Coincidence and Fixed Points for Multivalued Mappings in Incomplete Metric Spaces with Applications

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Abstract. In the present paper, firstly, we review the notion of \(R\)-complete metric spaces, where \(R\) is a binary relation (not necessarily a partial order). This notion lets us to consider some fixed point theorems for multivalued mappings in incomplete metric spaces. Secondly, as motivated by the recent work of Wei-Shih Du (On coincidence point and fixed point theorems for nonlinear multivalued maps, Topology and its Applications 159 (2012) 49–56), we prove the existence of coincidence points and fixed points of a general class of multivalued mappings satisfying a new generalized contractive condition in \(R\)-complete metric spaces which extends some well-known results in the literature. In addition, this article consists of several non-trivial examples which signify the motivation of such investigations. Finally, we give an application to the nonlinear fractional boundary value equations.

1. Introduction and Preliminaries

Throughout this paper, \(\mathbb{N}, \mathbb{Q}\) and \(\mathbb{R}\) denote, respectively, the sets of all natural numbers, rational numbers and real numbers.

Let \((X, d)\) be a metric space. We denote by \(CB(X)\) the class of all nonempty closed and bounded subsets of \(X\), and \(K(X)\) the class of all nonempty compact subsets of \(X\).

For \(A, B \in CB(X)\) and \(x \in X\), define

\[
D(x, A) := \inf\{d(x, a); a \in A\}
\]

and

\[
H(A, B) := \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}.
\]

The function \(H\) is a metric on \(CB(X)\) and is called a Pompeiu-Hausdorff metric induced by \(d\). It is well known that if \(X\) is a complete metric space, then so is the metric space \((CB(X), H)\).

Let \(f : X \to X\) be a self-mapping and \(T : X \to CB(X)\) be a multivalued map. A point \(x \in X\) is a coincidence point of \(f\) and \(T\) if \(fx \in Tx\). If \(f = id\), the identity mapping, then \(x = fx \in Tx\) and we call \(x\) a fixed point of \(T\). The set of fixed points of \(T\) and the set of coincidence points of \(f\) and \(T\) are denoted by \(F(T)\) and \(COP(f, T)\), respectively.

In 1969, Nadler [15] extended the Banach contraction principle to multivalued mappings as follows.

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Theorem 1.1. Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that $H(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in T(z)$.

Inspiring from the results of Nadler the fixed point theory of multivalued contraction was further developed in different directions by many authors, in particular, by Reich [18], Berinde-Berinde [7], Mizoguchi and Takahashi [14], Du [11], Daffer et al. [9, 10], Amini-Harandi [2], Boonsri et al. [8], Petrusel et al.[16] and many others.


Theorem 1.2. Let $(X, d)$ be a complete metric space. Let $T : X \rightarrow CB(X)$ be a multivalued mapping, $f : X \rightarrow X$ be a continuous self-mapping and $\beta : [0, \infty) \rightarrow [0, 1)$ be a function such that $\lim \sup_{s \rightarrow t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that

\begin{enumerate}[(a1)]
  \item for each $x \in X$, $(f y : y \in Tx) \subseteq Tx$;
  \item there exists a function $\hat{h} : X \rightarrow [0, \infty)$ such that
\end{enumerate}

\[ H(Tx, Ty) \leq \beta(d(x, y))d(x, y) + \hat{h}(f y)D(f y, Tx) \]

for each $x, y \in X$. Then $COP(f, T) \cap F(T) \neq \emptyset$.

In the following, we state Berinde-Berinde’s fixed point theorem [7].

Theorem 1.3. Let $(X, d)$ be a complete metric space. Let $T : X \rightarrow CB(X)$ be a multivalued mapping and $\beta : [0, \infty) \rightarrow [0, 1)$ be a function such that $\lim \sup_{s \rightarrow t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that

\[ H(Tx, Ty) \leq \beta(d(x, y))d(x, y) + L.D(y, Tx) \]

for each $x, y \in X$, where $L \geq 0$. Then $F(T) \neq \emptyset$.

Notice that, if we let $L = 0$ in above theorem, then we can obtain Mizoguchi-Takahashi’s fixed point theorem [14] which is a partial answer of Problem 9 in [18]. Indeed, Reich established the following:

Theorem 1.4. Let $(X, d)$ be a complete metric space. Let $T : X \rightarrow K(X)$ be a multivalued mapping and $\beta : [0, \infty) \rightarrow [0, 1)$ be a function such that $\lim \sup_{s \rightarrow t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that

\[ H(Tx, Ty) \leq \beta(d(x, y))d(x, y) \]

for each $x, y \in X$. Then $F(T) \neq \emptyset$.

Reich [18] posed the question whether above theorem is also true for a mapping $T : X \rightarrow CB(X)$. Mizoguchi and Takahashi [14] in 1989 responded to this conjecture and proved the following theorem which additionally is more general than Nadler’s theorem.

Theorem 1.5. Let $(X, d)$ be a complete metric space. Let $T : X \rightarrow CB(X)$ be a multivalued mapping and $\beta : [0, \infty) \rightarrow [0, 1)$ be a function such that $\lim \sup_{s \rightarrow t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that

\[ H(Tx, Ty) \leq \beta(d(x, y))d(x, y) \]

for each $x, y \in X$. Then $F(T) \neq \emptyset$.

In 2011, Amini-Harandi [2] introduced the concept of a set-valued quasi-contraction and proved the following interesting fixed point theorem.

Theorem 1.6. Let $(X, d)$ be a complete metric space. Let $T : X \rightarrow CB(X)$ be a multivalued mapping. Assume that

\[ H(Tx, Ty) \leq k.\max\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\} \]

for each $x, y \in X$, where $0 < k < \frac{1}{2}$. Then $F(T) \neq \emptyset$. 
On the other hand, Boonsri and Saejung in [8] showed that the conclusion of Daffer and Kaneno[9] remains true without assuming the lower semicontinuity of the function \( x \mapsto D(x, Tx) \). In the following, we state Boonsri-Saejung’s fixed point theorem.

