I_Q\) the corresponding monic Jacobi matrices. Then, the recurrence relations for such SMOPs read
\[
\begin{align*}
xP &= I_P P, & xR &= I_R R, & x\tilde{R} &= I_{\tilde{R}} \tilde{R}, & xQ &= I_Q Q.
\end{align*}
\]

On the other hand, from (3.4) and (3.9) we have the matrix representations
\[
\begin{align*}
R &= L_1 P, & (x - \alpha)P &= U_1 R, \\
\tilde{R} &= L_2 \tilde{R}, & (x - \beta)R &= U_2 \tilde{H}, \\
Q &= L_3 \tilde{R}, & (x - \gamma)\tilde{R} &= U_3 Q,
\end{align*}
\]
where \(L_1, L_2\) and \(L_3\) are three lower bidiagonal matrices with 1 as entries in the diagonal and \(U_1, U_2\) and 
\(U_3\) are upper bidiagonal matrices with 1 as entries in the upper diagonal given explicitly by
\[
\begin{align*}
L_1 &= \begin{pmatrix}
1 & a_1 & 1 & \cdots \\
a_2 & 1 & a_3 & \cdots \\
a_3 & \cdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}, & L_2 &= \begin{pmatrix}
1 & b_1 & 1 & \cdots \\
b_2 & 1 & b_3 & \cdots \\
b_3 & \cdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}, & L_3 &= \begin{pmatrix}
1 & s_1 - a_1 - b_1 & 1 & \cdots \\
s_1 - a_1 & 1 & s_2 - a_2 - b_2 & \ddots \\
s_2 - a_2 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}, \\
U_1 &= \begin{pmatrix}
d_0 & 1 & \cdots \\
d_1 & 1 & \cdots \\
d_2 & 1 & \cdots \\
d_3 & \cdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}, & U_2 &= \begin{pmatrix}
d'_0 & 1 & \ddots \\
d'_1 & 1 & \cdots \\
d'_2 & 1 & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots
\end{pmatrix}, & U_3 &= \begin{pmatrix}
d''_0 & 1 & \ddots \\
d''_1 & 1 & \ddots \\
d''_2 & 1 & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots
\end{pmatrix},
\end{align*}
\]
Notice that from (3.10) and (3.11), we get

\[
\begin{align*}
J_P - aI &= U_1L_1, \\
J_R - aI &= L_1U_1, \\
J_R - \beta I &= U_2L_2, \\
\tilde{J}_R - \beta I &= L_2U_2, \\
J_R - \gamma I &= U_3L_3, \\
\tilde{J}_Q - \gamma I &= L_3U_3.
\end{align*}
\]  

(3.12)  
(3.13)  
(3.14)  
(3.15)  
(3.16)  
(3.17)

As a consequence we can summarize the process as follows.

Step 1. Given \( J_P \), from (3.12) we get \( U_1 \).

Step 2. From (3.13) we get \( j_R \).

Step 3. Given \( J_R \), from \( L_2 \) and (3.14) we get \( U_2 \).

Step 4. From (3.15) we get \( \tilde{J}_R \).

Step 5. Given \( \tilde{J}_R \), from \( L_3 \) and (3.16) we get \( U_3 \).

Step 6. From (3.17) we get \( \tilde{J}_Q \).

Notice that this is essentially the iteration of canonical Geronimus transformations (see [18]).

3.2. The split of a 1-4 relation in two relations of types 1-2 and 1-3.

We have to consider two subcases:

3.2.1. 1-2 relation and then 1-3 relation.

Proposition 3.3. Let \( \{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0} \) and \( \{c_n\}_{n \geq 0} \) be sequences of complex numbers, \( a_n \neq 0, n \geq 1 \) and \( c_n \neq 0, n \geq 2 \). The representation (2.1) can be written as

\[
\begin{align*}
R_n &= P_n + a_nP_{n-1}, \quad n \geq 1, \\
Q_n &= R_n + b_nR_{n-1} + c_nR_{n-2}, \quad n \geq 2,
\end{align*}
\]  

(3.18)

where \( \{R_n\}_{n \geq 0} \) is an SMOP if and only if there exists a complex number \( \alpha \) such that

\[
D_n := \beta_n - a_{n+1} - \frac{\gamma_n}{a_n} = \alpha, \quad n \geq 1,
\]  

(3.19)

and

\[
\begin{align*}
t_{n+2} &= a_{n+1}(s_{n+2} - a_{n+1}) + \frac{\beta_{n+2}}{a_n}, \quad n \geq 1, \\
b_n &= s_n - a_n, \quad n \geq 1, \\
c_n &= t_n - (s_n - a_n)a_{n-1}, \quad n \geq 2,
\end{align*}
\]  

(3.20)

with \( t_2 \neq a_1(s_2 - a_2) \).

Furthermore, \( \alpha \) is a zero of \( q(x) \) and \( \frac{q(x)}{x-\alpha} \) is the corresponding linear functional of the SMOP \( \{R_n\}_{n \geq 0} \).

Proof. From Theorem 3.1 and the first equation of (3.18), we have (3.19) and

\[
(x - \alpha)w = -\lambda u,
\]  

