ps-Drazin Inverses in Banach Algebras

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Abstract. An element $a$ in a Banach algebra $A$ has ps-Drazin inverse if there exists $p^2 = p \in \text{comm}^2(a)$ such that $(a - p)^k \in J(A)$ for some $k \in \mathbb{N}$. Let $A$ be a Banach algebra, and let $a, b \in A$ have ps-Drazin inverses. If $a^2b = ab$ and $b^2a = bab$, we prove that

1. $ab \in A$ has ps-Drazin inverse.
2. $a + b \in A$ has ps-Drazin inverse if and only if $1 + a^2b \in A$ has ps-Drazin inverse.

As applications, we present various conditions under which a $2 \times 2$ matrix over a Banach algebra has ps-Drazin inverse.

1. Introduction

Let $A$ be a Banach algebra with an identity. The commutant of $a \in A$ is defined by $\text{comm}(a) = \{ x \in A \mid xa = ax \}$. The double commutant of $a \in A$ is defined by $\text{comm}^2(a) = \{ x \in A \mid xy = yx \text{ for all } y \in \text{comm}(a) \}$. An element $a$ in a Banach algebra $A$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $b \in \text{comm}^2(a)$ such that $b = bab$, $a - a^2b \in A^{qnil}$. The preceding $b$ is unique, if such element exists, and called the g-Drazin inverse of $a$ and denote $b$ by $a^\pi$. Also, $a^\pi = 1 - aa^d$ is called spectral idempotent of $a$. As is known, $a \in A$ has g-Drazin inverse if and only if there exists $e^\pi = e \in \text{comm}^2(a)$ such that $a + e \in U(A)$ and $ae \in A^{qnil}$. Here, $A^{qnil} = \{ x \mid 1 - xr \in U(A) \text{ for any } r \in \text{comm}(x) \}$. Following [10], an element $a \in A$ has p-Drazin inverse (i.e., pseudo Drazin inverse) if there exists $b \in A$ such that

$$b = bab, b \in \text{comm}^2(a), a - a^2b \in J(A)$$

for some $k \in \mathbb{N}$. Evidently, $a \in A$ has p-Drazin inverse if and only if there exists $e^\pi = e \in \text{comm}^2(a)$ such that $a + e \in U(A)$ and $(ae)^k \in J(A)$ for some $k \in \mathbb{N}$, if and only if there exists $b \in A$ such that

$$b = bab, b \in \text{comm}^2(a), (a - a^2b)^k \in J(A)$$

for some $k \in \mathbb{N}$. Following [4], an element $a \in A$ has gs-Drazin inverse if there exists $b \in A$ such that

$$b = bab, b \in \text{comm}^2(a), a - ab \in A^{qnil}.$$
These generalized inverses in a Banach algebra have extensively studied from different points of view, e.g., [1]-[8], [13] and [14].

Motivating by g-Drazin, p-Drazin and gs-Drazin inverses, we introduce a new kind of generalized inverses in a Banach algebra. An element $a$ in a Banach algebra $A$ has ps-Drazin inverse if there exists $p^2 = p \in \text{comm}^2(a)$ such that $(a - p)^k \in J(A)$ for some $k \in \mathbb{N}$. As in the proof of [8, Lemma 2.2], we easily prove that $a \in A$ has ps-Drazin inverse if and only if there exists $b \in A$ such that

$$b = bab, b \in \text{comm}^2(a), (a - ab)^k \in J(A)$$

for some $k \in \mathbb{N}$.

The purpose of this paper is to investigate further algebraic properties of ps-Drazin inverses. Let $a, b \in A$ have ps-Drazin inverses. In Section 2, we investigate when the product of $a$ and $b$ has ps-Drazin inverse in a Banach algebra. If $a^2b = aba$ and $b^2a = bab$, we prove that $ab \in A$ has ps-Drazin inverse. In Section 3, we determine when the sum of $a$ and $b$ has ps-Drazin inverse. We prove that $a + b \in A$ has ps-Drazin inverse if and only if $1 + a^2b \in A$ has ps-Drazin inverse. Finally, in the last section, we present various conditions under which a $2 \times 2$ matrix over a Banach algebra has ps-Drazin inverse.