**Theorem 1.7.** Let \((X, d)\) be a complete metric space. Let \( T : X \to CB(X) \) be a multivalued mapping. Assume that

\[
H(Tx, Ty) \leq k \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\}
\]

for each \( x, y \in X \), where \( 0 < k < 1 \). Then \( F(T) \neq \emptyset \).

As motivated by these works, we define a new type of monotone multivalued mappings and prove some coincidence point and fixed point theorems under a new generalized contractive condition which are different from Nadler’s theorem, Berinde-Berinde’s theorem, Boonsri-Saejung’s theorem, Mizoguchi-Takahashi’s theorem, Du’s theorem and Amini-Harandi’s theorem for nonlinear multivalued contractive mappings. Our results compliment and extend some important fixed point theorems for multivalued contractive mappings.

### 2. Basic Definitions and Notations

Very recently, Eshaghi Gordji et al. [12] and Baghani et al. [4] introduced the notation of orthogonal sets and gave a real generalization of the Banach fixed point theorem in incomplete metric spaces. The notion helps them to find the solution of an integral equation in incomplete metric spaces. For more details, we refer the reader to [1, 3, 5, 6, 17].

To set up our results in the next sections, we need to introduce some definitions that play a major role in further sections.

Let \( X \) be a nonempty set, \( A, B \subseteq X \) and \( R \) be an arbitrary binary relation on \( X \). The binary relations strongly relation (briefly, SR) and weakly relation (briefly, WR) are defined between \( A \) and \( B \) as follows.

1. \( A \ (\text{SR}) \ B \) if \( a R b \), for all \( a \in A \) and \( b \in B \).
2. \( A \ (\text{WR}) \ B \) if for each \( a \in A \) there exists \( b \in B \) such that \( a R b \).

It is clear that the relation SR implies the relation WR. Example 2.2 shows that the converse of the statement is not true in general. Now, we introduce a type of monotone multivalued mappings by using the relation SR.

**Definition 2.1.** Let \((X, d)\) be a metric space endowed a relation \( R \) on \( X \) and \( T : X \to CB(X) \). Then \( T \) is said to be a monotone mapping of type SR if

\[
x, y \in X, x \ R \ y \Rightarrow Tx \ (\text{SR}) \ Ty.
\]

**Example 2.2.** Let \( X = \left\{ \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots \right\} \cup \{0, 1\} \), \( d(x, y) = |x - y| \) for all \( x, y \in X \), and relation \( R \) be defined on \( X \) by

\[
x \ R \ y \iff \begin{cases} \frac{x}{2} \in \mathbb{N}, \\ or \ x = y = 0. \end{cases}
\]

Let \( T : X \to CB(X) \) be defined by

\[
Tx = \begin{cases} \left\{ \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots \right\}, & \text{if } x = \frac{1}{2^n}, n = 1, 2, \cdots, \\ \{0\}, & \text{if } x = 0, \\ \{1, \frac{1}{2}, \frac{1}{4}\}, & \text{if } x = 1. \end{cases}
\]

It is easy to see that \( T \) is not monotone of type SR.
Example 2.3. Let $X = [0, 1)$ be equipped with the Euclidean metric. Define relation $R$ on $X$ by $x R y$ if and only if $x = 0$ or $y = 0$. Let $T : X \rightarrow CB(X)$ be a mapping defined by

$$T(x) = \begin{cases} \left\{ \frac{1}{2}x^2, x \right\}, & \text{if } x \in \mathbb{Q} \cap X, \\ [0], & \text{if } x \in \mathbb{Q}^c \cap X. \end{cases}$$

It is easy to see that $T$ is monotone of type $SR$.

Definition 2.4. Let $X \neq \emptyset$ and $R \subseteq X \times X$ be a relation. A sequence $\{x_n\}$ is called an $R$-sequence if

$$(\forall n, k \in \mathbb{N}) : x_n Rx_{n+k}.$$ 

Definition 2.5. Let $(X, d)$ be a metric space and $R$ be a relation on $X$. Then $X$ is said to be $R$-regular if for each $R$-sequence $\{x_n\}$ with $x_n \rightarrow x$ for some $x \in X$, there exists $n_0 \in \mathbb{N}$ such that

$$(\forall n \geq n_0) : x_n Rx.$$ 

Definition 2.6. Let $(X, d)$ be a metric space and $R$ be a relation on $X$. Then $X$ is said to be $R$-complete if every Cauchy $R$-sequence is convergent (briefly, $(X, d, R)$ is called an $R$-complete metric space).

Example 2.7. Consider $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ equipped with the Euclidean metric. Define relation $R$ on $X$ by $R = \{(0, 0), (0, 1), (1, 0), (1, 1), (0, 2)\}$. It is easy to see that $(X, d, R)$ is an $R$-complete (not complete) metric space. We are going to show that $(X, d, R)$ is an $R$-regular metric space. Take $R$-sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} x_n = x$. Since $\{x_n\}$ is an $R$-sequence then for each $n \in \mathbb{N}$, $(x_n, x_{n+1}) \subseteq \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ which gives rise to $\{x_n\} \subseteq [0, 1]$. As $\{0, 1\}$ is closed, we have $x_n Rx$ for all $n \in \mathbb{N}$.

Example 2.8. Let $X$ be a linear subspace of a Hilbert space $H$. For all $x, y \in X$, define $x R y$ if $\| \langle x, y \rangle \| = \| x \| \| y \|$. We claim that $(X, \| \|, R)$ is an $R$-complete metric space which is not $R$-regular. Let $\{x_n\} \subseteq X$ be a Cauchy $R$-sequence. Then $\{x_n\}$ converges to some $x \in H$. Our aim is to show that $x$ is an element of $X$. The relation $R$ ensures that for all $n \in \mathbb{N}$,

$$\exists \alpha_n \text{ s.t. } x_n = \alpha_n x_{n+1} \text{ or } x_{n+1} = \alpha_n x_n.$$  

We distinguish two cases.