(3.21)

where \( \alpha \) and \( \lambda \) are certain complex numbers, \( \lambda \neq 0 \), and \( w \) is the linear functional with respect to which \( \{R_n\}_{n \geq 0} \) is orthogonal. Replacing \( R_n \) in (2.1), we get

\[
\begin{align*}
Q_1 &= R_1 + s_1 - a_1, \\
Q_2 &= R_2 + (s_2 - a_2)R_1 + t_2 - (s_2 - a_2)a_1, \\
Q_n &= R_n + (s_n - a_n)R_{n-1} + [t_n - (s_n - a_n)a_{n-1}]R_{n-2} + [r_n - (t_n - (s_n - a_n)a_{n-1})a_{n-2}]P_{n-3}, \quad n \geq 3.
\end{align*}
\]
Then we have \( r_n - (t_n - (s_n - a_n)a_{n-1})a_{n-2} = 0 \) for all \( n \geq 3 \) and the conditions \( t_n \neq a_{n-1}(s_n - a_n) \) for each \( n \geq 3 \). Hence (3.20) follows and, furthermore \( t_2 \neq (s_2 - a_2)a_1 \) holds. Moreover, since \( \{R_n\}_{n \geq 0} \) is an SMOP with respect to \( w \), being \( \{Q_n\}_{n \geq 0} \) an SMOP with respect to \( v \), then using Theorem 2.2 in [2] we find

\[
(x^2 + \beta x + \gamma)v = \mu w,
\]

where \( \beta, \gamma \) and \( \mu \) are complex numbers, \( \mu \neq 0 \). Thus,

\[
q(x)v = \frac{k}{\mu \lambda} (x^2 + \beta x + \gamma)(x - \alpha)v.
\]

Since \( v \) is regular, this gives \( k = \mu \lambda \) and \( \alpha \) is a zero of \( q(x) \).

Conversely, given \( \{a_n\}_{n \geq 1} \) in the above conditions, from Theorem 3.1, (3.19) implies that the sequence \( \{R_n\}_{n \geq 0} \) defined by \( R_n = P_n + a_nP_{n-1}, \ n \geq 1, \) is an SMOP with respect to a linear functional \( w \) such that \((x - \alpha)w = ku \) where \( k \in \mathbb{C} - \{0\} \).

Taking \( r_n = (t_n - (s_n - a_n)a_{n-1})a_{n-2}, \ n \geq 3 \), we have \( t_n \neq (s_n - a_n)a_{n-1}, \ n \geq 2 \). So, we can write

\[
Q_n = P_n + s_nP_{n-1} + t_nP_{n-2} + (t_n - (s_n - a_n)a_{n-1})a_{n-2}P_{n-3}
\]

\[
= R_n + (s_n - a_n)R_{n-1} + (t_n - (s_n - a_n)a_{n-1})R_{n-2}, \ n \geq 2.
\]

A matrix interpretation. In the sequel, we present a matrix interpretation of these results in terms of the monic Jacobi matrices associated with the SMOPs \( \{P_n\}_{n \geq 0}, \{R_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \), respectively.

Let \( \mathcal{P} = (P_0, P_1, \ldots) \), \( \mathcal{R} = (R_0, R_1, \ldots) \) and \( \mathcal{Q} = (Q_0, Q_1, \ldots) \) be the column vectors associated with these orthogonal families, and \( J_{P}, J_{R}, \) and \( J_{Q} \) the corresponding monic Jacobi matrices. Then, the recurrence relations for such SMOPs read \( x\mathcal{P} = J_{P}x\mathcal{P} \), \( x\mathcal{R} = J_{R}x\mathcal{R} \) and \( x\mathcal{Q} = J_{Q}x\mathcal{Q} \).

If \( w \) denotes the corresponding linear functional for \( \{R_n\}_{n \geq 0} \), given by (3.21), then it is well known (see [7]) that

\[
(x - \alpha)P_n = R_{n+1} + d_nR_n, \ n \geq 0, \ \text{with} \ d_n \neq 0.
\]

Then, from the first equation of (3.18), we get

\[
\mathcal{R} = L\mathcal{P}, \quad (x - \alpha)\mathcal{P} = U\mathcal{R},
\]

where \( L \) is a lower bidiagonal matrix with 1 as diagonal entries and \( U \) is an upper bidiagonal matrix with 1 as entries in the upper diagonal given explicitly by

\[
L = \begin{pmatrix}
1 & & \\
a_1 & 1 & \\
a_2 & a_1 & 1 \\
& a_3 & 1 & \\
& & & \ddots
\end{pmatrix}
\]

and

\[
U = \begin{pmatrix}
d_0 & 1 & & \\
& d_1 & 1 & \\
& & d_2 & 1 \\
& & & \ddots
\end{pmatrix}.
\]

Thus, we get

\[
J_{P} - \alpha I = UL
\]

and

\[
J_{R} - \alpha I = LU.
\]

The previous process is known as Darboux transformation and \( J_{R} \) is said to be the Darboux transform of \( J_{P} \) (see [5]).
On the other hand, from (3.22) and the classical Christoffel formula (see [7]) we can express \((x^2 + \beta x + \gamma)R\) using the matrix representation
\[
(x^2 + \beta x + \gamma)R = NQ,
\]
where \(N\) is a banded upper triangular matrix such that \(n_{k,k+2} = 1\) and \(n_{k,j} = 0\) for \(j - k > 2\). Next, we will describe a method to find the matrix \(IQ\) using the matrix \(JR\) and the polynomial \(x^2 + \beta x + \gamma\). From the first equation of (3.18), we may write \(Q = MR\) where
\[
M = \begin{pmatrix}
1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 1 & 0 & 0 \\
\end{pmatrix}
\]

with, \(b_n = s_n - a_n, \ n \geq 1\) and \(c_n = t_n - (s_n - a_n)a_{n-1}, \ n \geq 2\), then \(xMR = IQMR\) and, as a consequence, 
\[
JRMR = M^\dagger IQMR.
\]
Thus, we get
\[
MJR = IQM.
\]

Thus \((x^2 + \beta x + \gamma)R = NM\), and then
\[
\begin{align*}
\begin{pmatrix}
\delta_1 & \beta_1 & \gamma_1 \\
\delta_2 & \beta_2 & \gamma_2 \\
\vdots & \vdots & \vdots \\
0 & 0 & \gamma_n \\
\end{pmatrix}
+ \begin{pmatrix}
1 & \beta & \gamma \\
& 1 & \beta \\
& & 1 \\
& & & \ddots & \beta & \gamma \\
& & & & 1 & \beta \\
& & & & & 1 \\
\end{pmatrix}
\end{align*}
\]

As a conclusion, we can summarize our process as follows.

