A Generalization of the $m$-Topology on $C(X)$
Finer than the $m$-Topology

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Abstract. It is well known that the component of the zero function in $C(X)$ with the $m$-topology is the ideal $C_0(X)$. Given any ideal $I \subseteq C_0(X)$, we are going to define a topology on $C(X)$ namely the $m^I$-topology, finer than the $m$-topology in which the component of 0 is exactly the ideal $I$ and $C(X)$ with this topology becomes a topological ring. We show that compact sets in $C(X)$ with the $m^I$-topology have empty interior if and only if $X \setminus \cap I$ is infinite. We also show that nonzero ideals are never compact, the ideal $I$ may be locally compact in $C(X)$ with the $m^I$-topology and every Lindelöf ideal in this space is contained in $C_0(X)$. Finally, we give some relations between topological properties of the spaces $X$ and $C_m(X)$. For instance, we show that the set of units is dense in $C_m(X)$ if and only if $X$ is strongly zero-dimensional and we characterize the space $X$ for which the set $r(X)$ of regular elements of $C(X)$ is dense in $C_m(X)$.

1. Introduction

Throughout this paper we denote by $C(X)$ ($C^*(X)$) the ring of all (bounded) real-valued continuous functions on a completely regular Hausdorff space $X$. The $m$-topology on $C(X)$ is defined by taking the set of the form

$$B(f,u) = \{g \in C(X) : |f(x) - g(x)| < u(x), \forall x \in X\}$$

as a base for a neighborhood system at $f$, for each $f \in C(X)$ and $u \in U^*(X)$, where $U^*(X)$ is the set of all positive elements of $C(X)$. $C(X)$ endowed with the $m$-topology is denoted by $C_m(X)$ which is a Hausdorff topological ring. The $m$-topology is first introduced in the late 40s in [8] and later the research in this area became active over the last 20 years, for example, the works in [2], [3], [6] and [10].

Compact sets and connected sets in $C_m(X)$ are investigated in [2] and it is shown that the component of 0 in $C_m(X)$ is the ideal $C_0(X)$. Clearly the connected sets (component of 0) in $C(X)$ with a topology finer that the $m$-topology are also connected in $C_m(X)$ (is contained in $C_0(X)$). In this paper, for a given ideal $I$ contained in $C_0(X)$, we define a topology on $C(X)$, namely the $m^I$-topology, in which the component of 0 is exactly the ideal $I$. This topology is finer than the $m$-topology and makes $C(X)$ a topological ring. We denote the space $C(X)$ with the $m^I$-topology by $C_{m^I}(X)$. More generally, if $I$ is an arbitrary ideal in $C(X)$, the $m^I$-topology is defined similarly and we show that the component of 0 in the space $C_{m^I}(X)$ is $C_0(X) \cap I$. We

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also investigate compact sets in \( C(X) \) with the \( m^l \)-topology and it turns out that whenever \( I \not\subseteq C_p(X) \), every compact set in \( C_m(X) \) has an empty interior.

For each \( f \in C(X) \), the set zeros of \( f \) is called the zero-set of \( f \) and is denoted by \( Z(f) \), \( \text{coz} f = X \setminus Z(f) \) and \( \text{cl}_X \text{coz} f \) is called the support of \( f \). We also denote the sets \( \{ x \in X : f(x) > 0 \} \) and \( \{ x \in X : f(x) < 0 \} \) by \( \text{pos} f \) and \( \text{neg} f \) respectively. An ideal \( I \) in \( C(X) \) is called a \( z \)-ideal if whenever \( f \in I \), \( g \in C(X) \) and \( Z(f) \subseteq Z(g) \), then \( g \in I \). We recall that \( C_p(X) \) is a \( z \)-ideal in \( C(X) \) consisting of all functions with pseudocompact (compact) support and it is well-known that \( f \in C_p(X) \) if and only if \( X \setminus Z(f) \) is relatively pseudocompact, i.e., every function in \( C(X) \) is bounded on \( X \setminus Z(f) \), see Theorem 2.1 in [11]. It is well-known that \( C_p(X) = \gamma X \) and \( C_p(X) = M^X \), where \( \gamma X \) is the Stone-Čech compactification and \( vX \) is the Hewitt realcompactification of \( X \), see [5]. For terminology and notations, the reader is referred to [4] and [5].

