INTERACTION OF HEAT TRANSFER AND PERISTALTIC PUMPING OF FRACTIONAL SECOND GRADE FLUID THROUGH A VERTICAL CYLINDRICAL TUBE

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Abstract

This paper deals with a theoretical investigation of interaction of heat transfer with peristaltic pumping of a fractional second grade fluid through a tube, under the assumption of low Reynolds number and long wave length approximation. Analytical solution of problem is obtained by using Caputo’s definition. Effect of different physical parameters, material constant, amplitude ratio, friction force, temperature and heat transfer on pumping action and frictional force are discussed with particular emphasis. The computed results are presented in graphical form.

Keywords: Peristalsis, Fractional second grade model, Heat transfer, pressure, friction force, Caputo’s fractional derivative.

1. Introduction

Peristalsis is well known mechanism for mixing and transporting fluid. Peristaltic flow can be generated by the propagation of waves along flexible wall of channel or tube. Peristalsis is used by a living body to propel or to mix the contents of the tube such as transport of urine through ureter, food mixing and chyme movement in intestines, transport in bile duct etc. Peristaltic flows have attracted a number of researchers because of wide applications in physiology and industry. The theoretical work on peristaltic transport primarily with inertia free Newtonian flow driven by sinusoidal transverse wave of small amplitude by Fung and Yih [1]. Burns and Parkes [2] studied the peristaltic motion of a viscous fluid through a pipe and a channel by considering sinusoidal variation at the walls. Abd El Naby and El-Misiery [3] picked up peristaltic pumping of a Carreau fluid in presence of an endoscope. Abd El Hakeem et al. [4] studied hydro magnetic flow of generalized Newtonian fluid through a uniform tube with peristalsis. Recently a study of ureteral peristalsis in cylindrical tube through porous medium has been discussed by Rathod et al. [5].

Existing literature indicates that little efforts is made to explain the heat transfer on peristaltic transport. Some authors [6-9] have analyzed the interaction of peristalsis with heat transfer. Peristaltic mechanism in an asymmetric channel with heat transfer has been presented by Hayat et al. [10]. Recently, Vasudeva et al. [11] have studied peristaltic flow of a Newtonian fluid through a porous medium in vertical tube under the effect of magnetic field. The influence of heat transfer on peristaltic transport of Jeffrey fluid in vertical porous stratum has been studied by vajarvelu et al. [12]. Vasudeva et al. [13] have investigated the effect of heat transfer on peristaltic flow of a Jeffrey fluid through a porous medium in a vertical annulus. Srinivas and Kothandapani [14] have presented the peristaltic transport in an asymmetric channel with heat transfer.

In the last few decades fractional calculus is increasingly used in the modeling of various physical and dynamical system. Fractional calculus has encountered much success in the description of viscoelastic
characteristics. The starting point of the fractional derivative model of non-Newtonian model is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann–Liouville fractional calculus operators. This generalization allows to define precisely non-integer order integrals or derivatives. Fractional second grade model is the model of viscoelastic fluid. In general, fractional second grade model is derived from well known second grade model by replacing the ordinary time derivatives to fractional order time derivatives and this plays an important role to study viscoelastic properties. In the past few decades, both mathematicians and physicists have made significant progress in this direction. Mention may be made to some recent investigations [15-20] that deals with fractional Oldroyd-B model, unsteady flows of viscoelastic fluid with fractional Maxwell model, fractional Burgers’ model and fractional generalized Burger’s model through channel/tube/annulus. Very recent works on fractional second grade fluids have been made by Tripathi et al. [21] solutions for velocity field and pressure are obtained by homotopy perturbation method and Adomian decomposition methods. Further, numerical study on peristaltic flow of generalized burgers’ fluids in uniform tubes in the presence of an endoscope has been studied by Tripathi et al. [22]. Peristaltic flow of a fractional second grade fluid through a cylindrical tube has been studied by Tripathi et al. [23]. Some important works [24-29] such as; Numerical and analytical simulation of peristaltic flow of generalized Oldroyd-B fluids, mathematical model for the peristaltic flow of chyme movement in small intestine, peristaltic transport of fractional Maxwell fluids in uniform tubes: applications in endoscopy, peristaltic transport of a viscoelastic fluid in a channel, numerical study on peristaltic transport of fractional bio fluids model, a mathematical model for swallowing of food bolus through the oesophagus under the influence of heat transfer .Tripathi et al. [30] have studied the peristaltic transport of a generalized Burgers’ fluid: Application to the movement of chyme in small intestine. Heat and mass transfer effects on the peristaltic flows in annulus and endoscope [31-32] are very important because of its practical engineering applications, such as food processing and blood pumps in heart lungs machines. Endoscopic and heat transfer effects on the peristaltic flow of a third-order fluid with chemical reactions have been studied by Nadeem and Akbar [33]. Peristaltic flow of a Williamson fluid in an inclined asymmetric channel with partial slip and heat transfer has been studied by Akbar et al. [34].