\textbf{Proposition 3.4.} Given three sequences of complex numbers \(\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}\) and \(\{c_n\}_{n \geq 0}, \ b_n \neq 0, \ n \geq 2\) and \(c_n \neq 0, \ n \geq 1\), then (2.1) can be written as
\[
\begin{align*}
R_n &= P_n + a_nP_{n-1} + b_nP_{n-2}, \ n \geq 2, \\
Q_n &= R_n + c_nR_{n-1}, \ n \geq 1,
\end{align*}
\]

where \(\{R_n\}_{n \geq 0}\) is a SMOP, if and only if \(\gamma_1 + b_1 - b_{i+1} + a_i(\beta_{n-1} - \beta_i - a_i + a_{i+1}) \neq 0, \ \text{for} \ i = 1, \ 2 \ \text{and exist two complex numbers} \ \alpha, \ \beta, \ \text{such that}
\]
\[
\begin{align*}
D_n &:= \frac{a_n}{b_{n+1}}[\gamma_{n+1} + b_{n+1} - b_{n+2} + a_{n+1}(\beta_n - \beta_{n+1} - a_{n+1} + a_{n+2})] + a_n - \beta_{n-1} - \beta_n \\
&= a_n, \ n \geq 1, \\
E_n &:= \frac{1}{b_{n+1}}[\gamma_{n+1} + b_{n+1} + a_{n+1}(\beta_n - \beta_{n+1} - a_{n+1} + a_{n+2})](\gamma_n + b_n - b_{n+1} + a_n(a_{n+1} - \beta_n)) \\
&+ b_n - \gamma_{n-1} + (a_{n+1} - \beta_n)(a_n - \beta_{n-1}) = \beta, \ n \geq 1,
\end{align*}
\]

3.2.2. 1-3 relation and then 1-2 relation.
and
\[
\begin{align*}
a_{n+1} &= s_{n+1} - \frac{r_{n+1}}{s_n}, \quad n \geq 2, \\
b_{n+1} &= t_{n+1} - (a_{n+1} - s_{n+1})a_n, \quad n \geq 1, \\
c_n &= s_n - a_n, \quad n \geq 1,
\end{align*}
\]
(3.30)
with \(a_1 \neq s_1\), \(a_2 \neq s_2\).

In this case \(q(x) = (x - x_1)(x^2 + ax + \beta)\) where \(x_1 \in \mathbb{C}\) and \((x - x_1)v\) is the linear functional associated with the SMOP \(\{R_n\}_{n \geq 0}\).

**Proof.** From Theorem 2.2 in [2], we have (3.29) and
\[
(x^2 + ax + \beta)w = \lambda u,
\]
where \(\alpha\), \(\beta\) and \(\lambda\) are certain complex numbers, \(\lambda \neq 0\), and \(w\) is the regular functionals with respect to which \(\{R_n\}_{n \geq 0}\) is orthogonal. Replacing \(R_n\) in (2.1), we get
\[
\begin{align*}
Q_1 &= R_1 + s_1 - a_1, \\
Q_2 &= R_2 + (s_2 - a_2)R_1 + t_2 - b_2 - (s_2 - a_2)a_1, \\
Q_n &= R_n + (s_n - a_n)R_{n-1} + [t_n - b_n - (s_n - a_n)a_{n-1}]P_{n-2} + [r_n - (s_n - a_n)b_{n-1}]P_{n-3}, \quad n \geq 3.
\end{align*}
\]
Then
\[
t_n - b_n = (s_n - a_n)P_{n-1} = 0 \text{ for all } n \geq 2 \text{ and } r_n = (s_n - a_n)b_{n-1} = 0 \text{ for all } n \geq 3. \quad \text{Hence (3.30) follows and, furthermore, } a_1 \neq s_1 \text{ and } a_2 \neq s_2 \text{ hold. Using the second equation of (3.28) and Theorem 3.1, we have}
\]
\[
(x - \gamma)v = -k_1w,
\]
(3.32)
where \(\gamma\) and \(k_1\) are complex numbers, \(k_1 \neq 0\). Thus, since by hypothesis we also have
\[
q(x)v = \frac{k}{k_1\lambda}(x - \gamma)(x^2 + ax + \beta)v.
\]
This gives \(k = k_1\lambda\) and then \(\gamma\) is one of the zeros of \(q(x) := x^3 + ax^2 + bx + c\) because \(v\) is regular.

Conversely, given \(\{a_n\}_{n \geq 0}\) and \(\{b_n\}_{n \geq 2}\) in the above conditions, from Theorem 2.2 in [2], (3.29) implies that the sequence \(\{R_n\}_{n \geq 0}\) defined by \(R_0 = 1, R_1 = P_1 + a_1P_0, R_n = P_n + a_nP_{n-1} + b_nP_{n-2}, n \geq 2\), is a SMOP with respect to a regular linear functional \(v\) such that \((x^2 + ax + \beta)v = ku\) where \(k \in \mathbb{C} - \{0\}\).

Taking \(t_n = b_n + (s_n - a_n)a_{n-1}, n \geq 2\) and \(r_n = (s_n - a_n)b_{n-1}, n \geq 3\). So, we can write
\[
Q_n = P_n + s_nP_{n-1} + (b_n + (s_n - a_n)a_{n-1})P_{n-2} + (s_n - a_n)b_{n-1}P_{n-3} = R_n + (s_n - a_n)R_{n-1}, n \geq 1.
\]
\(\square\)

**When \(\{P_n\}_{n \geq 0}\) is symmetric.**

Assume that the sequence \(\{P_n\}_{n \geq 0}\) is orthogonal with respect to a symmetric linear functional \(u\) (i.e. \((u)_{2n+1} = 0, n \geq 0\)). Then \(\beta_n = 0, n \geq 0\), and there exist two polynomial sequences \(\{V_n\}_{n \geq 0}\) and \(\{V_n^*\}_{n \geq 0}\) such that for all \(n, P_{2n}(x) = V_n(x^2)\) and \(P_{2n+1} = xV_n^*(x^2)\).