2. \( m^l \)-Topology on \( C(X) \)

Let \( I \) be an ideal (not necessarily proper) of \( C(X) \). For each \( f \in C(X) \) and \( u \in U^+(X) \), we define the subset \( B(f, I, u) \) of \( C(X) \) as follows:

\[
B(f, I, u) = \{ g \in C(X) : |f - g| < u \text{ and } g \equiv f \text{ (mod} I\text{)} \}.
\]

We define the \( m^l \)-topology on \( C(X) \) by taking the family \( \{ B(f, I, u) : u \in U^+(X) \} \) as a base for a neighborhood system at \( f \) for each \( f \in C(X) \). The set \( C(X) \) endowed with the \( m^l \)-topology is denoted by \( C_m(X) \). To see that \( \{ B(f, I, u) : u \in U^+(X) \} \) is a base at \( f \), it is evident that \( f \in B(f, I, u) \), \( B(f, I, u \cap v) \subseteq B(f, I, u) \cap B(f, I, v) \), for all \( u, v \in U^+(X) \) and whenever \( g \in B(f, I, u) \) for some \( u \in U^+(X) \), then \( B(g, I, v) \subseteq B(f, I, u) \), where \( v = u - |f - g| < u \). If \( I = C(X) \), then the \( m^l \)-topology and the \( m \)-topology coincide and whenever \( f \in \bigcup I \) are two ideals in \( C(X) \), it is clear that the \( m^l \)-topology is finer than the \( m \)-topology. This implies that for each ideal \( I \) in \( C(X) \), the \( m^l \)-topology is finer than the \( m \)-topology.

**Proposition 2.1.** The space \( C_m(X) \) is a topological ring.

**Proof.** If \( u \in U^+(X) \) and \( B(f + g, I, u) \) is a neighborhood at \( f + g \), then we consider the neighborhoods \( B(f, I, \frac{u}{2}) \) and \( B(g, I, \frac{u}{2}) \) at \( f \) and \( g \) respectively. Now suppose that \( h \in B(f, I, \frac{u}{2}) \) and \( k \in B(g, I, \frac{u}{2}) \), then we have \( |h - f| < \frac{u}{2} \), \( |k - g| < \frac{u}{2} \), \( h - f \in I \) and \( k - g \in I \). Hence we have \( |h + k| - (f + g) \leq u \) and \( |h + k| - (f + g) \in I \), i.e., the function \( + \) is continuous. For the continuity of \( \times \), let \( B(fg, I, u) \) be a neighborhood at \( fg \) and take \( v = \frac{u}{2|f|} \) and \( w = \frac{u}{2|g|} \). Now if \( h \in B(f, I, v) \) and \( k \in B(g, I, w) \), then \( |h| < u |f| \) and \( |k| < u |g| \) imply that \( |h| |k| |g| < \frac{u}{2} \) and \( |h| |k| |g| < \frac{u}{2} \). On the other hand \( f(k - g) \in I \) and \( k(h - f) \in I \) imply that \( k(k - g) \in I \) and we are through. \( \square \)

We need the following results in the sequel.

**Proposition 2.2.** The following statements hold.

(a) Every ideal containing \( I \) is a closed-open set in \( C_m(X) \).
(b) If \( I \) is a \( z \)-ideal and \( J \) is a closed ideal in \( C_m(X) \), then \( I \cap J \) is also a \( z \)-ideal.
(c) Every maximal ideal is closed in \( C_m(X) \).
(d) \( C^*(X) \cap I \) is a closed-open set in \( C_m(X) \).
(e) The closure of every proper ideal in \( C_m(X) \) is a proper ideal.

**Proof.** (a) Let \( I \subseteq J \) and \( f \in C_m(J) \), where \( C_m(J) \) means the closure of \( J \) in \( C_m(X) \). Hence there exists \( j \in J \) such that \( j \in B(f, I, J) \). Thus \( f - j \in I \subseteq J \), so \( f \in J \). This implies that \( J \) is closed. On the other hand if \( g \in J \), then \( B(g, I, u) \subseteq J \), for all \( u \in U^+(X) \), i.e., \( J \) is open.

(b) Let \( Z(g) \subseteq Z(f) \) and \( g \in I \cap J \). Since \( I \) is a \( z \)-ideal, it is enough to show that \( f \in J \). For each \( u \in U^+(X) \), we define
\[ h(x) = \begin{cases} \frac{(x+u(x))}{g(x)} & f(x) \leq -u(x) \\ 0 & |f(x)| \leq u(x) \\ \frac{(x-u(x))}{g(x)} & u(x) \leq f(x). \end{cases} \]

Clearly \( h \in C(X) \) and \(|f - gh| < u \). Since \(|f - gh| \in I, f \in cl\m I \). But \( I \) is closed, hence \( f \in I \).

(c) In fact every closed ideal in \( C_m(X) \) is also closed in \( C_m^\ast(X) \).

(d) For each \( f \in C^\ast(X) \cap I \), we have \( B\langle f, I, 1 \rangle \subseteq C^\ast(X) \cap I \). Now if \( h \in cl\m (C^\ast(X) \cap I) \), then there exists \( f \in B\langle f, I, 1 \rangle \cap C^\ast(X) \cap I \). Hence \(|f - h| < 1 \) and \( f - h \in I \), whence \( h \in C^\ast(X) \cap I \).

(e) If \( I \) is an ideal, \( f, g \in cl\m I \) and \( u \in U^\ast(X) \), then there are \( h, k \in C(X) \) such that \( h \in B\langle f, I, \frac{2}{3} \rangle \cap I \), \( k \in B\langle g, I, \frac{2}{3} \rangle \cap I \) and \( h - f, k - g \in I \). Hence it is clear that \( h + k \in B\langle f + g, I, u \rangle \cap I \) and hence \( f + g \in cl\m I \).