Throughout the paper, all Banach algebras are complex with an identity. We use $J(A)$ and $U(A)$ to denote the Jacobson radical of $A$ and the set of all units in $A$. $A^{zd}$ and $A^p$ denote the sets of all elements having p-Drazin and ps-Drazin inverses in the Banach algebra $A$, respectively. N stands for the set of all natural numbers.

2. Multiplicative property

In this section, we investigate multiplicative property of ps-Drazin inverses. We begin with the relation between ps-Drazin and p-Drazin inverse, which will be used frequently in the sequel.

**Theorem 2.1.** Let $A$ be a Banach algebra, and let $a \in A$. Then $a \in A$ has ps-Drazin inverse if and only if

1. $a \in A^{zd}$;
2. $(a - a^2)^k \in J(A)$ for some $k \in \mathbb{N}$.

**Proof.** $\implies$ Write $a = e + w$ with $a^2 = e \in \text{comm}^2(a), w^k \in J(A)$ for some $k \in \mathbb{N}$. Then $a + (1-e) = 1 + w \in U(A)$ and $(a(1-e))^k = (1-e)w^k \in J(A)$. Therefore, $a$ has p-Drazin inverse. Moreover, $(a - a^2)^k = (1 - 2e - w)^k w^k \in J(A)$, as desired.

$\iff$ Since $a \in A$ has p-Drazin inverse, we can find some $b \in \text{comm}^2(a)$ such that $b = bab$ and $(a - a^2b)^k \in J(A)$. We check that $(a - 1 + ab)(b - 1 + ab) = 1 - (a - a^2b) \in U(A)$. Hence, $a - 1 + ab \in U(A)$. Set $e = 1 - ab$. Then $e^2 = e \in \text{comm}^2(a)$ and $u := a - e \in U(A)$. Hence, $a - a^2 = (e + u) - (e + u)^2 = -u(2e + u - 1)$. This shows that $a - (1 - e) = -u^{-1}(a - a^2)$. This implies that $(a - (1 - e))^k \in J(A)$. This completes the proof.

**Corollary 2.2.** Let $A$ be a Banach algebra, and let $a \in A$. If $a \in A$ has ps-Drazin inverse, then $a \in A$ has p-Drazin inverse.

We note that the converse of Corollary 2.2 is not true, in general. Let $C$ be the field of all complex numbers. Then $2 \in C$ has p-Drazin inverse. But it has no ps-Drazin inverse, as $(2^2 - 2)^k = 2^k \notin J(C)$ for all $k \in \mathbb{N}$.

**Lemma 2.3.** (see [12, Lemma 2.6]) Let $A$ be a Banach algebra with $a^2b = aba$ and $b^2a = bab$. Then, the following hold for any integer $k \in \mathbb{N}$.

1. $(ab)^k = a^k b^k$.
2. $(a + b)^k = \sum_{i=0}^{k-1} C_i^{k-1} (a^i b + b^i a)$.

**Lemma 2.4.** (see [12, Theorem 2.8]) Let $A$ be a Banach algebra and $a, b \in A$ have p-Drazin inverse. If $a^2b = aba$ and $b^2a = bab$, then $ab$ has p-Drazin inverse.
Lemma 3.1. Let $J$ be a Banach algebra, and let $a, b \in J$ have ps-Drazin inverses. If $a^2 b =aba$ and $b^2 a = bab$, then $a \in J$ has ps-Drazin inverse.

Proof. Let $a$ and $b$ have ps-Drazin inverses. Then there exists $m, n \in \mathbb{N}$ such that $(a - a^2)^m \in J(J)$ and $(b - b^2)^n \in J(J)$ by Theorem 2.1. Let $c = a - a^2$.

\[
c^2 b = (a - a^2)^2 b = (a^2 - 2a^3 + a^4)b = aba - 2aba^2 + aba^3 = (a - a^2)(b(a - a^2)) = cbc.
\]