It is known (see [6]) that \(\{V_n\}_{n \geq 0}\) and \(\{V_n^*\}_{n \geq 0}\) are SMOPs with respect to the linear functionals \(a_\alpha\) and \(x\bar{\alpha}\) where \((a_\alpha, x^m) = (u, x^m), n \geq 0\).

It’s clear that the polynomials \(Q_n\) defined by (2.1) can not be symmetric because \(r_n \neq 0\) for all \(n \geq 3\). Suppose the sequence \(\{Q_n\}_{n \geq 0}\) is orthogonal with respect to a linear functional \(v\) such that \(xv\) is symmetric and regular, then \(v\) is said to be quasi-antisymmetric (for more information about these linear functionals please see [14] and [16]). From (2.16), we obtain \((ax + \beta)\bar{v} = 0\) then \(a = c = 0\) because \(\bar{v}\) is regular. Therefore, the relation between the linear functionals \(u\) and \(v\) is \((x^2 + b)v = -kv\). Noting \(w = xv\), then \((x^2 + b)v = -kv\) and from Proposition 2.1 in [13], there exists a symmetric sequence \(\{R_n\}_{n \geq 0}\) orthogonal
with respect to \( w \) and satisfying (3.28). Thus \( a_n = 0 \). From Proposition 3.4, we obtain \( b_n = t_n \) and \( s_{n+1} = \frac{t_{n+1}}{t_n} \). Furthermore, there exist \( \{G_n\}_{n \geq 0} \) and \( \{G'_n\}_{n \geq 0} \) SMOPs with respect to \( \sigma_v \) and \( x_\sigma_v \), respectively, satisfying
\[
Q_{2n}(x) = G_n(x^2) + \lambda_n x G'_{n-1}(x^2), \quad Q_{2n+1}(x) = \lambda_n x G_n(x^2) + x G'_n(x^2), \quad n \geq 0,
\]
with \( \theta_n \neq 0 \) and \( \lambda_n \neq 0 \), \( n \geq 0 \). In this case, from (3.28), we have for \( n \geq 0 \), \( Q_n = R_n + s_n R_{n-1} \), \( \theta_n = s_{2n} \), \( \lambda_n = s_{2n+1} \) and
\[
R_{2n}(x) = G_n(x^2), \quad R_{2n+1}(x) = x G'_n(x^2),
\]
where
\[
G_n(x) = V_n(x) + t_{2n} V_{n-1}(x), \quad G'_n(x) = V'_n(x) + t_{2n+1} V'_{n-1}(x).
\]
The coefficients \( s_n \), \( t_{2n} \) and \( t_{2n+1} \) can be computed using Theorem 3.1.

Moreover, the parameters \( \beta_n \) and \( \gamma_n \) of the recurrence relation of the sequence \( \{Q_n\}_{n \geq 0} \) are defined by
\[
\beta_n = s_n - s_{n+1}, \quad n \geq 0,
\]
\[
\gamma_n = -s^2_n, \quad n \geq 1.
\] (3.33) (3.34)

Indeed, taking \( \beta_n = 0 \) in (2.5) and \( a = 0 \) in (2.13) , we get (3.33) and
\[
\frac{t_{n+2}}{r_{n+2}} = -s_{n+3}.
\]

Using (2.3) for \( n + 2 \) instead of \( n \), we obtain
\[
\frac{1}{r_{n+2}}[t_{n+1} \gamma_{n+2} - r_{n+2}(s_{n+3} - s_{n+2})] = -s_{n+3},
\]
and introducing \( t_{n+1} = \frac{t_{n+2}}{r_{n+2}} \), that is, for \( n \geq 1 \)
\[
\gamma_{n+2} = -s^2_{n+2}.
\] (3.35)

From (2.23) and (3.35), for \( n=1 \), we obtain
\[
-s^2_3 \tilde{\gamma}_1 \tilde{\gamma}_2 - s^2_3 s^2_1 \tilde{\gamma}_3 - s^2_3 (t_2 - \tilde{\gamma}_1) \frac{t_3}{r_3} - s^3_3 \tilde{\gamma}_1 + r_3.
\]

Inserting \( t_2 = \frac{r_3}{s_3} \), we obtain
\[
-s^2_3 \frac{r_3}{s_3} \tilde{\gamma}_3 [\tilde{\gamma}_1 + s^2_3] = 0.
\]

Then, we deduce \( \tilde{\gamma}_1 = -s^2_3 \).

Using (2.15) and (2.2)-(2.6), we get
\[
\frac{\tilde{\gamma}_2 \tilde{\gamma}_3}{t_4} + (s_2 - \beta_1) \frac{s_2}{r_4} \tilde{\gamma}_4 - [s_3 \beta_1 + \gamma_1 + \gamma_2 - t_3 - \beta_1 \beta_2] \frac{t_3}{r_4} \tilde{\gamma}_4
\]
\[
+ (\beta_2 - \beta_0 + \beta_3 - s_4) \tilde{\gamma}_1 - (s_4 - \beta_3) \gamma_2 + s_3 \beta_1 - \beta_1 t_4 + \beta_1 \beta_2 s_4 - \beta_1 \beta_2 \beta_3 + r_4 = C_2.
\]

From \( \beta_n = 0 \), \( \tilde{\gamma}_3 = -s^2_3 \), \( \tilde{\gamma}_4 = -s^2_4 \) and \( t_3 = \frac{r_3}{s_3} \), we obtain
\[
\frac{\tilde{\gamma}_3 s^2_2}{t_4} + \frac{s^2_2 s^2_2}{r_4} = 0.
\]

Therefore \( \tilde{\gamma}_2 = -s^2_2 \).
A matrix interpretation. We will describe a method to find the matrix $J_R$ using the matrix $J_P$ and the polynomial $x^2 + ax + \beta$.