Whenever \( f \in cl\m I \), \( g \in C(X) \) and \( u \in U^\ast(X) \), then there exists \( h \in B\langle f, I, \frac{u}{1+u} \rangle \cap I \), i.e., \(|h - f| < \frac{u}{1+u} \) and \( h - f \in I \). Hence \(|gh - fg| < \frac{|u|}{1+u} < u \) and \( gh - fg \in I \) which means that \( B\langle fg, I, u \rangle \cap I \neq \emptyset \), so \( fg \in cl\m I \).

Remark 2.3. Whenever \( S \subseteq C(X) \) and \( C_m(X) \) is endowed with the \( m \)-topology (\( m^\ast \)-topology), then \( S \) may be considered as a subspace of \( C_m(X) \) with the relative topology. We should emphasize here that \( m \)-topology and \( m^\ast \)-topology on \( I \) coincide. In fact whenever \( u \in U^\ast(X) \) and \( f \in I \), we have \( B\langle f, I, u \rangle \cap I = B\langle f, I, u \rangle = B\langle f, u \rangle \cap I \).

3. Connectedness in \( C_m(X) \)

In this section we characterize the components of \( C_m(X) \) and investigate the disconnectedness of \( C_m(X) \). To this end, we need the following lemmas.

Lemma 3.1. \( f \in C_\psi(X) \cap I \), if and only if the function \( \varphi_f : \mathbb{R} \rightarrow C_m^\ast(X) \) defined by \( \varphi_f(r) = rf \), for all \( r \in \mathbb{R} \), is continuous.

Proof. Let \( u \in U^\ast(X) \). Since \( f \in C_\psi(X) \), \( u \) is bounded away from zero on \( X \setminus Z(f) \) for, \( \frac{1}{r} \) is bounded on \( X \setminus Z(f) \). Thus we may assume that \( u(x) > a > 0 \), for all \( x \in X \setminus Z(f) \). If \( |f| < M \), then \( (r - \frac{1}{M}, r + \frac{1}{M}) \subseteq \varphi_f^{-1}(B(rf, I, u)) \). So whenever \( |r - s| < \frac{1}{M} \), then we have \(|rf - sf| < \frac{1}{M}|f| < a < u \). It is also evident that \( rf - sf \in I \) for, \( f \in I \), hence \( \varphi_f \) is continuous. Conversely suppose that \( \varphi_f \) is continuous. Hence for every \( u \in U^\ast(X) \), there exists \( \delta > 0 \) such that \((-\delta, \delta) \subseteq \varphi_f^{-1}(B(0, I, u)) \). This means that for each \( 0 \neq s \in (-\delta, \delta) \), we have \(|sf| < u \) and \( sf \in I \), so \( f \in I \). Now whenever \( g \in C(X) \), by taking \( u = \frac{1}{1+|g|} \in U^\ast(X) \), we have \(|gf| < \frac{1}{1+|g|} |f| < \frac{1}{2} \), for each \( 0 \neq s \in (-\delta, \delta) \). This implies that \( fg \in C(X) \), for all \( g \in C(X) \) and hence by Lemma 2.10 in [7], we have \( f \in C_\psi(X) \), therefore \( f \in C_\psi(X) \cap I \).

Corollary 3.2. \( f \in C_\psi(X) \), if and only if the function \( \varphi_f : \mathbb{R} \rightarrow C_m(X) \) defined by \( \varphi_f(r) = rf \) is continuous, for all \( r \in \mathbb{R} \).

Whenever every element of an ideal in \( C(X) \) is bounded, we call it a bounded ideal. The largest bounded ideal in \( C(X) \) exists by the following result, see Corollary 3.10 in [2] for its proof.

Lemma 3.3. The largest bounded ideal in \( C(X) \) is \( C_\psi(X) \).

The following theorem shows that the component of \( 0 \) in \( C(X) \) with the \( m \)-topology is \( C_\psi(X) \), see also [2]. This theorem also shows that whenever \( I \subseteq C_\psi(X) \), then the component of \( 0 \) in \( C_m(X) \) is \( I \).

Theorem 3.4. The component of \( 0 \) in \( C_m(X) \) is \( C_\psi(X) \cap I \).

Proof. For each \( f \in C_\psi(X) \cap I \), the function \( \varphi_f \) is continuous by Lemma 3.1. Hence \( \varphi_f(\mathbb{R}) \) is connected. But \( C_\psi(X) \cap I = \bigcup_{f \in C_\psi(X) \cap I} \varphi_f(\mathbb{R}) \) means that \( C_\psi(X) \cap I \) is also connected. Now suppose that \( I \) is a connected ideal in \( C_m(X) \). Since by (d) of Proposition 2.2, \( C^\ast(X) \cap I \) is a closed-open set in \( C_m(X) \), we have \( I \subseteq C^\ast(X) \cap I \). This implies that \( I \) is a bounded ideal and hence \( I \subseteq C_\psi(X) \) by Lemma 3.3. Therefore \( I \subseteq C_\psi(X) \cap I \), i.e., \( C_\psi(X) \cap I \) is the component of \( 0 \).
Since the ideal I is an open-closed set in $C_m(X)$, by Proposition 2.2, the following corollary is evident.