Also, $b^2 c = b^2(a - a^2) = b^2 a - b^2 a a = bab - bab a = bab - ba^2 b = b(a - a^2) b = bcb$. Thus, for any integer $k \geq 0$, $((a - a^2)b)^k = (a - a^2)^kb$ by Lemma 2.3. Similarly, $(a^2(b - b^2))l = (a^2)^l(b - b^2)^l$ for any integer $l \geq 0$. Now, let $x = (a - a^2)b$ and $y = a^2(b - b^2)$. We show that $x^2 y = yx y$.

\[
x^2 y = (a - a^2)(b(a - a^2)ba(b - b^2)) = (a - a^2)(bab - bab a)(a - a^2)(b - b^2) = (a - a^2)baba(a - a^2)(b - b^2) = (a - a^2)baba (a - a^2) b = x y x.
\]

Also,

\[
y^2 x = a^2(b - b^2)a^2(b - b^2)(a - a^2)b = a^2(b - b^2)a^2(bab - a^2 b - bab + ba^2 b) = a^2(b - b^2)a^2(bab - bab - bab a + b^2 aba) = a^2(b - b^2)a^2(bab - bab - bab a + b^2 aba) = a^2(b - b^2)a^2(bab - a^2 b - ab^2 + abba) = a^2(b - b^2)a^2(bab - a^2 b - ab^2 + abba) = a^2(b - b^2)a^2(bab - a^2 b - ab^2 + a^2 b^2) = a^2(b - b^2)a^2(bab - a^2 b - ab^2 + a^2 b^2) = a^2(b - b^2)a^2(b(1 - a) b(a - b^2) = a^2(b - b^2)(a - a^2)ba(b - b^2) = y x y.
\]

Hence, $(a - (ab)^2)^{m+n+1} = (x + y)^{m+n+1} = \sum_{i=0}^{m+n+1} C_{m+n}^{i}(x^{m+n+1-i}y + y^{m+n+1-i}x)$ by Lemma 2.3. As we proved, $x^k = ((a - a^2)b)^k = (a - a^2)^kbk$ for any integer $k \geq 0$ and $y^l = (a^2(b - b^2))^l = (a^2)^l(b - b^2)^l$ for any integer $l \geq 0$. Also, $(a - a^2)^m \in J(J)$ and $(b - b^2)^n \in J(J)$ for some $m, n \in \mathbb{N}$, and so we have $(ab - (ab)^2)^{m+n+1} \in J(J)$. Therefore, $ab$ has ps-Drazin inverse by Theorem 2.1 and Lemma 2.4. □

Corollary 2.6. Let $J$ be a Banach algebra, and let $a, b \in J$ have ps-Drazin inverses. If $ab = ba$, then $a \in J$ has ps-Drazin inverse.

Proof. It is clear by Theorem 2.5, since the condition $ab = ba$ implies that $a^2 b = aba$ and $b^2 a = bab$. □

3. Additive property

In this section, we concern on the additive properties of ps-Drazin inverses. For the convenience, we use $J^n(J)$ to denote the set of all elements $x$ with $x^n \in J(J)$ for some $n \in \mathbb{N}$. We now derive

Lemma 3.1. Let $J$ be a Banach algebra, and let $a, b \in J$ have ps-Drazin inverses. If $ab = ba = 0$, then $a + b$ has ps-Drazin inverse.

Proof. In view of [10, Theorem 5.4], \( a + b \) has p-Drazin inverse. We easily checks that \( a + b - (a + b)^2 = (a - a^2) + (b - b^2) \in J^s(A) \). This completes the proof by Theorem 2.1. \( \square \)

Lemma 3.2. Let \( A \) be a Banach algebra, and let \( a, b \in J^s(A) \). If \( a^2b = ab(a+b) \) and \( b^2a = bab \), then \( a + b \in J^s(A) \).

Proof. Write \( a^m, b^n \in J(A) \) for some \( m, n \in \mathbb{N} \). According to Lemma 2.3, we see that \( (a + b)^{m+n} \in J(A) \), as desired. \( \square \)

Lemma 3.3. Let \( A \) be a Banach algebra, and let \( a, b \in A \) have ps-Drazin inverses. If \( ab = ba \) and \( 1 + d^2a \in A \) has ps-Drazin inverse, then \( b + a \) has ps-Drazin inverse.