Taking into account the first equation of (3.28) we may write $R = MP$ where $M = (m_{k,j})$ is a banded lower triangular matrix such that $m_{k,k} = 1$, and $m_{k,j} = 0$ for $k - j > 2$.

$$M = \begin{pmatrix} 1 & 0 & 0 & \ldots & \ldots & \ldots \\ a_1 & 1 & 0 & \ldots & \ldots & \ldots \\ b_2 & a_2 & 1 & \ldots & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\ \vdots & \ddots & b_n & a_n & 1 & 0 \\ 0 & \ldots & 0 & \ldots & \ldots & \ldots \end{pmatrix}$$

then $xMP = J_RM$ and, as a consequence, $J_P = M^{-1}J_RM$. Thus, we get

$$MJ_P = J_RM.$$ 

On the other hand, from (3.31) and the classical Christoffel formula (see [7]) we can express $(x^2 + ax + \beta)P$ using the matrix representation

$$(x^2 + ax + \beta)P = NQ,$$

where $N$ is a banded upper triangular matrix such that $n_{k+2} = 1$ and $n_{k,j} = 0$ for $j - k > 2$.

Thus $(x^2 + ax + \beta)P = NMP$, and then

$$J_R^2 + \alpha J_R + \beta I = NM.$$ 

(3.36)

But, from $(x^2 + ax + \beta)R = MN$, we get

$$J_R^2 + \alpha J_R + \beta I = MN.$$ 

(3.37)

By (3.32), it is well known (see [7]) that

$$(x - \gamma)R_n = Q_{n+1} + d_nQ_n, \quad n \geq 0, \quad \text{with} \quad d_n \neq 0.$$ 

Then, from the second equation of (3.28), we obtain

$$Q = LR, \quad (x - \gamma)R = UQ,$$ 

(3.38)

where

$$L = \begin{pmatrix} 1 & c_1 & 1 & \ldots & \ldots & \ldots \\ c_2 & 1 & c_1 & \ldots & \ldots & \ldots \\ c_3 & 1 & \ldots & \ldots & \ldots & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & \ddots \\ & & & & & & \ddots \\ & & & & & & & \ddots \end{pmatrix}$$

and

$$U = \begin{pmatrix} d_0 & 1 & & & & \\ d_1 & d_0 & 1 & & & \\ & d_1 & d_0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}.$$

Thus, we get

$$J_R - \gamma I = UL$$ 

(3.39)

and

$$J_Q - \gamma I = LU.$$ 

(3.40)

As a conclusion, we can summarize our process as follows.
Step 1. Given $f_P$, we find the polynomial matrix $J_P^2 + af_P + bI$.

Step 2. From $M$ and (3.36) we find $N$.

Step 3. From (3.37) we obtain the polynomial matrix $J_R^2 + aJ_R + bI$.

Step 4. Taking into account that $f_R$ is a tridiagonal matrix, from step 3 we can deduce $J_R$, since $(J_R + \frac{a}{4}I)^2 = MN - (b - \frac{a^2}{4})I$.

Step 5. Given $J_R$, from $L$ and (3.39) we get $U$.

Step 6. From (3.40) we get $J_Q$.

4. Illustrative Examples

(I) Let $\{P_n = L_n^{(a)}\}_{n \geq 0}$ be the sequence of monic Laguerre polynomials orthogonal with respect to the linear functional $u$ defined by the weight function $x^a e^{-x} \chi_{[0, +\infty)}$ with $\alpha > 1$. We can take the auxiliary polynomials $R_n(x) = L_n^{(a-1)}(x)$ and $\tilde{R}_n(x) = L_n^{(a-2)}(x)$ orthogonal, respectively with respect to $\omega_1$ and $\omega_2$. These polynomials satisfy $R_n(x) = L_n^{(a)}(x) + nL_{n-1}^{(a)}(x)$, and $\tilde{R}_n(x) = L_n^{(a-1)}(x) + nL_{n-1}^{(a-1)}(x)$ (see [6]).

Then, the new sequence $\{Q_n\}_{n \geq 0}$ such that $Q_n(x) = \tilde{R}_n(x) + c_nR_{n-1}(x)$ is orthogonal with respect to the linear functional $v$ satisfying $xv = (\alpha - 1 - c_1)\omega_2$. Thus

$$x^2v = ku, \quad k = \alpha(\alpha - 1)(\alpha - 1 - c_1).$$

According to Proposition 3.2, the polynomials $Q_n$ satisfy the relation (2.1) where $s_n = c_n + 2n$, $t_n = (2n - 2)s_n - 3n(n - 1)$, $n \geq 1$ and $v_n = (n - 1)(n - 2)(s_n - 2n)$, $n \geq 1$. It is well known that the recurrence coefficients of $L_n^{(a-2)}$, $\alpha > 1$, are $\beta_n = 2n + \alpha - 1$, $n \geq 0$ and $\gamma_n = n(n + \alpha - 2)$, $n \geq 1$ (see [6]).