**Corollary 3.5.** If I is an ideal in C(X) and $I \subseteq C_0(X)$, then the quasicomponent of 0 in $C_m(X)$ is I.

If $C_0(X) \neq \emptyset$, then $C_0(X) = M^M_{X\times X}$ is free and hence it is an essential ideal (i.e., intersects every nonzero ideal nontrivially) by Proposition 2.1 in [1]. Now if I is a nonzero ideal in C(X), then $C_0(X) \cap I \neq \emptyset$ and the following corollary is evident.

**Corollary 3.6.** The following statements hold.

(a) If $C_0(X)$ is a totally disconnected if and only if either $I = \emptyset$ or $C_0(X) = \{0\}$.

(b) If X is pseudocompact, then $C_0(X)$ is a totally disconnected space if and only if I = (0).

(c) Whenever I is a proper ideal in C(X), then $C_0(X)$ is never connected.

4. **Compactness in $C_m(X)$**

In this section we investigate compact subsets of $C_m(X)$. Using the next theorem, for an infinite space $X$, every compact subset of $C(X)$ with the $m$-topology has an empty interior. To prove the theorem, we first need the following lemma.

**Lemma 4.1.** Suppose that u is unit and I is an ideal in C(X).

(a) If $\{a_1, a_2, \ldots, a_k\} \subseteq X \setminus Z(I)$, for each $1 \leq i \leq k$, there exists $t_i \in I$ such that $|t_i| < u$, $t_i(a_i) = \frac{1}{2}u(a_i)$ and $t_i(a_j) = 0$, for all $j \neq i$.

(b) If $X \setminus Z(I)$ is finite, then the subspace I of $C_m(X)$ is homeomorphic to $\mathbb{R}^k$ for some $k \in \mathbb{N}$.

**Proof.** (a) Since $\{a_1, a_2, \ldots, a_k\}$ and $I \subseteq X \setminus Z(I)$, for each $1 \leq i \leq k$, there exists $s_i \in I$ such that $s_i(a_i) \neq 0$ and $s_i(a_j) = 0$. Without loss of generality, let $s_i(a_i) = 1$ and $s_i \geq 0$. Now consider the function $t_i = \frac{s_i}{2s_i + u}$. Clearly we have $t_i \in I$, $t_i(a_i) = \frac{1}{2}u(a_i)$, $t_i(a_j) = 0$ and $|t_i| < \frac{1}{2}u < u$.

(b) Let $X \setminus Z(I) = \{a_1, a_2, \ldots, a_k\}$. Clearly, each $a_i$ is an isolated point. First we show that $I = (e)$, where $e(a_1) = \cdots = e(a_k) = 1$ and $e(x) = 0$, otherwise. For each $1 \leq i \leq k$, there exists $f_i \in I$ such that $f_i(x) \neq 0$, for $x \in X \setminus Z(I)$. Now $h = f_1^2 + \cdots + f_k^2 \in I$ and $Z(h) = \bigcap Z[I]$. But $Z(h) = Z(e)$ is open, hence $e$ is a multiple of $h$, by 1D in [5], i.e., $e \in I$. On the other hand, for each $f \in I$, we have $Z(e) \subseteq Z(f)$ which means that $f \in (e)$ by 1D in [5] again, i.e., $I = (e)$.

Now corresponding to each $b = (b_1, b_2, \ldots, b_k) \in \mathbb{R}^k$, the function $f_b$ defined by $f_b(a_i) = b_i$, for all $i = 1, \ldots, k$ and $f_b(\bigcap Z[I]) = \{0\}$ belongs to I = (e). We define $\varphi : \mathbb{R}^k \rightarrow I \subseteq C_m(X)$ by $\varphi(b) = f_b$, for all $b \in \mathbb{R}^k$. Clearly, the function $\varphi$ is one to one and onto. The function $\varphi$ is also continuous. In fact for every $f_b \in I$, where $b = (b_1, \ldots, b_k) \in \mathbb{R}^k$ and for each positive unit u in C(X), we have $\varphi^{-1}(B(f_b, I, u) \cap I) = \prod_{i=1}^{k}(b_i - u(a_i), b_i + u(a_i))$.

Finally, $\varphi$ is open for, $\varphi(\overline{\{e_i\}}(b_i - \epsilon_i, b_i + \epsilon_i)) = \{f \in I : |f(a_i) - b_i| < \epsilon_i\}$ is open in I for each $i = 1, \ldots, k$ and $\epsilon_i > 0$. Therefore, I is homeomorphic to $\mathbb{R}^k$. □

**Theorem 4.2.** If I is an ideal in C(X), then every compact subset of $C_m(X)$ has an empty interior if and only if $X \setminus Z[I]$ is infinite.