Proof. Clearly, \((1 + d^2a)(1 - d^2a) = (1 + a^2d)(1 - a^2d) \) since \( a + b \) has ps-Drazin inverse. Hence, \( a^2d + (a^2d)^2 = (a - a^d)(1 - a^d) = 0 \), and \( a \) has ps-Drazin inverse. We easily check that \((a + b)^2 = a^2 + b^2 + 2ab \). Therefore, \( a^2 + b^2 = a^2b^2 = (a^2b)(a^2b) = 0 \). This completes the proof by Theorem 2.1. \( \square \)

Theorem 3.4. Let \( A \) be a Banach algebra, and let \( a, b \in A \) have ps-Drazin inverses. If \( a^2b = ba \) and \( b^2a = bab \), then \( a + b \in A \) has ps-Drazin inverse if and only if \( 1 + d^2a \in A \) has ps-Drazin inverse.

Proof. Let \( a + b \) has ps-Drazin inverses. Write \( a + b = x + y \) where \( x = 1 - a^d \) and \( y = a^d \). Let \( x, y \in \mathbb{N} \) and \( xy = 0 \). Moreover, we see that \( x^2 = a^d(1 - a^d) = a^d(1 - a^d) \), and \( a = \text{comm}(ab) \). We easily check that \( (a + b)^2 = a^2 + b^2 + 2ab \). Since \( a \) has ps-Drazin inverse, it has ps-Drazin inverse and we can find some \( k \in \mathbb{N} \) such that \( (a - a^d)^2 \in J(A) \). In view of [12, Theorem 2.3], \( a^2d \) has ps-Drazin inverse. We easily check that \( a^2d - (a - a^d)^2 = (a - a^d)^2 \), and so \( (a^2d - (a - a^d)^2)^2 = (a - a^d)^2 \). Hence, \( (a - a^d)(a + b) = x + y \in J(A) \). Choose \( p = a^d \), then \( a + b = \left( \begin{array}{ccc} p(a + b)p & 0 \\ (1 - p)(a + b)p & (1 - p)(a + b)(1 - p) \end{array} \right) \). Since \( p(a + b)p = (a + b)p \), \( a^d \) has ps-Drazin inverse. Therefore \( 1 + a^d \in \mathbb{N} \) by Lemma 3.1.

Step 1. Assume that \( a^d \in J(A) \). Then \( (1 - a^d)(a + b) = x + y \) where \( x = (a - a^d)^2 \) and \( y = (a - a^d)^2 \). Then \( x^2 = x^2y = y^2x = x^2y \). Also, \( x + a - a^2d^2 \in J(A) \) and \( y = 1 - a^d \in J(A) \). Hence, \( (1 - a^d)(a + b) = x + y \in J(A) \). Choose \( p = a^d \), then \( a + b = \left( \begin{array}{ccc} p(a + b)p & 0 \\ (1 - p)(a + b)p & (1 - p)(a + b)(1 - p) \end{array} \right) \). Since \( p(a + b)p = (a + b)p \), \( a^d \) has ps-Drazin inverse. Therefore \( 1 + a^d \in \mathbb{N} \) by Lemma 3.1.

Step 2. Choose \( p = bb^d \). Then \( a = \left( \begin{array}{ccc} a_1 \ast 0 \\ a_2 \ast b_2 \end{array} \right) \) and \( b = \left( \begin{array}{ccc} b_1 \ast 0 \\ b_2 \ast b_2 \end{array} \right) \) where \( a_1 = p(a + b) \), \( a_2 = (1 - p)(a + b) \), \( b_1 = b b^d \) and \( b_2 = b_2 + (1 - p)(b_2) \). Obviously, \( a_1, b_1 \in \mathbb{N} \) and \( a_1 b_1 = b_1 a_1 \). In Step 1, we see that \( 1 + a^d b_1 \in \mathbb{N} \). Therefore \( a_1 + b_1 \in \mathbb{N} \). By Lemma 3.3, it follows that \( a_1 a^d b_1 \) has ps-Drazin inverse. Moreover, we see that \( a^d b_1 \) has ps-Drazin inverse. Consequently, \( a + b \in \mathbb{N} \). \( \square \)
We see that the condition in Theorem 3.4 is a generalization of the commutativity of \( a \) and \( b \). But we have,

**Example 3.5.** Let \( a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), \( b = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \) \( \in M_2(\mathbb{Z}_2) \). Then \( a^2b = aba \), \( b^2a = bab \). In this case \( a, b, 1 + a^2b \in M_2(\mathbb{Z}_2) \) has ps-Drazin inverse and \( ab \neq ba \).