The present analysis is of viscoelastic fluid with fractional second grade model and heat transfer through a vertical cylindrical tube under the assumption of long wavelength and low Reynolds number. Caputo’s definition is used to find fractional differentiation and numerical results of problem for different cases are discussed graphically. This model is applied to study of movement of chyme through small intestine and also applicable in mechanical point of view.

**Caputo’s Definition**

Caputo’s definition [23] of the fractional order derivative is defined as

\[
D^{\alpha_i} f(t) = \frac{1}{\Gamma(n-\alpha_i)} \int_{b}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{n+\alpha_i-n}} d\tau \quad (n-1, \Re(\alpha_i) \leq n, \ n \in N)
\]

Where, \( \alpha_i \) is the order of derivative and is allowed to be real or even complex, \( b \) is the initial value of function \( f \). For the Caputo’s derivatives we have
$$D^{\alpha_t}t^{\beta_t} = \begin{cases} 0 & (\beta_t \leq \alpha_t - 1) \\ \frac{\Gamma(\beta_t + 1)}{\Gamma(\beta_t - \alpha_t + 1)} t^{\beta_t - \alpha_t} & (\beta_t \geq \alpha_t - 1) \end{cases}$$

2. Mathematical formulation

Consider an incompressible fractional second grade fluid in vertical tube (See Fig. 1) induced by sinusoidal wave trains propagating with constant speed $c$. Walls of the tube are maintained at temperature $T_0$ and $T_1$ respectively.

The constitutive equation for viscoelastic fluid with fractional second grade model is given by

$$\ddot{\vec{S}} = \mu \left( 1 + \lambda_1^{\alpha_t} \frac{\partial^\alpha_t}{\partial t^\alpha_t} \right) \dot{\gamma}, \quad (1)$$

where $\ddot{\vec{S}}$, $\ddot{\gamma}$, $\dot{\gamma}$, $\mu$, $\lambda_1$ and $\alpha_t$ are shear stress, time, material constants, viscosity and rate of shear strain respectively, $\alpha_t$ is fractional time derivative parameter such that $0 < \alpha_t \leq 1$. This model reduces to second grade models with $\alpha_t = 1$, and classical Navier Stokes model is obtain by substituting $\lambda_1 = 0$.