Case 1. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = 0$ for all $k$. This implies that $x = 0 \in X$.

Case 2. For all sufficiently large $n \in \mathbb{N}$, $x_n \neq 0$. Take $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $x_n \neq 0$. It follows from (1) that for all $n \geq n_0$ there exists $\alpha_n > 0$ such that $x_n = \alpha_n x_m$. In other words,

$$\| x_n - \alpha_m \| \| x_m \| = \| x_n - \alpha_m \| \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$ 

Therefore, $\{\alpha_n\}$ is a Cauchy sequence in $\mathbb{R}$. Assume that $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \alpha_n x_{n_0} = \alpha x_{n_0}$. This implies that $x \in X$.

Remark 2.9. Every complete metric space is $R$-complete, but Examples 2.8 and 2.7 show that the converse is not true in general.

Definition 2.10. Let $\Lambda$ denote the class of those functions $\phi(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \rightarrow \mathbb{R}$, which satisfy the following conditions

$(\Lambda_1)$ $\phi$ is increasing in $t_2, t_3, t_4$ and $t_5$;

$(\Lambda_2)$ $v < \phi(u, u, v, u + v, 0)$ implies that $v < u$, for each $u, v \in \mathbb{R}_+$;

$(\Lambda_3)$ If $t_{n+1} \rightarrow 0$ and $u_n \rightarrow \gamma > 0$, as $n \rightarrow \infty$, then we have $\limsup_{n \rightarrow \infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1}) \leq \gamma$;

$(\Lambda_4)$ $\phi(u, u, u, 2u, 0) \leq u$ for each $u \in \mathbb{R}^+ := [0, +\infty)$.

Many functions belong to the class $\Lambda$ as shown by the following examples.
Example 2.11. (I) 
\[ \phi_1(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma t_3 + \delta t_4 + \lambda t_5, \]
where \( \alpha, \beta, \gamma, \delta, \lambda \geq 0 \), \( \alpha + \beta + \gamma + 2\delta = 1 \) and \( \gamma \neq 1 \).

(II) 
\[ \phi_2(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2} \max(t_1, t_2, t_3, t_4, t_5). \]

(III) 
\[ \phi_3(t_1, t_2, t_3, t_4, t_5) = \max(t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)). \]

Example 2.12. Let \( \phi \in \Lambda \). Suppose \( \hat{\phi} : \mathbb{R}^5_+ \to \mathbb{R}_+ \) is defined by 
\[ \hat{\phi}(t_1, t_2, t_3, t_4, t_5) = \phi(t_1, t_2, t_3, t_4, t_5) + L t_5, \]
where \( L \geq 0 \). It is easy to see that \( \hat{\phi} \in \Lambda \).

Definition 2.13. Let \((X, d)\) be a metric space and \( R \) be a relation on \( X \). A mapping \( f : X \to X \) is \( R \)-continuous at \( a \in X \) if for each \( R \)-sequence \( \{a_n\} \) in \( X \), if \( a_n \to a \), then \( f(a_n) \to f(a) \). Also, \( f \) is \( R \)-continuous on \( X \) if \( f \) is \( R \)-continuous at each \( a \in X \).

Example 2.14. Let \( X = [0, 1] \) with the Euclidean metric. Assume \( x, y \in \mathbb{R} \) and \( x \neq y \) is only if \( xy = 0 \). Define \( f : X \to X \) by
\[ f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1], \\ x, & \text{if } x \in \mathbb{Q}^c \cap [0, 1]. \end{cases} \]

Notice that \( f \) is not continuous but we can see that \( f \) is \( R \)-continuous. If \( \{x_n\} \) is a \( R \)-sequence in \( X \) which converges to \( x \in X \). Applying definition \( R \) we obtain \( x_n = 0 \). This implies that \( 1 = f(x_n) \to f(x) = 1 \).

3. Main Results

In below, we state and prove the main theorem of this manuscript in \( R \)-complete metric spaces. This theorem helps us to find coincidence points and fixed points for multivalued mappings in incomplete metric spaces.

Theorem 3.1. Let \((X, d, R)\) be an \( R \)-complete (not necessarily complete) and \( R \)-regular metric space. Let \( T : X \to CB(X) \) be a multivalued mapping, \( f : X \to X \) be an \( R \)-continuous self-mapping and \( \varphi : [0, \infty) \to [0, 1) \) be a function such that \( \limsup_{s \to t} \varphi(s) < 1 \) for each \( t \geq 0 \). Assume that
(a1) for each \( x \in X \), \( \{fy : y \in Tx\} \subseteq Tx \);
(a2) there exist functions \( \hat{h} : X \to [0, \infty) \) and \( \phi \in \Lambda \) such that
\[ H(Tx, Ty) \leq \varphi(d(x, y))D(x, Tx, D(y, Ty), D(x, Ty), D(y, Tx)) + \hat{h}(y)D(fy, Tx) \]
for each \( x, y \in X \). Suppose that
(i) \( T \) is monotone of type \( SR \);
(ii) there exists \( x_0 \in X \) such that for each \( x \in X \), \( \{x_0\} \) (WR) \( Tx \).
Then \( COP(f, T) \cap F(T) \neq \emptyset \).

Proof. By (a1), we note that, for each \( x \in X \), \( D(fy, Tx) = 0 \) for all \( y \in Tx \). Also, it is easy to see that, if \( x' \in T(x') \), then \( x' \in COP(f, T) \cap F(T) \). For this reason we suppose that \( T \) has no fixed point, i.e., \( D(x, Tx) > 0 \) for all \( x \in X \).