4. Splitting in Banach algebras

The goal of this section is to use splitting approach to determine when an element in a Banach algebra has ps-Drazin inverse. We derive

**Lemma 4.1.** Let \( \mathcal{A} \) be a Banach algebra. If \( a, d \in \mathcal{A} \) have ps-Drazin inverses, then \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) \( \in M_2(\mathcal{A}) \) has ps-Drazin inverse.

**Proof.** In view of Theorem 2.1, \( a, d \in \mathcal{A} \) have p-Drazin inverse and \( (a - a^2)^k, (b - b^2)^k \in J(\mathcal{A}) \) for some \( k \in \mathbb{N} \).

In view of [10, Theorem 5.3], \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathcal{A}) \) has p-Drazin inverse. On the other hand, we have some \( z \in \mathcal{A} \) such that

\[
\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^2 \, 2^k = \begin{pmatrix} (a-a^2)^k & z \\ 0 & (d-d^2)^k \end{pmatrix}^2
\]

\[
= \begin{pmatrix} (a-a^2)^k & (a-a^2)^k z + z(d-d^2)^k \\ 0 & (d-d^2)^k \end{pmatrix}^2
\]

\( \in J(M_2(\mathcal{A})). \)

According to Theorem 2.1, we complete the proof. \( \square \)

**Theorem 4.2.** Let \( \mathcal{A} \) be a Banach algebra, and let \( a, d \in \mathcal{A} \) have ps-Drazin inverses. If \( bc = dc = 0 \), then \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A}) \) has ps-Drazin inverse.

**Proof.** Clearly, we have \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \overline{=} p + q \), where

\[
p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.
\]

In view of Lemma 4.1, \( p \in M_2(\mathcal{A}) \) has ps-Drazin inverse. As \( q^2 = 0 \), we easily see that \( q \in M_2(\mathcal{A}) \) has ps-Drazin inverse. Moreover,

\[
q^2 p = 0 = \begin{pmatrix} 0 & 0 \\ c b c & 0 \end{pmatrix} = q p q,
\]

and

\[
p^2 q = \begin{pmatrix} a b c + b d c & 0 \\ d c a & 0 \end{pmatrix} = 0 = \begin{pmatrix} b c a & b c b \\ d c a & d c b \end{pmatrix} = p q p.
\]

Clearly, \( q^4 = 0 \), and so \( 1 + q^4 p = 1 \) has ps-Drazin inverse. Therefore, \( p + q \in \mathcal{A} \) has ps-Drazin inverse, by Theorem 3.4. This completes the proof. \( \square \)
Corollary 4.3. Let $\mathcal{A}$ be a Banach algebra, and let $a, d \in \mathcal{A}$ have ps-Drazin inverses. If $bc = 0$ and $dc = c$, then 
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A}) \text{ has ps-Drazin inverse.}
\]

Proof. Since $dc = c$, $-(1-d)c = 0$. So in light of Theorem 4.2, $I_2 - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$ has ps-Drazin inverse since $bc = 0$ and $-(1-d)c = 0$. Thus, we can find an idempotent $E \in \text{comm}(I_2 - \begin{pmatrix} a & b \\ c & d \end{pmatrix})$ such that 
\[
(I_2 - \begin{pmatrix} a & b \\ c & d \end{pmatrix} - E)^k \in J(M_2(\mathcal{A})) \text{ for some } k \in \mathbb{N},
\]
and so 
\[
\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} - (I_2 - E) \right)^k \in J(M_2(\mathcal{A})).
\]
Clearly, $I_2 - E \in \text{comm}^2\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$. This completes proof.

Next we consider another splitting of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and get the alternative results.