Using formula (3.2) for this case, having $x_1 = 0$ and $c_2 = 1 + \alpha - \frac{\alpha^2}{\alpha + \frac{1}{\alpha}}$, by induction we can derive that, for $\alpha > 1$, and $\alpha \neq 2$, the values of the parameters $c_n$ in terms of $c_1$ are

$$c_n = n \frac{\Gamma(\alpha - 1)(\alpha - 1 - c_1) + (c_1 - 1)\frac{\Gamma(n + \alpha - 1)}{\Gamma(n + 1)}}{\Gamma(\alpha - 1)(\alpha - 1 - c_1) + (c_1 - 1)\frac{\Gamma(n - 2 + \alpha)}{\Gamma(n - 1)}}, \quad n \geq 1, \quad (4.1)$$

and then $v$ is regular if and only if

$$\Gamma(n)\Gamma(\alpha - 1)(\alpha - 1 - c_1) + (c_1 - 1)\Gamma(n - 2 + \alpha) \neq 0, \quad n \geq 1.$$

Notice that if $\alpha \in \mathbb{N} - \{2\}$ then $c_n$ is a rational function of $n$, namely,

$$c_n = n \frac{\Gamma(\alpha - 1)(\alpha - 1 - c_1) + (c_1 - 1)(\alpha + n - 2)...(n + 1)}{\Gamma(\alpha - 1)(\alpha - 1 - c_1) + (c_1 - 1)(\alpha + n - 3)...n}, \quad n \geq 1.$$

If $\alpha = 2$, then, by induction, we can also obtain, for $n \geq 2$

$$c_n = n \frac{(c_1 - 1)(1 + \frac{1}{2} + c_1)}{(c_1 - 1)(1 + \frac{1}{2} + ... + \frac{1}{n-1})} + 1$$

and $v$ is regular if and only if

$$(c_1 - 1)(1 + \frac{1}{2} + ... + \frac{1}{n}) + 1 \neq 0, \quad n \geq 1.$$

We have

$$v = (\alpha - c_1 - 1)x^{-1}\omega_2 + \delta_0, \quad (4.3)$$
In particular for $\alpha > 2$, we can write
\[(\alpha - 2)v = (\alpha - c_1 - 1)w_3 + (c_1 - 1)\delta_0,\] (4.4)
where $w_3$ is the corresponding linear functional for monic Laguerre polynomials $\{L^{(\alpha-3)}_n\}_{n \geq 0}$.

**A matrix interpretation.** If $P_n = L^{(\alpha)}_n$, $\alpha > 1$, and $a_n = b_n = n$, we obtain
\[
(J_P)_{n+1} = \begin{pmatrix}
\alpha + 1 & 1 & 0 & \cdots & 0 \\
\alpha + 1 & \alpha + 3 & \cdots & \cdots & \vdots \\
0 & 2(\alpha + 2) & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & n(n + \alpha) & 2n + \alpha + 1
\end{pmatrix},
\]
(4.5)
and
\[
(L_1)_{n+1} = (L_2)_{n+1} = \begin{pmatrix}
1 \\
1 \\
1 \\
\vdots \\
n \\
1
\end{pmatrix}.
\]
(4.6)

From (3.12), we obtain
\[
(U_1)_{n+1} = \begin{pmatrix}
\alpha & 1 \\
\alpha + 1 & 1 \\
\alpha + 2 & 1 \\
\vdots & \ddots \\
\alpha + n
\end{pmatrix},
\]
thus,
\[
(J_R)_{n+1} = (L_1)_{n+1}(U_1)_{n+1} = \begin{pmatrix}
\alpha & 1 \\
\alpha & 1 \\
\alpha + 2 & 1 \\
\vdots & \ddots \\
\vdots & \ddots \\
n(n + \alpha - 1) & \alpha + 2n
\end{pmatrix}.
\]
Using (3.14), we obtain
\[
(U_2)_{n+1} = \begin{pmatrix}
\alpha - 1 & 1 \\
\alpha & 1 \\
\alpha + 1 & 1 \\
\vdots & \ddots \\
\alpha + n - 1
\end{pmatrix},
\]
Then by (3.15), we get
\[
(J_R)_{n+1} = (L_2)_{n+1}(U_2)_{n+1} = \begin{pmatrix}
\alpha - 1 & 1 \\
\alpha - 1 & 1 \\
\alpha - 1 & 1 \\
\vdots & \ddots \\
\vdots & \ddots \\
n(n + \alpha - 2) & 1
\end{pmatrix}.
\]
From (3.16), we have

$$(U_3)_{n+1} = \begin{pmatrix}
\alpha - 1 - c_1 & 1 \\
\alpha + 1 - c_2 & \ddots \\
& \ddots & 1 \\
2n + \alpha - 1 - c_{n+1}
\end{pmatrix}.$$  

With $c_n = s_n - 2n$ satisfies $c_{n+1} = 2n + \alpha - 1 - \frac{n(n+1-2)}{c_n}$, $n \geq 1$. Then $c_n$ is defined by (4.1) and (4.2). Using (3.17), we get

$$(JQ)_{n+1} = (L_3)_{n+1}(U_3)_{n+1} = \begin{pmatrix}
\alpha - 1 - c_1 & 1 \\
c_1(\alpha - 1 - c_1) & c_1 + \frac{n+1}{c_n} & 1 \\
& \ddots & \ddots \\
& & c_n(2n + \alpha - 3 - c_n) & c_n + \frac{n(n+1-2)}{c_n}
\end{pmatrix}.$$  

(2) Let $\{P_n = U_n\}_{n \geq 0}$ be the sequence of monic Chebyshev polynomials of the second kind, orthogonal with respect to the linear functional $v = \mathcal{U}$ defined by the weight function $(1 - x^2)^{1/2} \chi_{[-1,1]}(x)$ with the recurrence coefficients $\beta_n = 0$, $n \geq 0$, and $\gamma_n = \frac{1}{2}$, $n \geq 1$. Consider the SMOP $\{R_n = T_n\}_{n \geq 0}$ orthogonal with respect to the Chebyshev linear functional of first kind $w = T$. We have (see [6])

$$R_n = U_n - \frac{1}{4}U_{n-2}, \quad n \geq 2,$$

and $(x^2 - 1)w = -\frac{1}{2}u$. The new polynomials $Q_n$, such that $Q_n = R_n + s_nR_{n-1}$, $n \geq 1$, satisfy the relation (2.1) with $t_n = -\frac{1}{4}$ and $r_n = -\frac{1}{2}s_n$. Thus $wx = -2w$, and $v$ is quasi-antisymmetric. It is well known that the recurrence coefficient of $R_n$ are $\beta_n^w = 0$, $n \geq 0$, $\gamma_n^w = \frac{1}{2}$, $n \geq 2$, and $\gamma_1^w = \frac{1}{2}$ (see [6]). Using Theorem 3.1 for this case, since $x_1 = 0$, by induction, the values of the parameters $s_n$, $n \geq 2$, are

$$\begin{align*}
\beta_0 - x_1 - s_1 &= -2, \text{ i.e. } s_1 = 2, \\
s_{2n} &= \frac{1}{2s_1} = \frac{1}{4}, \quad n \geq 1, \\
s_{2n+1} &= s_1 = 1, \quad n \geq 1.
\end{align*}$$  