By properties of functions \( \varphi \), for each \( t > 0 \), there exist \( k(t) > 0 \) and \( \delta(t) > 0 \) such that
\[ \varphi(s) \leq k(t) < 1 \quad \text{for all } s \in (t, t + \delta(t)). \]
Since \( \{x_0\} \) (WR) \( TX_0 \), there exists \( x_1 \in TX_0 \) such that \( x_0 \) R \( x_1 \). If \( x_0 = x_1 \), then \( x_0 = x_1 \in TX_0 \) and this is a contradiction. So, we may assume that \( x_0 \neq x_1 \). Moreover by monotonicity of \( T \), we have \( TX_0 \) (SR) \( TX_1 \). Put \( t_1 = D(x_1, TX_1) \). It is clear that \( D(x_1, TX_1) \leq d(x_1, y) \) for all \( y \in TX_1 \). The following cases are considered:

**Case 1.** \( D(x_1, TX_1) < d(x_1, y) \) for all \( y \in TX_1 \). Select positive number \( d(t_1) \) such that

\[
d(t_1) < \min[\delta(t_1), (\frac{1}{k(t_1)} - 1)t_1],
\]

and put

\[
e(x_1) = \min[1, \frac{d(t_1)}{t_1}].
\]

Then there exists \( x_2 \in TX_1 \) such that \( x_1 \) R \( x_2 \) and

\[
d(x_1, x_2) < D(x_1, TX_1) + e(x_1)D(x_1, TX_1) = (1 + e(x_1))D(x_1, TX_1).
\]

By the hypotheses that \( T \) no fixed point, we have \( x_1 \neq x_2 \). On the other hand by (2) and \((\Lambda_1)\), we can write

\[
D(x_2, TX_2) \leq H(TX_1, TX_2)
\]

\[
\leq \varphi(d(x_1, x_2)) \varphi(d(x_1, x_2), D(x_1, TX_1), D(x_2, TX_2), D(x_1, TX_2), D(x_2, TX_1))
\]

\[
\leq \varphi(d(x_1, x_2)) \varphi(d(x_1, x_2), d(x_1, x_2), D(x_2, TX_2), D(x_1, TX_2), 0)
\]

\[
\leq \varphi(d(x_1, x_2)) \varphi(d(x_1, x_2), d(x_1, x_2), D(x_2, TX_2), d(x_1, x_2) + D(x_2, TX_2), 0)
\]

\[
\leq \varphi(d(x_1, x_2), d(x_1, x_2), D(x_2, TX_2), d(x_1, x_2) + D(x_2, TX_2), 0).
\]

Now by above relation, \((\Lambda_2)\), \((\Lambda_1)\) and \((\Lambda_4)\), we conclude that

\[
D(x_2, TX_2) \leq \varphi(d(x_1, x_2))d(x_1, x_2).
\]

Therefore

\[
D(x_1, TX_1) - D(x_2, TX_2) \geq D(x_1, TX_1) - \varphi(d(x_1, x_2))d(x_1, x_2)
\]

\[
> (\frac{1}{1 + e(x_1)} - \varphi(d(x_1, x_2)))d(x_1, x_2).
\]

By (4), (5) and (6)

\[
t_1 = D(x_1, TX_1) \leq d(x_1, x_2) < D(x_1, TX_1) + e(x_1)D(x_1, TX_1) \leq t_1 + d(t_1) < t_1 + \delta(t_1).
\]

This implies by (3) that \( \varphi(d(x_1, x_2)) \leq k(t_1) < 1 \). Since \( e(x_1) \leq \frac{d(t_1)}{t_1} < \frac{1}{k(t_1)} - 1 \), we have

\[
\frac{1}{1 + e(x_1)} - \varphi(d(x_1, x_2)) > 0.
\]

It follows (8) that \( D(x_2, TX_2) < D(x_1, TX_1) \).

**Case 2.** \( D(x_1, TX_1) = d(x_1, x_2) \) for some \( x_2 \in TX_1 \). Since \( TX_0 \) (SR) \( TX_1 \), then \( x_1 \) R \( x_2 \) and also

\[
D(x_1, TX_1) - D(x_2, TX_2) \geq (1 - \varphi(d(x_1, x_2)))d(x_1, x_2) > 0.
\]

Therefore \( D(x_2, TX_2) < D(x_1, TX_1) \).