**Proof.** Let $X \setminus Z[I]$ be infinite and $F$ be a compact subset of $C_m(X)$. Suppose that $f \in \text{int}_{m}(F)$, then there exists $u \in U^*(X)$ such that $B(f, I, u) \subseteq F$. Since F is compact, there are $g_1, g_2, \ldots, g_n \in F$ such that $F \subseteq \bigcup_{i=1}^{n} B(g_i, I, \frac{\epsilon}{4})$. Since $X \setminus Z[I]$ is an infinite set, we may produce a set $\{x_1, x_2, \ldots, x_{n+1}\} \subseteq X \setminus Z[I]$ with distinct elements. Now by invoking Lemma 4.1, for $i \in \{1, 2, \ldots, n+1\}$, we define the function $t_i \in I$ with $|t_i| < u$, where $t_i(x_i) = \frac{1}{2}u(x_i)$ and $t_i(x_j) = 0$, for all $j \neq i$. If we take $h_i = f_i + t_i$, then we have $h_i - f = t_i \in I$ and $|h_i - f| = |t_i| < u$, for all $i = 1, 2, \ldots, n+1$. Therefore $h_i \in B(f, I, u) \subseteq \bigcup_{i=1}^{n} B(g_i, I, \frac{\epsilon}{4})$, for all $k = 1, \ldots, n+1$. This means that for some $1 \leq s \leq n+1$, $B(g_s, I, \frac{\epsilon}{4})$ contains at least two of $h_i$'s. Let $h_i, h_j \in B(g_s, I, \frac{\epsilon}{4})$, for $i \neq j$. Thus we have $|h_i - h_j| < \frac{\epsilon}{2}$ which implies that $|h_i - t_j| < \frac{\epsilon}{2}$. But $t_i(x_j) = 0$ implies that $\frac{1}{4}u(x_j) < \frac{1}{2}u(x_j)$, a contradiction. Conversely, suppose that $X \setminus Z[I]$ is a finite set, say $\{a_1, a_2, \ldots, a_k\}$. By what we have already shown in the proof of Lemma 4.1, the function $\varphi : \mathbb{R}^k \rightarrow I \subseteq C_m(X)$, defined by $\varphi(b) = f_b$, for all $b \in \mathbb{R}^k$ is continuous. Now consider $S = \{f \in I : |f| \leq 1\}$. Clearly $B(0, I, 1) \subseteq S$, implies that $\text{int}_{m}(S) = \emptyset$ and $\varphi([1_{i=1}^{k}(-1, 1)]) = S$ implies that S is compact and the proof is complete. □
Proposition 4.3. If $I$ is an ideal in $C(X)$, then $I$ is a locally compact subspace of $C_m(X)$ if and only if $X \setminus \bigcap Z[I]$ is finite.

Proof. If $I$ is locally compact, then by Proposition 2.2 and Theorem 4.2, $X \setminus \bigcap Z[I]$ is finite. On the other hand, whenever $X \setminus \bigcap Z[I]$ is finite, then by Lemma 4.1, $I$ as a subspace of $C_m(X)$ is homeomorphic to $\mathbb{R}^k$ for some $k \in \mathbb{N}$, so $I$ is locally compact. \qed

By Lemma 3.3 and Theorem 4.2, the following result is evident. We note that whenever $f \in I \setminus C_\psi(X)$, then $f$ is unbounded, by Lemma 3.3 and hence $X \setminus \bigcap Z[I]$ must be infinite.

Corollary 4.4. If $I \not\subseteq C_\psi(X)$, then every compact subset of $C_m(X)$ has an empty interior.

The following result is also an immediate consequence of our Theorem 4.2 and Proposition 2.1 in [1], see also Proposition 3.2 in [12] for more general case.

Corollary 4.5. If $I$ is an essential ideal in $C(X)$, then every compact subset of $C_m(X)$ has an empty interior.

We conclude this section by the following proposition which investigates the compactness and Lindelöfness of ideals in $C_m(X)$. For an example of a Lindelöf ideal in $C_m(X)$ (which coincides with $C_m(X)$, where $I = C(X)$), see Example 4.7 in [2].

Proposition 4.6. Let $I$ be an ideal in $C(X)$.

(a) $I$ is never compact in $C_m(X)$,

(b) If $I$ is Lindelöf in $C_m(X)$, then $I \subseteq C_\psi(X)$.

Proof. (a) Let $I$ be compact and $u \in U^*(X)$. Since $I \subseteq \bigcup_{f \in I} B(f, I, u)$, there are $f_1, f_2, \ldots, f_n \in I$ such that $I \subseteq \bigcup_{j=1}^n B(f_j, I, u)$. Suppose that $x_0 \notin \bigcap_{j \in I} Z(f)$ and let $\alpha = \sup \{|f_1(x_0)| + u(x_0), \ldots, |f_n(x_0)| + u(x_0)|$. Take $f \in I$ such that $f(x_0) = \alpha$ (if $g \in I$ with $g(x_0) \neq 0$, consider $f = \alpha \frac{g}{\sup I} \in I$). Thus $f \in B(f, I, u)$ for some $1 \leq k \leq n$. Hence $|f| < |f_k| + u$ implies that $\alpha = |f(x_0)| < f_k(x_0) + u(x_0)$, a contradiction.

