Approximation of the Block Numerical Range of Block Operator Matrices

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Abstract. In this paper we obtain an approximation of the block numerical range of bounded and unbounded block operator matrices by projection methods.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. The spectra of linear operators play quite a relevant role in many branches of mathematics and in numerous applications. The classical tool to enclosed the spectrum of a linear operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ is the numerical range (see [1, 2]). In [4, 7], the notion of quadratic numerical range was introduced and it may give a better localization of the spectrum than the usual numerical range. In [5], the quadratic numerical range of a (finite) block matrix was approximated by projection methods.

This concept was generalized to block numerical range in [8]. Using the refinement of the decomposition of the space, it was shown that there exists a decreasing sequence of compact sets $\{\overline{\mathcal{W}}_k(\mathcal{A})\}_{k=1}^{\infty}$, such that $\sigma(\mathcal{A}) \subseteq \bigcap_{k=1}^{\infty} \overline{\mathcal{W}}_k(\mathcal{A})$ (see [8]). A total decomposition of $\mathcal{H}$ and an estimable decomposition of $\mathcal{H}$ for $\sigma(\mathcal{A})$ were introduced in [6]. By an estimable decomposition, one can approximate the spectrum of $\mathcal{A}$ by block numerical ranges of $\mathcal{A}$, i.e., there exist a decreasing sequence $\{\mathcal{W}_k(\mathcal{A})\}_{k=1}^{\infty}$, such that $\sigma(\mathcal{A}) = \bigcap_{k=1}^{\infty} \mathcal{W}_k(\mathcal{A})$. But, the existence on the estimable decomposition is, in general, hard to obtain and numerical approximations for the spectra may not be reliable, in particular, if the operator is not self-adjoint or normal. This paper arose from an attempt to gain a better understanding of the block numerical range. In contrast with the quadratic numerical range, we consider how to compute $\mathcal{W}_n(\mathcal{A})$ by projection methods, which reduce the problem to that of computing the block numerical range of a (finite) block matrix. When $\mathcal{A}$ is unbounded, we do assume either that $\mathcal{A}$ is diagonally dominant or off-diagonally dominant.

The organization of this paper is as follows: In section 2 we are going to introduce the related definition and lemma. In section 3 we will give an approximation of the block numerical range of bounded block
operator matrices. In section 4 we obtain the approximations of the block numerical range of unbounded block operator matrices which are diagonally (off-diagonally) dominant.

2. Preliminaries

The following notion of block numerical range for the bounded block operator matrix is due to Tretter and Wagenhofer [8].

Let \( \mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \) where \( \mathcal{H}_1, \ldots, \mathcal{H}_n \) are Hilbert spaces. With respect to this decomposition, the bounded linear operator \( A \) on \( \mathcal{H} \) has a block operator matrix representation:

\[
A := \begin{bmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{bmatrix},
\]

(1)

where \( A_{ij} \in B(\mathcal{H}_j, \mathcal{H}_i), i, j = 1, \ldots, n. \)

**Definition 2.1.** Let \( S^n := \{(x_1, \ldots, x_n)^t \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n : \|x_1\| = \cdots = \|x_n\| = 1\}. \) For \( x = (x_1, \ldots, x_n)^t \in S^n, \) define the \( n \times n \) matrix \( A_x \) as follows:

\[
A_x := \begin{bmatrix}
(A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\
\vdots & \ddots & \vdots \\
(A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n)
\end{bmatrix}.
\]

(2)

Let \( W^n(A) := \{ \lambda \in \mathbb{C} : \lambda \in \sigma(A_x), x \in S^n \} \) be block numerical range of the block operator matrix \( A \), which is defined by (1).

**Remark 2.2.** For \( n = 1 \), the block numerical range is just the usual numerical range, for \( n = 2 \), it is the quadratic numerical range.

In the following Lemma we state some properties for block numerical range of the bounded block operator matrix. (For details see [7, 8].)