Next, let \( t_2 = D(x_2, TX_2) \). Then \( D(x_2, TX_2) \leq d(x_2, y) \) for all \( y \in TX_2 \). Again we consider the following two cases:
Case A. $D(x_2, T x_2) < d(x_2, y)$ for all $y \in T x_2$. For $\delta(t_2)$ and $k(t_2)$, choose $d(t_2)$ with
\[
d(t_2) < \min(\delta(t_2), (\frac{1}{k(t_2)} - 1)t_2)
\]
and set
\[
e(x_2) = \min\{d(t_2), \frac{1}{t_2}, \frac{t_1}{t_2} - 1\}.
\]
By using the similar reason as above, we obtain $x_3 \in T x_2$ such that $x_2 \in R \setminus x_3$, $x_2 \neq x_3$, $d(x_2, x_3) < (1 + \epsilon(x_2))D(x_2, T x_2)$ and
\[
D(x_2, T x_2) - D(x_3, T x_3) \geq \left(\frac{1}{1 + \epsilon(x_2)} - \varphi(d(x_1, x_2))\right)d(x_2, x_3) > 0.
\]
Hence $D(x_3, T x_3) < D(x_2, T x_2)$. From $\epsilon(x_2) \leq \frac{b}{t_2} - 1$, it follows that
\[
d(x_2, x_3) < (1 + \epsilon(x_2))D(x_2, T x_2) \leq D(x_1, T x_1) \leq d(x_1, x_2).
\]
Case B. $D(x_2, T x_2) = d(x_2, x_3)$ for some $x_3 \in T x_2$. Since $T x_1 (SR) T x_2$, then $x_2 \in R = x_3$ and also by using the same method as above, we can show that
\[
D(x_2, T x_2) - D(x_3, T x_3) \geq (1 - \varphi(d(x_2, x_3)))d(x_2, x_3) > 0
\]
and
\[
d(x_2, x_3) = D(x_2, T x_2) < D(x_1, T x_1) \leq d(x_1, x_2).
\]
Hence, $D(x_3, T x_3) < D(x_2, T x_2)$ and $d(x_2, x_3) < d(x_1, x_2)$. Repeating this process, we find that there exists an R-sequence $\{x_n\}$ with $x_{n+1} \in T x_n$ such that $D(x_n, T x_n)$ and $d(x_n, x_{n+1})$ are decreasing sequences of positive numbers and for each $n \in \mathbb{N},$
\[
D(x_n, T x_n) - D(x_{n+1}, T x_{n+1}) \geq \left(\frac{1}{1 + \gamma(x_n)} - \varphi(d(x_n, x_{n+1}))\right)d(x_n, x_{n+1}),
\]
where $\gamma(x_n)$ is real number with $0 \leq \gamma(x_n) \leq \frac{1}{t}$. Since $\{d(x_n, x_{n+1})\}$ is decreasing sequence, there exists $t \in (0, \infty)$ such that $\lim_{n \to \infty} d(x_n, x_{n+1}) = t$.
Let $a_n := \frac{1}{1 + \gamma(x_n)} - \varphi(d(x_n, x_{n+1}))$ for all $n \in \mathbb{N}$, then
\[
\lim_{n \to \infty} a_n \geq \lim_{n \to \infty} \frac{1}{1 + \gamma(x_n)} - \lim_{n \to \infty} \sup_{n \to \infty} \varphi(d(x_n, x_{n+1}) > 0.
\]
This implies that from (10), there exists $b > 0$ such that
\[
D(x_n, T x_n) - D(x_{n+1}, T x_{n+1}) \geq b \cdot d(x_n, x_{n+1})
\]
for large enough $n$. Since $\{d(x_n, x_{n+1})\}$ is decreasing sequence, it is convergent. On the other hand, for each $n < m$, we have
\[
d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\
\leq \frac{1}{b} \sum_{i=n}^{m-1} [D(x_i, T x_i) - D(x_{i+1}, T_{i+1})] \\
= \frac{1}{b} [D(x_m, T x_n) - D(x_m, T x_m)] \to 0
\]
as \( n, m \to \infty \). Hence \( \{x_n\} \) is a Cauchy R-sequence. Since \( X \) is R-complete then \( \lim_{n \to \infty} x_n = x^\ast \), for some \( x^\ast \in X \). Since \( x_{n+1} \in TX_n \) it follows from \((a_1)\) that \( f x_{n+1} \in TX_n \) for each \( n \in N \). Since \( f \) is R-continuous and \( \lim_{n \to \infty} x_n = x^\ast \), we have

\[
\lim_{n \to \infty} f x_{n+1} = f x^\ast.
\]

By assumption R-regularity of \( X \), since \( x_{n+k} \in X \) for all \( n, k \in N \) and \( x_n \to x^\ast \), as \( n \to \infty \), then \( x_n \in X \) for \( n \geq n_0 \), for some \( n_0 \in N \). Thus, from \((2)\) with \( x = x_n \) and \( y = x^\ast \), we obtain

\[
D(x_{n+1}, Tx^\ast) = H(Tx, Tx^\ast)
\leq \phi(d(x_n, x^\ast)), \phi(d(x_n, x^\ast)), D(x^\ast, Tx^\ast), D(x_n, Tx^\ast), D(x^\ast, x_{n+1}), h(f x^\ast) \leq d(f x, f x_{n+1})
\]

for each \( n \in N \) with \( n \geq n_0 \).

Now since \( x^\ast \in Tx^\ast \) then by using \((11)\) and \((A3)\) we have

\[
D(x^\ast, Tx^\ast) = \limsup_{n \to \infty} D(x_{n+1}, Tx^\ast)
\leq \limsup_{n \to \infty} \left( \phi(d(x_n, x^\ast)), \phi(d(x_n, x^\ast)), D(x^\ast, x_{n+1}), D(x_n, x_{n+1}), h(f x^\ast) \leq d(f x, f x_{n+1}) \right)
< D(x^\ast, Tx^\ast).
\]

Then \( x^\ast \in Tx^\ast \) which is a contradiction because it is supposed that \( T \) has no fixed point. By \((a_1)\), \( f x^\ast \in Tx^\ast \). Hence \( x^\ast \in \text{COP}(f, T) \). This completes the proof. \( \Box \)

4. Some Consequences

Letting

\[
\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma t_3 + \delta t_4 + L t_5,
\]

where \( \alpha, \beta, \gamma, \delta, L \geq 0, \alpha + \beta + \gamma + 2L = 1 \) and \( \gamma \neq 1 \), we get a generalization of Theorem 2.2 of \([11]\), Theorem 4 of \([7]\) and Theorem 5 of \([14]\).

**Corollary 4.1.** Let \( (X, d, R) \) be an R-complete (not necessarily complete) and R-regular metric space. Let \( T : X \to CB(X) \) be a multivalued mapping, \( f : X \to X \) be an R-continuous self-mapping and \( \phi : [0, \infty) \to [0, 1) \) be a function such that \( \limsup_{s \to t} \phi(s) \leq 1 \) for each \( t \geq 0 \). Assume that

\((a_1)\) for each \( x \in X \), \( \{f y : y \in Tx \} \subseteq Tx \);

\((a_2)\) there exists a function \( \hat{h} : X \to [0, \infty) \) such that

\[
H(Tx, Ty) \leq \phi(d(x, y))\left( \alpha d(x, y) + \beta D(x, Tx) + \gamma D(y, Ty) + \delta D(x, Ty) + \lambda D(y, Tx) + \hat{h}(f y).D(f y, Tx) \right)
\]

for each \( x \in X \) with \( x \neq y \), where \( \alpha, \beta, \gamma, \delta, L \geq 0, \alpha + \beta + \gamma + 2L = 1 \) and \( \gamma \neq 1 \). Suppose that

\((i)\) \( T \) is monotone of type SR;

\((ii)\) there exists \( x_0 \in X \) such that for each \( x \in X \), \( \{x_0\} \) (WR) \( Tx \).