(4.8)

Then

$$
\begin{align*}
r_{2n} &= \frac{1}{16}, \quad r_{2n+1} = -\frac{1}{4}, \quad n \geq 1, \\
\tilde{\beta}_0 &= -s_1 = -2, \quad \tilde{\beta}_1 = s_1 - s_2 = \frac{9}{4}, \quad \tilde{\beta}_{2n} = -\frac{5}{4}, \quad \tilde{\beta}_{2n+1} = \frac{5}{4}, \quad n \geq 1.
\end{align*}$$  

(4.9)

From (3.33)-(3.34), and (4.9), we get

$$
\gamma_1 = \gamma_{2n}^2 = -4, \quad \gamma_{2n} = s_{2n}^2 = -\frac{1}{16}, \quad \gamma_{2n+1} = -s_{2n+1}^2 = -1, \quad n \geq 1.
$$  

(4.10)

The regular linear functional $v$ is given by

$$v = -2x^{-1}w + \delta_0.$$  

A matrix interpretation. From Proposition 3.4 where $P_n = U_n$, and $R_n = T_n$, we have $a_n = 0$, $n \geq 0$, and $b_n = -\frac{1}{4}$, $n \geq 2$. Then, the polynomials $\{Q_n\}_{n \geq 0}$ satisfy the relation (2.1) with, $t_n = -\frac{1}{4}$, $n \geq 2$, and
\( r_n = -\frac{1}{2} s_n, \ n \geq 3. \) Therefore
\[
(L)_{n+1} = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
s_1 & 1 & \ddots & \ddots & \\
0 & s_2 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & s_n & 1
\end{pmatrix}.
\]
From the above results, we have
\[
(JP)_{n+1} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
1/4 & 0 & \ddots & \ddots & \ddots \\
0 & 1/4 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 1/4 & 0
\end{pmatrix}, \quad (JR)_{n+1} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
1/2 & 0 & \ddots & \ddots & \ddots \\
0 & 1/4 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 1/4 & 0
\end{pmatrix}.
\]
Then
\[
[(JP)_{n+1}]^2 - I = \begin{pmatrix}
-3/4 & 0 & 1 & \ldots & 0 \\
0 & -1/2 & \ddots & \ddots & \ddots \\
1/16 & 0 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 1/16 & 0 & -1/2
\end{pmatrix}.
\]
From (3.36), we obtain
\[
(N)_{n+1} = \begin{pmatrix}
-1/2 & 0 & 1 & \ldots & 0 \\
0 & -1/4 & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & -1/4
\end{pmatrix}.
\]
From (3.39), we get
\[
(U)_{n+1} = \begin{pmatrix}
-s_1 & 1 & \ldots & 0 \\
-s_2 & 1 & \ddots & \ddots \\
-s_3 & 1 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
-s_{n+1}
\end{pmatrix},
\]
with \( s_n \) satisfies \( s_2 = -\frac{1}{2s_1} \) and \( s_{n+1} = \frac{1}{4s_n}, \ n \geq 2. \) Then \( s_n \) is again defined by (4.8).
Using again (3.40), we get
\[
(JQ)_{n+1} = (L)_{n+1} (U)_{n+1} = \begin{pmatrix}
-s_1 & 1 & 0 & \ldots & 0 \\
-s_2^2 & s_1 - s_2 & \ddots & \ddots & \ddots \\
0 & -s_2^2 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & -s_n^2 & s_n - s_{n+1}
\end{pmatrix}.
(3) Let \( \{P_n = B_n^a\}_{n \geq 0} \) be the sequence of monic Bessel polynomials orthogonal with respect to the linear functional \( u = \mathcal{B}^a \) defined by the weight function \( x^{2(n-1)}e^{-x} \int_x^{\infty} e^{-2\kappa e^{-x}} \sin(e^x) \, d\epsilon x \) with \( \kappa > 1 \) (see [15]). We can take the auxiliary polynomials \( R_n = B_n^{-1} \) satisfying

\[
R_n(x) = B_n^a(x) + \frac{n}{(n - 2 + \alpha)(n + \alpha - 1)} B_{n-1}^a(x) + \frac{n(n - 1)}{(2n + 2\alpha - 5)(n - 2 + \alpha)(2n + 2\alpha - 3)} B_{n-2}^a(x)
\]

orthogonal with respect to the linear functional \( w = \mathcal{B}^{-1} \).

This linear functional verifies \( x^2 w = \frac{4}{(\alpha - 1)(\alpha - 1)} u \) (see [6]).