Theorem 4.4. Let $\mathcal{A}$ be a Banach algebra, and let $a, d \in \mathcal{A}$ have ps-Drazin inverses. If $bc = cb = 0$ and $dc = ca$, then \[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A}) \text{ has ps-Drazin inverse.}
\]

Proof. We see that \[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = p + q,
\]
where 
\[
p = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.
\]
In view of Lemma 4.1, $p$ has ps-Drazin inverse. Since $q - q^2 = 0$ and $q \in \mathcal{A}^{pd}$, $q$ has ps-Drazin inverse by Theorem 2.1. Clearly, $q^2 = 0$, and so $1 + q^2p$ has ps-Drazin inverse. From $bc = cb = 0$ and $dc = ca$, we see that 
\[
pq = \begin{pmatrix} 0 & 0 \\ dc & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ca & cb \end{pmatrix} = qp.
\]
In light of Lemma 3.3, $p + q$ has ps-Drazin inverse, as asserted.

Example 4.5. Let $A, B, C$ be operators, acting on separable Hilbert space $l_2(\mathbb{N})$, defined as follows respectively:

\[
\begin{align*}
A(x_1, x_2, x_3, x_4, \cdots) &= (x_1, x_2, x_3, x_4, \cdots), \\
B(x_1, x_2, x_3, x_4, \cdots) &= (x_1, -x_2, 0, 0, \cdots), \\
C(x_1, x_2, x_3, x_4, \cdots) &= (0, x_1 + x_2, x_3, x_4, \cdots), \\
D(x_1, x_2, x_3, x_4, \cdots) &= (-x_1, x_2, x_3, x_4, \cdots).
\end{align*}
\]

Then we easily check that $BC = CB = 0$ and $DC = CA$. In light of Theorem 4.4, the operator matrix \[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
has ps-Drazin inverse. In this case, $DC \neq 0$.

Lemma 4.6. Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$ have ps-Drazin inverse. If $e^2 = e \in \text{comm}(a)$, then $ea \in \mathcal{A}$ has ps-Drazin inverse.
Proof. Since \( e \in \mathcal{A}^p \), we easily obtain the result by Theorem 2.5. \( \square \)

Let \( a \in \mathcal{A} \) have ps-Drazin inverse. Then it has g-Drazin inverse. We use \( a^\pi \) to denote the spectral idempotent of \( a \), i.e., \( a^\pi = 1 - aa^d \). We now derive

**Theorem 4.7.** Let \( \mathcal{A} \) be a Banach algebra, and let \( a, d \in \mathcal{A} \) have ps-Drazin inverses. If \( bc = cb = 0 \), \( ca(1 - a^\pi) = d^\pi dc \) and \( a^\pi ab = bd(1 - d^\pi) \), then \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A}) \) has ps-Drazin inverse.

**Proof.** Let

\[
p = \begin{pmatrix} a(1 - a^\pi) & b \\ 0 & d^\pi \end{pmatrix}, \quad q = \begin{pmatrix} a^\pi & 0 \\ c & d(1 - d^\pi) \end{pmatrix}.
\]

Then \( M = p + q \). In view of Lemma 4.1, \( p \) has ps-Drazin inverse. Likewise, \( q \) has ps-Drazin inverse. It is easy to verify that

\[
pq = \begin{pmatrix} 0 & bd(1 - d^\pi) \\ d^\pi c & 0 \end{pmatrix} = \begin{pmatrix} 0 & aa^\pi b \\ ca(1 - a^\pi) & 0 \end{pmatrix} = qp.
\]

One easily checks that

\[
p^d = \begin{pmatrix} (a(1 - a^\pi))^d & x \\ 0 & d^d d^\pi \end{pmatrix} = \begin{pmatrix} a^d & x \\ 0 & 0 \end{pmatrix}
\]

where \( x = (a^d)^2 \sum_{n=0}^{\infty} (a^d)^n b(d^\pi)^n \). Hence,

\[
p^d q = \begin{pmatrix} a^d & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^\pi & 0 \\ c & d(1 - d^\pi) \end{pmatrix} = \begin{pmatrix} xc & xd(1 - d^\pi) \\ 0 & 0 \end{pmatrix}
\]

where \( xc = (a^d)^2 (b + \sum_{n=1}^{\infty} (a^d)^n b(d^\pi)^n) c = 0 \) as \( bc = 0 \), \( b(d^\pi)^n c = 0 \). Moreover, we have