Then \( \text{COP}(f, T) \cap F(T) \neq \emptyset \).

**Proof.** Define a function \( \beta \) from \([0, \infty) \) into \([0, 1) \) by \( \beta(t) = \frac{\phi(t)}{2} \) for \( t \in [0, \infty) \). Then the following hold:

1. \( \limsup_{t \to t} \beta(s) < 1 \) for all \( t \in [0, \infty) \).
2. \( \phi(t) < \beta(t) \) for all \( t \in [0, \infty) \).
3. \( \beta(t) \geq \frac{1}{2} \) for all \( t \in [0, \infty) \).
Now we have
\[ H(Tx, Ty) \leq \varphi(d(x, y))\left(\max\{d(x, y), D(y, Ty)\} + \frac{1}{2}D(x, Ty)\right) + \frac{1}{2}D(y, Ty) + L\cdot D(y, Tx) + \hat{h}(y)D(fy, Tx) \]
for each \( x \neq y \).

Therefore by applying Theorem 2 and Example 2.11-I, we can see the results.

Letting
\[ \phi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2} \max\{t_1, t_2, t_3, t_4, t_5\}, \]
we get a generalization of Theorem 2.2 of [2].

**Corollary 4.2.** Let \((X, d, R)\) be an R-complete (not necessarily complete) and R-regular metric space. Let \( T : X \to CB(X) \) be a multivalued mapping, \( f : X \to X \) be an R-continuous self-mapping and \( \varphi : [0, \infty) \to [0, 1) \) be a function such that \( \limsup_{t \to \infty} \varphi(s) < \frac{1}{\beta} \) for each \( t \geq 0 \). Assume that

(a1) for each \( x \in X \), \( \{fy : y \in Tx\} \subseteq Tx \);

(a2) there exists a function \( \hat{h} : X \to [0, \infty) \) such that
\[ H(Tx, Ty) \leq \varphi(d(x, y))\left(\max\{d(x, y), D(y, Ty)\} + \frac{1}{2}D(x, Ty)\right) + \frac{1}{2}D(y, Ty) + L\cdot D(y, Tx) + \hat{h}(y)D(fy, Tx) \]
for each \( x \neq y \), where \( L \geq 0 \). Suppose that

(i) \( T \) is monotone of type SR;

(ii) there exists \( x_0 \in X \) such that for each \( x \in X \), \( \{x_0\} \) (WR) \( Tx \).

Then \( \text{COP}(f, T) \cap F(T) \neq \emptyset \).

**Proof.** We can prove this corollary by Example 2.11-II, Example 2.12 and the technique has been used in Corollary 4.1.

Letting
\[ \phi(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, t_4, t_5\}, \]
we get a generalization Theorem 1 of [8], Theorem 2.2 of [11] and Theorem 4 of [7].

**Corollary 4.3.** Let \((X, d, R)\) be a R-complete (not necessarily complete) and R-regular metric space. Let \( T : X \to CB(X) \) be a multivalued mapping, \( f : X \to X \) be an R-continuous self-mapping and \( \varphi : [0, \infty) \to [0, 1) \) be a function such that \( \limsup_{t \to \infty} \varphi(s) < \frac{1}{\beta} \) for each \( t \geq 0 \). Assume that

(a1) for each \( x \in X \), \( \{fy : y \in Tx\} \subseteq Tx \);

(a2) there exists a function \( \hat{h} : X \to [0, \infty) \) such that
\[ H(Tx, Ty) \leq \varphi(d(x, y))\left(\max\{d(x, y), D(y, Ty)\} + \frac{1}{2}D(x, Ty)\right) + \frac{1}{2}D(y, Ty) + L\cdot D(y, Tx) + \hat{h}(y)D(fy, Tx) \]
for each \( x \neq y \), where \( L \geq 0 \). Suppose that

(i) \( T \) is monotone of type SR;

(ii) there exists \( x_0 \in X \) such that for each \( x \in X \), \( \{x_0\} \) (WR) \( Tx \).

Then \( \text{COP}(f, T) \cap F(T) \neq \emptyset \).

**Proof.** We can prove this corollary by Example 2.11-III, Example 2.12 and the technique has been used in Corollary 4.1.
5. Some Examples

The following simple examples show the generality of our main theorem over Theorem 1 of [8], Theorem 2.2 of [11], Theorem 4 of [7], Theorem 5 of [14] and Theorem 2.2 of [2].

**Example 5.1.** Consider the sequence \( \{S_n\} \) as follows:

\[
\begin{align*}
S_1 &= 1 \times 2, \\
S_2 &= 1 \times 2 + 2 \times 3, \\
S_3 &= 1 \times 2 + 2 \times 3 + 3 \times 4, \\
&\vdots \\
S_n &= 1 \times 2 + 2 \times 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}, n \in \mathbb{N}.
\end{align*}
\]

Let \( X = \{S_n : n \in \mathbb{N}\} \) and \( d(x,y) = |x - y|, \) \( x, y \in X. \) For all \( S_n, S_m \in X \) define \( S_n \ R \ S_m \) if and only if \( (1 = n \leq m). \) Hence \((X,d,R)\) is an \( R\)-complete and \( R\)-regular metric space. Define a multivalued mapping \( T : X \to CB(X) \) by the formulae:

\[
T_x = \begin{cases} 
\{S_{n-1}, S_{n+1}\}, & \text{if } x = S_n, n = 3, 4, \ldots , \\
\{S_1\}, & \text{if } x = S_1, S_2.
\end{cases}
\]

It is easy to see that \( T \) is monotone of type \( SR \) and \( \{S_1\} \) (WR) \( TS_n \) for each \( n \in \mathbb{N}. \) Now since,

\[
\lim_{n \to \infty} \frac{H(T(S_n), T(S_1))}{d(S_n, S_1)} = 1,
\]

then \( T \) is not contraction.