According to Proposition 3.4, the new polynomials \( Q_n \) such that

\[
Q_n(x) = R_n(x) + c_n R_{n-1}(x), \quad n \geq 1,
\]

satisfy the relation (2.1) with

\[
s_n = c_n + \frac{n(n-1)}{(2n+2\alpha - 5)(2n+2\alpha - 3)}, \quad n \geq 1,
\]

\[
t_n = \frac{n(n-1)}{(2n+2\alpha - 5)(2n+2\alpha - 3)}, \quad n \geq 2,
\]

and

\[
r_n = \frac{n(n-1)}{(2n+2\alpha - 5)(2n+2\alpha - 3)} c_n, \quad n \geq 3.
\]

It is well known that the recurrence coefficients of \( B_n^{-1} \) are

\[
\beta_0^w = -\frac{1}{\alpha - 1}, \quad \beta_n^w = \frac{2 - \alpha}{n(n + 2\alpha - 4)}, \quad n \geq 1,
\]

\[
\gamma_n^w = -\frac{2n + 2\alpha - 5}{(2n + 2\alpha - 4)(2n + 2\alpha - 3)}, \quad n \geq 1.
\]

Using formula (3.2) for this case, having \( x_1 = 0 \), and taking into account (4.12), we can deduce by induction

\[
c_n = -\frac{n + 2\alpha - 4}{(n + 2\alpha - 2)(2n + 2\alpha - 5)} x_n, \quad n \geq 1,
\]

where

\[
\left\{ \begin{array}{l}
\alpha \neq \frac{3}{2} : \quad x_n = (\lambda - \frac{2}{2} \frac{\Gamma(2\alpha - 3)}{\Gamma(2\alpha - 3)} - \frac{(1)^n\Lambda(2\alpha - 3)\Gamma(n + 1)}{\Gamma(n + 2\alpha - 3)}, \quad n \geq 0, \\
\alpha = \frac{3}{2} : \quad x_n = 1 + (-1)^n \frac{\lambda}{2}, \quad n \geq 0,
\end{array} \right.
\]

with \( \lambda = \frac{1}{\alpha - 1} + c_1 \).

The linear functional \( v \) is regular for every \( c_1 \) such that \( x_n \neq 0, \quad n \geq 0 \), and it is given by

\[
v = -\frac{1}{\alpha - 1} x^\alpha w + \delta_0.
\]

In particular for \( \alpha > 2 \), we can write

\[
v = -\frac{(2\alpha - 3)(\alpha - 2)}{2} \left( \frac{1}{\alpha - 1} + c_1 \right) x^\alpha w - \frac{(2\alpha - 3)}{2} \left( \frac{1}{\alpha - 1} + c_1 \right) - 1 \delta_0.
\]

A matrix interpretation. We have \( P_n = B_n^a, \alpha > 1 \), and \( R_n = B_n^{-1} \), then

\[
(I_P)_{n+1} = \begin{pmatrix}
\beta_0 & 1 & 0 & \cdots & 0 \\
\gamma_1 & \beta_1 & \ddots & \ddots & \vdots \\
0 & \gamma_2 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \gamma_n & \beta_n
\end{pmatrix}, \quad (I_R)_{n+1} = \begin{pmatrix}
\beta_0^w & 1 & 0 & \cdots & 0 \\
\gamma_1^w & \beta_1^w & \ddots & \ddots & \vdots \\
0 & \gamma_2^w & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \gamma_n^w & \beta_n^w
\end{pmatrix}.
\]
where \( \beta_n, \gamma_n \) and \( \beta_n^w, \gamma_n^w \) are the recurrence coefficients of \( B_n^\alpha \) and \( B_n^{\alpha-1} \), respectively. Thus, \( R = MP \), where

\[
(M)_{n+1} = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
a_1 & 1 & \ddots & \ddots & \vdots \\
b_2 & a_2 & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & b_n & a_n & 1
\end{pmatrix},
\]

with

\[
a_n = \frac{n}{(n-2+\alpha)(n+\alpha-1)}, \quad n \geq 1,
\]

\[
b_n = \frac{n(n-1)}{(2n+2\alpha-5)(n-2+\alpha)(2n+2\alpha-3)}, \quad n \geq 2.
\]

The polynomials \( Q_n, n \geq 0 \), satisfy \( Q = LR \), with

\[
(L)_{n+1} = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
c_1 & 1 & \ddots & \ddots & \vdots \\
0 & c_2 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & c_n & 1
\end{pmatrix},
\]

and \( c_n \) defined by (4.13)-(4.14).

Using (3.39) for this case, we obtain

\[
(U)_{n+1} = \begin{pmatrix}
d_0 & 1 & \ddots & & \\
d_1 & 1 & \ddots & & \\
\vdots & \ddots & \ddots & \ddots & \\
d_n & \ddots & \ddots & \ddots & 1
\end{pmatrix},
\]

where

\[
d_0 = \beta_0^w - c_1 = -\frac{1}{\alpha-1} - c_1,
\]

\[
d_n = \beta_n^w - c_{n+1} = \frac{\gamma_n^w}{c_n} = \frac{n}{(n+\alpha-2)(2n+2\alpha-3)\frac{x_{n-1}}{x_n}}, \quad n \geq 1.
\]

From (3.40), we get

\[
(JQ)_{n+1} = (L)_{n+1} (U)_{n+1} = \begin{pmatrix}
d_0 & 1 & 0 & \ldots & 0 \\
c_1d_0 & c_1 + d_1 & \ddots & \ddots & \vdots \\
0 & c_2d_1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \ldots & 0 & c_n d_{n-1} & c_n + d_n
\end{pmatrix}.
\]
Then
\[ \tilde{\beta}_0 = d_0 = -\frac{1}{\alpha - 1} - c_1, \]
\[ \tilde{\beta}_{n+1} = c_{n+1} + d_{n+1} = -\frac{n + 2\alpha - 3}{(n + \alpha - 1)(2n + 2\alpha - 3)} x_{n+1} + \frac{n + 1}{(n + \alpha - 1)(2n + 2\alpha - 1)} x_n, \quad n \geq 0, \]
\[ \tilde{\gamma}_1 = -c_1\left(\frac{1}{\alpha - 1} + c_1\right), \]
\[ \tilde{\gamma}_{n+1} = -\frac{n(n + 2\alpha - 3)}{(n + \alpha - 1)(n + \alpha - 2)(2n + 2\alpha - 3)^2} x_{n-1} x_{n+1}, \quad n \geq 1. \]

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References