(b) Let $I \not\subseteq C_\psi(X)$. To prove that $I$ is not Lindelöf, it is enough to show that every open cover of $I$ is uncountable. Suppose that $I \subseteq \bigcup_{n=1}^\infty B(f_n, I, u_n)$, where $f_n \in C(X)$ and $u_n \in U^*(X)$, for all $n \in \mathbb{N}$. Since $I \not\subseteq C_\psi(X)$, there is an unbounded $f \in I$. Now using 1.20 in [5], there exists a copy of $\mathbb{N}$, say a sequence $\{x_n\}$ in $X$, C-embedded in $X$ on which $f$ is unbounded. Without loss of generality, we suppose that $|f(x_n)| > 1$, for all $n \in \mathbb{N}$. But $\{x_n\}$ is C-embedded, so a function $g \in C(X)$ exists such that $g(x_n) = |f(x_n)| + u(x_n)$. Now $fg \in I \subseteq \bigcup_{n=1}^\infty B(f_n, I, u_n)$ implies that $fg \in B(f_m, I, u_m)$ for some $m \in \mathbb{N}$. Therefore $|g(x_n)| < |f(x_n)||g(x_m)| < |f_m(x_m)| + u_m(x_m)$, a contradiction. \qed

5. Characteristics of the Space $X$ via Properties of Some Subspaces of $C_m(X)$

We devote this section to the special case $I = C(X)$ of $m^I$-topology on $C(X)$, i.e., to the $m$-topology on $C(X)$. In this section we investigate some relations between topological spaces $X$ and $C_m(X)$. The set $U(X)$ of units, the set $D(X)$ of zero-divisors, the set $r(X)$ of regulars (nonzerodivisors) and ideals of $C(X)$ are important subspaces of $C_m(X).$ We show that some properties of these subspaces completely determine the space $X$. For example, we show that $U(X)$ is dense in $C_m(X)$ if and only if $X$ is strongly zero-dimensional and $D(X)$ is closed in $C_m(X)$ if and only if $X$ is an almost $P$-space. First we recall that a space $X$ is strongly zero-dimensional if for every pair $A, B$ of completely separated subsets of the space $X$, there exists an open-closed set $G$ such that $A \subseteq G \subseteq X \setminus B$, see Theorem 6.2.5 in [4]. We also recall that a space $X$ is called an almost $P$-space if every nonempty $\mathcal{G}_0$-set (zero-set) in $X$ has a nonempty interior. Characterization of the space $X$ for which $r(X)$ ($C_X(X)$) is dense (closed) in $C_m(X)$ is also given in this section.

Proposition 5.1. $U(X)$ is dense in $C_m(X)$ if and only if $X$ is strongly zero-dimensional.

Proof. Let $X$ be strongly zero-dimensional, $f \in C(X)$ and $u$ be a positive unit in $C(X)$. Suppose that
Since $G$ and $H$ are two disjoint zero-sets and $X$ is strongly zero-dimensional, there exists an open-closed set $K$ in $X$ such that $G \subseteq K \subseteq X \setminus H$. Now define

$$v(x) = \begin{cases} 
  f(x) + \frac{1}{2} u(x) & x \in K \\
  f(x) - \frac{1}{2} u(x) & x \notin K.
\end{cases}$$

Clearly $v$ is unit, in fact if $x \in K$, then $x \notin H$ and hence $f(x) > -\frac{1}{2} u(x)$, i.e., $v(x) = f(x) + \frac{1}{2} u(x) > 0$ and if $x \notin K$, then $x \notin G$, so $f(x) - \frac{1}{2} u(x) = v(x) < 0$. Moreover, $|f - v| = \frac{1}{2} u < u$, i.e., $U(X)$ is dense in $C_m(X)$.

Conversely, let $U(X)$ be dense in $C_m(X)$ and $Z_1$ and $Z_2$ be two disjoint zero-sets. Suppose that $f \in C(X)$ such that $f(Z_1) = [-1]$ and $f(Z_2) = [1]$. Consider $u = \frac{1}{2}$, then there exists a unit $v \in B(f, \frac{1}{2})$, i.e., $|f - v| < \frac{1}{2}$. Let $K = \{x \in X : v(x) < 0\}$. Since $v$ is unit, $K$ is open-closed. Clearly $Z_1 \subseteq K \subseteq X \setminus Z_2$ which means that $X$ is strongly zero-dimensional. \(\square\)

**Proposition 5.2.** The set $D(X)$ of zerodivisors of $C(X)$ is closed in $C_m(X)$ if and only if $X$ is an almost $P$-space.