First, observe that

\[
S_n \ R \ S_m, \ T(S_n) \neq T(S_m) \iff (1 = n, m > 2).
\]

On the other hand, for every \( m \in \mathbb{N}, m > 2 \) we have

\[
H(TS_1, TS_m) \leq \varphi(d(S_1, S_m))(\alpha d(S_1, S_m)) + L.D(S_m, TS_1),
\]

where \( \alpha = 1, L = \frac{\varphi}{9} \) and \( \varphi : [0, \infty) \to [0, 1) \) is defined by \( \varphi(t) = \frac{1}{2}, t \in [0, 1). \) Hence by Corollary 4.1, for any function \( h : X \to [0, \infty) \) and any \( R\)-continuous self-mapping \( f : X \to X \) satisfying condition \( (a_1) \) of Corollary 4.1, we conclude that \( COP(f, T) \cap \tau(T) \neq \emptyset. \)

Notice that the mapping \( T \) does not satisfy the assumptions of Theorem 1 of [8], Theorem 2.2 of [11], Theorem 4 of [7], Theorem 5 of [14] and Theorem 2.2 of [2]. For this reason take \( x = S_3 \) and \( y = S_4. \)

**Example 5.2.** Let \( \ell^\infty \) be the Banach space consisting of all bounded real sequences with supremum norm and let \( \{e_n\} \) be the canonical basis of \( \ell^\infty. \) Let \( \{\tau_n\} \) be a bounded, strictly increasing sequence in \((0, \infty)\) satisfying \( \tau_n < 2\tau_n \) for all \( n \in \mathbb{N} \) (for example, let \( \tau_n = \frac{n^2}{2^n}, n \in \mathbb{N} \).) Put \( x_n = \tau_n e_n \) for each \( n \in \mathbb{N}. \) Define a bounded, complete subset \( X \) of \( \ell^\infty \) by \( X = \{x_1, x_2, x_3, \ldots \} \) and a mapping \( T \) from \( X \) into \( CB(X) \) by

\[
T_{x_0} = \begin{cases} 
\{x_{n-1}, x_{n+1}\}, & \text{if } n = 2, 3, \ldots , \\
\{x_1\}, & \text{if } n = 1.
\end{cases}
\]

For all \( x_n, x_m \in X \) define \( x_n \ R x_m \) if and only if \( (1 = n \leq m). \) Hence \((X,d,R)\) is an \( R\)-complete and \( R\)-regular metric space. It is easy to see that \( T \) is monotone of type \( SR \) and \( \{x_1\} \) (WR) \( TX_n \) for each \( n \in \mathbb{N}. \) On the other hand, for every \( m \in \mathbb{N} \) we have

\[
H(Tx_1, Tx_m) \leq \varphi(d(x_1, x_m))(\alpha d(x_1, x_m)) + L.D(x_m, Tx_1),
\]
where $\alpha = 1$, $L = \frac{3}{2}$ and $\varphi : [0, \infty) \to [0, 1]$ is defined by $\varphi(t) = \frac{1}{t}$, $t \in [0, \infty)$. Hence by Corollary 4.1, for any function $h : X \to [0, \infty)$ and any $R$-continuous self-mapping $f : X \to X$ satisfying condition $(a_1)$ of Corollary 4.1, we conclude that $\text{COP}(f, T) \cap \text{F}(T) \neq \emptyset$.

Notice that the mapping $T$ does not satisfy the assumptions of Theorem 1 of [8], Theorem 2.2 of [11], Theorem 4 of [7], Theorem 5 of [14] and Theorem 2.2 of [2]. For this reason take $x = x_4$ and $y = x_5$.

Below we explain a simple proof of Example A and Example B of [11].

**Example 5.3.** [11] Let $\ell^n$ be the Banach space consisting of all bounded real sequences with supremum norm and let $\{e_n\}$ be the canonical basis of $\ell^n$. Let $\{\tau_n\}$ be a sequence of positive real numbers satisfying $\tau_1 = \tau_2 = \tau_n < \tau_1$ for $n \geq 2$ (for example, let $\tau_1 = \frac{1}{2}$ and $\tau_n = \frac{1}{n}$ for $n \geq 2$). Put $x_n = \tau_n e_n$ for each $n \in \mathbb{N}$. Define a bounded, complete subset $X$ of $\ell^n$ by $X = \{x_1, x_2, x_3, \ldots\}$ and a mapping $T$ from $X$ into $\text{CB}(X)$ by

$$Tx_n = \begin{cases} [x_1, x_2], & \text{if } n = 1, 2, \\ [x_1, x_3, \ldots, x_n, x_{n+1}], & \text{if } n \geq 3. \end{cases}$$

For all $x_n, x_m \in X$ define $x_{nm} \in \mathbb{R}$ if and only if $(1 \leq n \leq m)$. Hence $(X, d, R)$ is a $R$-complete and $R$-regular metric space. It is easy to see that $T$ is monotone of type SR and $\{x_1\}$ (WR) $T x_n$ for each $n \in \mathbb{N}$. On the other hand, for every $m \in \mathbb{N}$ we have

$$H(TX_1, TX_m) \leq \varphi(d(x_1, x_m))(\alpha.d(x_1, x_m)) + L.D(x_m, TX_1),$$

where $\alpha = 1$, $L = 3$ and $\varphi : [0, \infty) \to [0, 1]$ is defined by $\varphi(t) = \frac{1}{t}$, $t \in [0, \infty)$. Hence by Corollary 4.1, for any function $h : X \to [0, \infty)$ and any $R$-continuous self-mapping $f : X \to X$ satisfying condition $(a_1)$ of Corollary 4.1, we conclude that $\text{COP}(f, T) \cap \text{F}(T) \neq \emptyset$. In particular, let $f : X \to X$ be defined by

$$f x_n = \begin{cases} x_{2m} & \text{if } n = 1, 2, \\ x_{m+1} & \text{if } n \geq 3, \end{cases}$$

then $\text{COP}(f, T) \cap \text{F}(T) \neq \emptyset$.