**Lemma 2.3.** Let \( A \) as in (1) be a block operator matrix on \( \mathcal{H} \). Then

1. \( \sigma_p(A) \subseteq W^n(A) \), where \( \sigma_p(A) \) is the point spectrum of \( A \).
2. \( \sigma(A) \subseteq W^n(A) \), where \( \sigma(A) \) is the spectrum of \( A \).
3. \( W^n(A) \subseteq W(A) \).
4. \( W^n(A^*) := \{ \overline{\lambda} : \lambda \in W^n(A) \} \).
5. \( W^n(\mathcal{A}) \subseteq W^n(A), \) where \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \) is a refinement (see [7], Definition 1.11.12) of \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \).

3. Convergence Theorems for Bounded Operator

**Theorem 3.1 (For bounded operator).** Let

\[
\mathcal{A} := \begin{bmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{bmatrix},
\]
be a bounded operator in $H = H_1 \oplus \cdots \oplus H_n$. For $i = 1, \ldots, n$, let $(U_i^\alpha)_{\alpha = 1}^\infty$ be nested families of space in $H_\alpha$, given by $U_i^\alpha := \text{span}[a_i^1, \ldots, a_i^{\alpha}]$, where $(a_i^\alpha)_{\alpha = 1}^\infty$ is orthonormal. Let $\mathbb{N}_+ := \{1, 2, 3, \cdots \}$, and multi-index $k := (k_1, \ldots, k_n) \in \mathbb{N}_+^n$. Consider

$$A_k := \begin{bmatrix} A_{k_1} & \cdots & A_{k_1} \\ & \ddots & \vdots \\ A_{k_n} & \cdots & A_{k_n} \end{bmatrix},$$

where $(A_{k_i})_{s,t} := (A_{k_i}^s a_i^t, a_i^t)$, $s = 1, \ldots, k_i; t = 1, \ldots, k_i$. Then $W^n(A_k) \subseteq W^n(\mathcal{A}).$

Proof. Let $\lambda \in W^n(A_k)$, there then exists $\beta := (\beta_1, \ldots, \beta_n)^t$, where $\beta_i \in \mathbb{C}^{k_i}$, with $\|\beta_i\| = 1, i = 1, \ldots, n$, such that $\lambda$ is an eigenvalue of

$$\begin{bmatrix} (A_{k_1} \beta_1, \beta_1) & \cdots & (A_{k_1} \beta_n, \beta_1) \\ \vdots & \ddots & \vdots \\ (A_{k_n} \beta_1, \beta_n) & \cdots & (A_{k_n} \beta_n, \beta_n) \end{bmatrix}$$

Define isometries $\pi_i^k : U_i^\alpha \to \mathbb{C}^{k_i}$, by $\pi_i^k(\beta_1 a_i^1 + \cdots + \beta_n a_i^n) := (\beta_1, \ldots, \beta_n)^t := \beta_i$, for $i = 1, \ldots, n$.

Choose $x = (x_1, \ldots, x_n)^t$, such that $\pi_i^k(x_i) = \beta_i, \|x_i\| = 1$, for $i = 1, \ldots, n$. By a simple calculation, it then follows that $(A_{k_i})_\beta = A_{k_i}$. Hence $\lambda \in W^n(\mathcal{A}).$

Lemma 3.2. Let $(U_i^\alpha)_{\alpha = 1}^\infty$ and $A_k$ be as in Theorem 3.1. Suppose that $\tilde{k}, k \in \mathbb{N}_+^n$ and $\tilde{k} \geq k$, in the sense that, $\tilde{k}_i \geq k_i$, for all $i = 1, \ldots, n$. Then $W^n(A_k) \subseteq W^n(A_{\tilde{k}})$.

Proof. This result is an immediate consequence of the fact that $C^k$ is a subspace of $\tilde{C}^k$ for $\tilde{k}_i \geq k_i, i = 1, \ldots, n$.

Remark 3.3. In the proof of Theorem 3.1 and Lemma 3.2, the boundedness of operators is less important than one might expect. In fact, the same results also hold, if $\mathcal{A}$ is an unbounded operator in $H = H_1 \oplus \cdots \oplus H_n$, and let $(U_i^\alpha)_{\alpha = 1}^\infty$ be nested families of space in $D_i := \bigcap_{j=1}^n D_{ij}$, where $D_{ij}$ is the domain of $A_{ij}$ for $i, j = 1, \ldots, n$.