**Proof.** It is enough to show that $cl_m D(X) = C_m(X) \setminus U(X)$. Clearly $U(X)$ is open in $C_m(X)$ for, if $u \in U(X)$, then $B(u, \pi) \subseteq U(X)$, where $\pi = \frac{u|}{\pi}$. In fact if $f \in B(u, \pi)$, then $|f - u| < \frac{u}{2}$ implies that $Z(f) = \emptyset$, i.e., $f \in U(X)$. Thus $C_m(X) \setminus U(X)$ is closed and hence $cl_m D(X) \subseteq C_m(X) \setminus U(X)$. Now suppose that $f \in C_m(X) \setminus U(X)$ and $\pi$ is positive unit. We show that $B(f, \pi) \cap D(X) \neq \emptyset$. Define

$$h(x) = \begin{cases} 
  f(x) + \frac{1}{2} \pi(x) & f(x) \leq -\frac{1}{2} \pi(x) \\
  0 & |f(x)| < \frac{1}{2} \pi(x) \\
  f(x) - \frac{1}{2} \pi(x) & |f(x)| \geq \frac{1}{2} \pi(x).
\end{cases}$$

Clearly $h \in C(X)$ and $|f - h| < \pi$, i.e., $h \in B(f, \pi)$. On the other hand $G = \{x \in X : |f(x)| < \frac{1}{2} \pi(x)\}$ is a nonempty open set in $X$, for $\emptyset \neq Z(f) \subseteq G$. Since $G \subseteq Z(h)$, the interior of $Z(h)$ is nonempty and hence $h \in D(X)$, i.e., $B(f, \pi) \cap D(X) \neq \emptyset$. \(\square\)

In the following proposition we characterize spaces $X$ for which the subset $r(X)$ of $C(X)$ is dense in $C_m(X)$. This proposition shows that for space $X = \mathbb{R}$ and more generally for a perfectly normal space $X$, the set $r(X)$ is dense in $C_m(X)$. First we prove the following lemma.

**Lemma 5.3.** Let $A$ and $B$ be two disjoint sets. $A$ and $B$ can be separated by disjoint cozero-sets whose union is dense if and only if there exists $g \in r(X)$ such that $A \subseteq posg$ and $B \subseteq negg$.

**Proof.** If there is such $g \in r(X)$, then $posg$ and $negg$ are cozero-sets whose union is dense for, $int_X Z(g) = \emptyset$. Conversely, suppose that $A$ and $B$ are separated by disjoint cozero-sets $cozh$ and $cozk$ respectively whose union is dense. Define

$$g(x) = \begin{cases} 
  |h(x)| & x \in cozh \\
  0 & x \in Z(h) \cap Z(k) \\
  -|k(x)| & x \in cozk.
\end{cases}$$

Clearly $g \in C(X)$, $int_X Z(g) = \emptyset$ (i.e., $g \in r(X)$), $A \subseteq posg$ and $B \subseteq negg$. \(\square\)

**Proposition 5.4.** $r(X)$ is dense in $C_m(X)$ if and only if disjoint zero-sets in $X$ can be separated by disjoint cozero-sets whose union is dense in $X$.

**Proof.** Suppose that $r(X)$ is dense in $C_m(X)$ and $Z(f) \cap Z(g) = \emptyset$. Consider $h \in C(X)$ such that $|h| \leq a$, $a > 0$ and $h(Z(f)) = [a]$, $h(Z(g)) = [-a]$. Since $r(X)$ is dense, there exists $k \in r(X) \cap B(h, a)$. Hence $h - a < k < h + a$ and $int_X Z(k) = \emptyset$. If $x \in Z(f)$, then $k(x) > h(x) - a = -a = 0$ and if $x \in Z(g)$, then $k(x) < h(x) + a = -a + a = 0$, i.e., $Z(f) \subseteq posk$, $Z(g) \subseteq negk$. Now by our lemma, we are through.
Conversely, suppose that disjoint zero-sets can be separated by disjoint cozero-sets whose union is dense in $X$. Let $f \in C(X)$ and $\pi$ be a positive unit in $C(X)$. By our lemma, there exists $g \in r(X)$ such that $\{x \in X : f(x) \geq \frac{-\pi}{2}(x)\} \subseteq posg$ and $\{x \in X : f(x) \leq -\frac{\pi}{2}(x)\} \subseteq negg$ and we consider $|g| \leq \frac{\pi}{2}$. Now define $h = \{(f + \frac{\pi}{2}) \land g\} \lor (f - \frac{\pi}{2})$. Clearly $h \geq f - \frac{\pi}{2}$. Hence for each $x \in X$, either $h(x) = f(x) - \frac{\pi(x)}{2} \leq f(x) + \frac{\pi(x)}{2}$ or $h(x) = \{(f(x) + \frac{\pi(x)}{2}) \land g\} \leq f(x) + \frac{\pi(x)}{2}$. Therefore $f - \frac{\pi}{2} \leq h \leq f + \frac{\pi}{2}$ and hence $h \in B(f, \pi)$. On the other hand, if $h(x) = 0$, then $f(x) \neq \frac{\pi(x)}{2}$. Whenever $f(x) = -\frac{\pi}{2}(x)$, then $g(x) < 0$, so $h(x) = g(x) \lor (f(x) - \frac{\pi}{2}(x)) = g(x) \lor -\pi(x) = g(x) < 0$ (note that $g(x) \geq -\frac{\pi}{2}(x)$). If $f(x) = \frac{\pi}{2}(x)$, then $g(x) > 0$, $f(x) + \frac{\pi}{2}(x) = \pi(x) > -\frac{\pi}{2}(x) \geq g(x)$ and hence $h(x) = g(x) \lor (f(x) - \frac{\pi}{2}(x)) = g(x) \lor 0 = g(x) > 0$. Also, $f(x) < -\frac{\pi}{2}(x) \land f(x) > \frac{\pi}{2}(x)$ do not happen. In fact $f(x) < -\frac{\pi}{2}(x)$ implies $g(x) < 0$, hence $h(x) < 0$ and $f(x) > \frac{\pi}{2}(x)$ implies $g(x) > 0$, so $h(x) > 0$. Therefore $f(x) - \frac{\pi(x)}{2} \leq h(x) < f(x) + \frac{\pi(x)}{2}$ and this means that $g(x) = 0$. Consequently, $Z(h) \subseteq Z(g)$ and hence $\text{int}_X Z(h) = \emptyset$, since $g \in r(X)$. This implies that $B(f, \pi) \cap r(X) \neq \emptyset$, i.e., $r(X)$ is dense in $C_m(X)$. \qed