\[
xd(1 - d^\pi) = (a^d)^2 (b + \sum_{n=1}^{\infty} (a^d)^n b(d^\pi)^n) d(1 - d^\pi) = (a^d)^2 (b + bd(1 - d^\pi)) = (a^d)^2 (b + a^\pi ab) = (a^d)^2 b
\]

and so \( p^d q = \begin{pmatrix} 0 & (a^d)^2 b \\ 0 & 0 \end{pmatrix} \). Thus, \( 1 + p^d q \) is invertible. So, it has p-Drazin inverse. Further, we have

\[
(1 + p^d q) - (1 + p^d q)^2 = -p^d q(1 + p^d q) = \begin{pmatrix} 0 & -(a^d)^2 b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & (a^d)^2 b \\ 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & -(a^d)^2 b \\ 0 & 0 \end{pmatrix} \in \mathcal{I}(\mathcal{A}).
\]

In light of Theorem 2.1, \( 1 + p^d q \in \mathcal{A}^p \). Therefore, we complete the proof by Theorem 3.4. \( \square \)

Finally, we concern on the ps-Drazin inverse for a operator matrix \( M \) has ps-Drazin inverse. Here,

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

where \( A, D \in L(X) \) has ps-Drazin inverses and \( X \) is a complex Banach space. Then \( M \) is a bounded linear operator on \( X \oplus X \).
Lemma 4.8. Let $\mathcal{A}$ be a Banach algebra, and let $A \in M_{m \times n}(\mathcal{A})$, $B \in M_{n \times m}(\mathcal{A})$ and $k \in \mathbb{N}$. Then $AB \in M_n(\mathcal{A})$ has ps-Drazin inverse if and only if $BA \in M_m(\mathcal{A})$ has ps-Drazin inverse.

Proof. Suppose that $AB \in M_n(\mathcal{A})$ has ps-Drazin inverse. Then $AB \in M_n(\mathcal{A})$ has p-Drazin inverse and $(BA - (AB)^2)^k \in M_n(J(\mathcal{A}))$. In light of [10, Theorem 3.6], $BA$ has p-Drazin inverse. One easily checks that $(BA - (AB)^2)^{k+1} = B(AB - (AB)^2)^k(A - ABA) \in M_n(J(\mathcal{A}))$.

According to Theorem 2.1, $BA \in M_m(\mathcal{A})$ has ps-Drazin inverse, as asserted. \qed

Lemma 4.9. Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$. If $a, b$ have ps-Drazin inverses and $ab = 0$, then $a + b \in \mathcal{A}$ has ps-Drazin inverse.

Proof. Let $A = (1, b)$ and $B = \begin{pmatrix} a \\ 1 \end{pmatrix}$. By the similar technique to the Lemma 4.1, $BA = \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}$ has ps-Drazin inverse. By virtue of Lemma 4.8, $AB = a + b \in \mathcal{A}$ has ps-Drazin inverse, as asserted. \qed

Theorem 4.10. Let $A \in L(X)$ has ps-Drazin inverse, $D \in L(X)$ and $M$ be given by (4.1). Let $W = AA^d + A^dBCA^d$.

If $AW$ has ps-Drazin inverse, then $M$ has ps-Drazin inverse.

Proof. We easily see that $M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$

where $P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}$, $Q = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}$.

By hypothesis, we verify that $QP = 0$. Clearly, $Q$ has ps-Drazin inverse. Furthermore, we have

$P = P_1 + P_2$, $P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}$, $P_2 = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}$

and $P_2P_1 = 0$. By virtue of Theorem 4.2, $P_2$ has ps-Drazin inverse. Obviously, we have

$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A \\ AA^dB \end{pmatrix}$.

By hypothesis, we see that

$\begin{pmatrix} A & AA^dB \\ C & CA^d \end{pmatrix} = AW$

has ps-Drazin inverse. In light of Lemma 4.8, $P_1$ has ps-Drazin inverse. Thus, $P$ has ps-Drazin inverse by Lemma 4.9. According to Lemma 4.9, $M$ has ps-Drazin inverse. Therefore, we complete the proof. \qed

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