Roughly speaking, the proof of Theorem 3.1 and Lemma 3.2 also yield for unbounded operators.

Theorem 3.4. Let $\mathcal{A}, A_k$ and $(U_i^\alpha)_{\alpha = 1}^\infty$ be as in Theorem 3.1. Suppose that $(a_i^\alpha)_{\alpha = 1}^\infty$ is orthonormal basis of $H_\alpha$, for $i = 1, \ldots, n$. Then

$$\bigcup_{k \in \mathbb{N}_+^n} W^n(A_k) = \bigcup_{m \in \mathbb{N}_+^n} W^n(A_m) = W^n(\mathcal{A}),$$

where $m := (m_1, \ldots, m_n) \in \mathbb{N}_+^n$.

Proof. By Lemma 3.2, it is immediate that

$$\bigcup_{k \in \mathbb{N}_+^n} W^n(A_k) \subseteq \bigcup_{m \in \mathbb{N}_+^n} W^n(A_m),$$

where $m := \max(k_1, \ldots, k_n)$. To see the other inclusion, consider $m := \min(k_1, \ldots, k_n)$. And hence proves that

$$\bigcup_{k \in \mathbb{N}_+^n} W^n(A_k) = \bigcup_{m \in \mathbb{N}_+^n} W^n(A_m).$$
To complete this proof, it therefore now remains to prove that \( W^n(\mathcal{A}) \subseteq \bigcup_{k \in \mathbb{N}_0^*} W^n(\mathcal{A}_k) \).

Let \( \lambda \in W^n(\mathcal{A}) \). There then exists \( x \in S^n \), such that \( \lambda \) is an eigenvalue of \( \mathcal{A}_n \), as defined in (2). Since \( (\alpha_i^k)_{k=1}^\infty \) is orthonormal basis of \( \mathcal{H}_i \), \( i = 1, \ldots, n \), there exists a sequence \( (x^i_k)_{k=1}^\infty \), with each \( x^i_k \in \text{span}(\alpha_i^1, \ldots, \alpha_i^k) \) for some \( k_i > 0 \), such that \( ||x^i - x^i_k|| \rightarrow 0 \), and \( ||A_{ij}^i x^i_k - A_{ij}^i x^i|| \rightarrow 0 \), as \( k \rightarrow \infty \), where \( x^i \) denotes the \( i \)-th component of \( x \) and \( j = 1, \ldots, n \). Let \( x_k = (x^1_k, \ldots, x^n_k) \), by a simple calculation, we then obtain that \( ||\mathcal{A}_k - \mathcal{A}_n|| \rightarrow 0 \), as \( k \rightarrow \infty \).

For a unbounded linear operator \( A \) in \( \mathcal{H} \) which admits a so-called block operator matrix representation:

\[
\mathcal{A} := \begin{bmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{bmatrix},
\]

where \( A_{ij} : \mathcal{H}_i \rightarrow \mathcal{H}_j \), is closable operators with dense domains \( D_{ij} \in \mathcal{H}_j \), for \( i, j = 1, \ldots, n \). We always suppose that \( \mathcal{A} \) with its natural domain \( D(\mathcal{A}) := D_1 \oplus \cdots \oplus D_n \), where \( D_i := \bigcap_{j=1}^n D_{ij} \in \mathcal{H}_j \), is also densely defined for \( i, j = 1, \ldots, n \).

\textbf{Remark 4.1.} It should be noted that, unlike bounded operators, unbounded linear operators, in general, do not admit a matrix representation (5), with respect to a given decomposition \( \mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \).

\textbf{Definition 4.2.} The block operator matrix \( \mathcal{A} \) in (5) is called

(1) diagonally dominant if \( A_{ij} \) is \( A_{jj} \)-bounded (see [7], Definition 2.1.2),

(2) off-diagonally dominant if \( A_{ij} \) is \( A_{n+1-j,j} \)-bounded, where \( i, j = 1, \ldots, n \).

The definition of the block numerical range for bounded linear operators (see [7], Definition 1.1.12.) generalizes as follows to unbounded block operator matrices \( \mathcal{A} \) of the form (1) with dense domain \( D(\mathcal{A}) \).