In the following result, we observe that for any space $X$ satisfying countable chain condition, i.e., for any space $X$ with countable cellularity $\chi$, the set $r(X)$ is also dense in $C_m(X)$. The smallest cardinal number $a \geq N_\omega$ such that every family of pairwise disjoint nonempty open subsets of $X$ has cardinality less than or equal to $a$, is called the cellularity of the space $X$ and is denoted by $\chi(X)$. If $\chi(X) = N_\omega$, we say $X$ satisfies the countable chain condition.

**Proposition 5.5.** If $\chi(X) = N_\omega$, then $r(X)$ is dense in $C_m(X)$.

**Proof.** Let $f \in C(X)$ and $\pi$ be a positive unit in $C(X)$. For every $a \in (0,1)$, we define $Z_a = \{x \in X : f(x) = a\}$. Clearly $Z_a \cap Z_b = \emptyset$, for all $a, b \in (0,1)$ and $a \neq b$. Since $C(X) = N_\omega$, then $\text{int}_X Z_a = \emptyset$ for some $a \in (0,1)$. Now we consider $h = f - a\pi$. Since $Z(h) = Z_a$, then $h \in r(X)$ and $|h - f| = a\pi < \pi$, i.e., $h \in B(f, \pi) \cap r(X)$. \qed

We conclude the paper with the following result which characterizes the space $X$ for which the ideal $C_K(X)$ is closed in $C_m(X)$. We recall that a space $X$ is called $\mu$-compact if $C_K(X) = I(X) := \bigcap_{f \in C(X)} M^\mu$, see [9] for more details of such spaces.

**Proposition 5.6.** The ideal $C_K(X)$ is closed in $C_m(X)$ if and only if $X$ is $\mu$-compact.

**Proof.** It is enough to show that $\text{cl}_m C_K(X) = I(X)$. Since $C_K(X) = \bigcup_{f \in C(X)} M^\mu$, we have $C_K(X) \subseteq \bigcup_{f \in C(X)} M^\mu = I(X)$. But $I(X)$ is closed, so $\text{cl}_m C_K(X) \subseteq I(X)$. Now suppose that $f \in I(X)$, then $\beta X \setminus X \subseteq \text{cl}_m Z(f)$. For every positive unit $\pi$ in $C(X)$, we must show that $B(f, \pi) \cap C_K(X) = \emptyset$. Consider the function $h$ defined in the proof of Proposition 5.2 and the zero-set $H = \{x \in X : f(x) \geq \frac{\pi(x)}{2}\}$, so $H = Z(g)$, for some $g \in C(X)$. Clearly $Z(f) \subseteq X \setminus Z(g) \subseteq Z(h)$, for if $f(x) = 0$, then $x \notin H$, hence $x \in X \setminus Z(g)$ and this implies that $|f(x)| < \frac{\pi(x)}{2}$, so $x \notin Z(h)$. Now $\text{cl}_m Z(h)$ is a neighborhood of $\text{cl}_m Z(f)$ and we have $\beta X \setminus X \subseteq \text{cl}_m Z(f) \subseteq \text{int}_X \text{cl}_m Z(h)$, therefore $h \in \bigcup_{f \in C(X)} M^\mu = C_K(X)$. On the other hand $|f - h| < \pi$, i.e., $h \in B(f, \pi)$ which means that $h \in B(f, \pi) \cap C_K(X)$. \qed

**References**