### 6. Application to the Nonlinear Fractional Boundary Value Equations

Let $X = \{u \in C[0, 1] : u(t) \geq 0, \forall t \in [0, 1]\}$ endowed with the metric $d$ induced by supremum norm. Consider the following nonlinear fractional boundary value equations

$$\begin{cases} D^{\alpha}_0 u(t) + \lambda f(t, u(t)) = 0, \\ u(0) = u'(0) = u''(0) = u'''(1) = 0, \end{cases}$$

where $0 < \lambda < 1$ is constant, $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function and $D^{\alpha}_0$ is the standard Riemann-Liouville fractional derivative.

Here, we consider the following hypotheses:

$(C_1)$ For all $u, v \in X$ with $u(t) v(t) \leq \max\{u(t), v(t)\}$ for each $t, t' \in [0, 1]$, we have

$$f(t, u(t)) f(t', v(t')) \leq \frac{1}{\lambda} f(t, v(t)), \forall t, t' \in [0, 1],$$

or

$$f(t, u(t)) f(t', v(t')) \leq \frac{1}{\lambda} f(t', v(t')), \forall t, t' \in [0, 1].$$
(C2) For all \(u,v \in X\) with \(u(t)v(t) \leq v(t)\) for each \(t \in [0,1]\), we have

\[
|f(t, u(t)) - f(t, v(t))| \leq \frac{||u - v||}{A},
\]

where \(\|u\| = \max_{t \in [0,1]} u(t)\) and \(A = \max_{0 < t < 1} \int_0^1 k(t, s)ds\), where \(k : [0,1] \times [0,1] \rightarrow [0,1]\) denotes the Green's function for the boundary value system (12).

Note that \(f : [0,1] \times [0,1] \rightarrow [0,\infty)\) is not necessarily Lipschitz from the given condition (C2) and there exist some functions satisfying in condition (C2) but not Lipschitz.

**Theorem 6.1.** Let the above conditions are satisfied. Then, the fractional boundary value problem (12) has a positive solution.

**Proof.** We define a operator equation \(T : X \rightarrow X\) as follows:

\[
Tu(t) = \lambda \int_0^1 k(t, s)f(s, u(s))ds,
\]  

(13)

where

\[
k(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-3}, & 0 \leq t \leq s \leq 1. \end{cases}
\]

We know that the differential equation has a positive solution if and only if \(T\) has a fixed point in \(X\) (see [13, Lemma 2.3]). We consider the following relation in \(X\):

\[
u, w \in X \quad \Rightarrow \quad u(t)v(t) \leq \max\{v(t), v(t')\},
\]

(14)

for all \(t, t' \in [0,1]\) and \(u, v \in X\). Since \((X, d)\) is a complete metric space, then \((X, d, R)\) is an R-complete and R-regular metric space. Now, we prove the following two steps to complete the proof.

**Step 1:** \(T\) is monotone of type SR. Let \(u, v \in X\) with \(uRv\). We must show that

\[Tu(t)Tu(t') \leq \max\{T(v(t)), T(v(t'))\}\]

for all \(t, t' \in [0,1]\). Applying (13), we have

\[
Tu(t)Tu(t') = \lambda^2 \int_0^1 \int_0^1 k(t, s)k(t', s')f(s, u(s))f(s', v(s'))ds'ds.
\]

Applying (C1), we have two cases:

1. \(f(s, u(s))f(s', v(s')) \leq \frac{1}{4}f(s, v(s))\) for each \(s, s' \in [0,1]\). Applying (13) and definition of \(k\), we have

\[
Tu(t)Tu(t') \leq \lambda^2 \int_0^1 \int_0^1 k(t, s)k(t', s')f(s, u(s))f(s', v(s'))ds'ds
\]

\[
= \lambda \int_0^1 k(t, s)f(s, u(s))ds
\]

\[
= T(v(t))
\]

\[
\leq \max\{T(v(t)), T(v(t'))\}.
\]
(2). $f(s, u(s))f(s', v(s')) \leq \frac{1}{k}f'(s', v(s'))$ for each $s, s' \in [0, 1]$. Applying (13) and definition of $k$, we have

$$Tu(t)Tv(t') = \lambda^2 \int_0^1 \int_0^1 k(t, s)k(t', s')f(s, u(s))f(s', v(s'))ds'ds$$

$$\leq \lambda \int_0^1 \int_0^1 k(t, s)k(t', s')f(s', v(s'))ds'ds$$

$$\leq \lambda \int_0^1 k(t', s')f(s', v(s'))ds'$$

$$= T(v(t'))$$

$$\leq \max(T(v(t)), T(v(t')))$$

These imply that $T$ is monotone of type SR.

Step 2: Show that for each elements $u, v \in X$ with $u \preceq v$, we have

$$d(Tu, Tv) \leq \lambda d(u, v).$$

Let $u, v \in X$ with $u \preceq v$. Then for all $t \in [0, 1]$, we have $u(t) \preceq v(t)$. Applying (C2), we obtain that

$$|Tu(t) - Tv(t)| = |\lambda \int_0^1 k(t, s)f(s, u(s))ds - \lambda \int_0^1 k(t, s)f(s, v(s))ds|$$

$$\leq \lambda \int_0^1 |k(t, s)|f(s, u(s)) - f(s, v(s))|ds$$

$$\leq \lambda \|u - v\|$$

for all $t \in [0, 1]$. Hence,

$$d(Tu, Tv) \leq \lambda d(u, v)$$

for all $u, v \in X$ with $u \preceq v$.

Applying Corollary 4.1, $T$ has a fixed point in $X$ which is a positive solution of the differential equation (12). \qed

References