\textbf{Definition 4.3.} Let \( S^n := \{(x_1, \ldots, x_n) : \|x_1\| = \cdots = \|x_n\| = 1\} \). For \( x = (x_1, \ldots, x_n) \in S^n \), define the \( n \times n \) matrix \( \mathcal{A}_x \) as follows:

\[
\mathcal{A}_x := \begin{bmatrix}
(A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\
\vdots & \ddots & \vdots \\
(A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n)
\end{bmatrix}.
\]

Let

\[
W^n(\mathcal{A}) := \{ \lambda \in \mathbb{C} : \lambda \in \sigma(\mathcal{A}_x), x \in S^n \}
\]

be block numerical range of the unbounded block operator matrix \( \mathcal{A} \), which is defined by (5).
Remark 4.4. For \( n = 1 \), the block numerical range is just the usual numerical range, for \( n = 2 \), it is the quadratic numerical range, as the bounded case.

The following result shows some important properties of the block numerical range of the unbounded block operator matrix.

**Proposition 4.5.** For an unbounded block operator matrix \( \mathcal{A} \), we have

1. \( \sigma_p(\mathcal{A}) \subseteq W^p(\mathcal{A}) \), where \( \sigma_p(\mathcal{A}) \) is the point spectrum of \( \mathcal{A} \).
2. \( W^p(\mathcal{A}) \subseteq W(\mathcal{A}) \).
3. \( W^p(\mathcal{A}) \subseteq W(\mathcal{A}) \), where \( \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_n \) is a refinement (see [7], Definition 1.11.12) of \( \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_n \).

**Proof.** The proofs are completely analogous to the proofs of the bounded case (see [7]) if we take \( x = (x_1, \ldots, x_n)^T \in S^n \).

In the following result we describe a property of convergence for unbounded operator.

**Theorem 4.6 (For unbounded operator).** Let

\[
\mathcal{A} := \begin{bmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{bmatrix}
\]

be an unbounded operator in \( \mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \). For \( i = 1, \ldots, n \), let \( (\mathcal{U}^i_k)_{k=1}^\infty \) be nested families of space in \( \mathcal{D}_i \), given by \( \mathcal{U}^i_k := \text{span}(\{x^i_k\}) \), where \( (x^i_k)_{k=1}^\infty \) is orthonormal. And \( A^i_k \) denotes as in Theorem 3.1. Suppose that \( \mathcal{A} \) is diagonally dominant, and \( (\mathcal{U}^i_k)_{k=1}^\infty \) is a core (see [3], Section III.3) of \( A^i_k \), \( i = 1, \ldots, n \). Then \( \bigcup_{i=1}^n \bigcap_{k=1}^\infty W^p(A^i_k) = \bigcap_{m \in \mathbb{N}_+} W^p(A_{\infty}^m) = W_p(\mathcal{A}) \), where \( m^i := (m, \ldots, m) \in \mathbb{N}_+^n \).

**Proof.** Since \( (\mathcal{U}^i_k)_{k=1}^\infty \) is a core of \( A^i_k \), for \( i = 1, \ldots, n \), there exists a sequence \( (x^i_k)_{k=1}^\infty \) with each \( x^i_k \in \mathcal{U}^i_k \) for some \( k_i > 0 \), such that \( \|x^i_k - x^i_0\| \to 0 \), and \( \|A^i_k x^i_k - A^i_i x^i_k\| \to 0 \). Because \( A^i_k \) is \( A^i_0 \)-bounded for \( j = 1, \ldots, n \), we have \( \|A^i_j x^i_k - A^i_j x^i_0\| \to 0 \), as \( k \to \infty \). The rest of proof is completely analogous to the proof of Theorem 3.4.

**Remark 4.7.** The same result holds if \( \mathcal{A} \) is off-diagonally dominant with \( (\mathcal{U}^i_k)_{k=1}^\infty \) being a core of \( A_{\infty-1-i} \), for \( i = 1, \ldots, n \).

**Remark 4.8.** Note that, the result of Theorem 2.3 in [5], is the \( n = 2 \) case of Theorem 4.6.

**Acknowledgment**

The authors are grateful to the referees for valuable comments on this paper.

**References**