![Flow geometry](image)

The governing equations of motion of viscoelastic fluid with fractional second grade model for axisymmetric flow are given by

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \ddot{u} \right) + \frac{\partial \ddot{u}}{\partial x} = 0 \quad (2)$$

$$\rho \left( \frac{\partial \ddot{u}}{\partial t} + \dot{u} \frac{\partial \ddot{u}}{\partial x} + \ddot{v} \frac{\partial \ddot{u}}{\partial r} \right) = \frac{\partial \ddot{p}}{\partial x} + \mu \left( 1 + \lambda_1^{\alpha_t} \frac{\partial^\alpha_t}{\partial t^\alpha_t} \right) \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \ddot{u}}{\partial r} \right) + \frac{\partial^2 \ddot{u}}{\partial x^2} \right) + \rho g a (T - T_0) \quad (3)$$

$$\rho \left( \frac{\partial \ddot{v}}{\partial t} + \dot{u} \frac{\partial \ddot{v}}{\partial x} + \ddot{v} \frac{\partial \ddot{v}}{\partial r} \right) = \frac{\partial \ddot{p}}{\partial r} + \mu \left( 1 + \lambda_1^{\alpha_t} \frac{\partial^\alpha_t}{\partial t^\alpha_t} \right) \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r \ddot{v}) \right) + \frac{\partial^2 \ddot{v}}{\partial x^2} \right) \quad (4)$$
\[
\rho c_p \left( \frac{\partial T}{\partial t} + \bar{u} \frac{\partial T}{\partial x} + \bar{v} \frac{\partial T}{\partial y} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial y^2} \right) + Q_0
\]  

(5)

In order to simplify the non-linear system of equation, it is necessary to introduce the following non-dimensional parameters:

\[
\begin{align*}
x &= \frac{x}{\lambda}, \quad r = \frac{r}{a}, \quad \lambda_1^{\alpha_1} = \frac{c \lambda_1^\alpha}{\lambda}, \quad u = \frac{\bar{u}}{c}, \quad v = \frac{\bar{v}}{c \delta} \\
\delta &= \frac{a}{\lambda}, \quad \phi = \frac{\phi}{a}, \quad p = \frac{\bar{p} a^2}{\mu c \lambda}, \quad R_e = \rho c a \delta / \mu \\
\theta &= \frac{T - T_0}{T_0}, \quad Gr = \frac{\rho g a a^3 T_0}{\mu}, \quad Pr = \frac{\mu c_p}{k}, \quad \beta = a^2 Q_0 / k T_0
\end{align*}
\]

(6)

where \( \rho \) is fluid density, \( \delta \) is wave number; \( p, c, v, u, t, r, \phi, \lambda \) and \( Q \) stands for pressure, wave velocity, axial velocities, radial velocities, time, radial coordinate, amplitude, wavelength and volume flow rate respectively, \( T \) is temperature, \( g \) is acceleration due to gravity, \( c_p \) is specific heat, \( Q_0 \) is constant heat addition/absorption, \( \beta \) is heat source/sink parameter, \( Gr \) is Grashof number, \( Pr \) is Prandtl number. Using the non-dimensional parameters (6) and taking long wavelength approximation and low Reynolds number, Eqs (2-5) reduces to

\[
\frac{\partial p}{\partial x} = \left( 1 + \lambda_1^{\alpha_1} \frac{\partial \phi}{\partial x} \right) \left[ \frac{\partial^2 u}{\partial x^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right] + Gr \theta,
\]

(7)

\[
\frac{\partial p}{\partial r} = 0,
\]

(8)

\[
\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \beta = 0.
\]

(9)

Boundary conditions are given by

\[\begin{align*}
\frac{\partial u}{\partial r} &= 0 \quad \text{at} \quad r = 0, \quad u = 0 \quad \text{at} \quad r = h, \\
\frac{\partial \theta}{\partial r} &= 0 \quad \text{at} \quad r = 0, \quad \theta = 0 \quad \text{at} \quad r = h.
\end{align*}\]

(10)

(11)

Solving Eq. (9) using the boundary conditions (11), we get

\[\theta = \frac{\beta}{4} (h^2 - r^2).\]

(12)

Substituting Eq. (12) in to the Eq. (7) and solving with boundary condition (10) we obtain

\[
\left( 1 + \lambda_1^{\alpha_1} \frac{\partial \phi}{\partial x} \right) u = \frac{1}{4} \frac{\partial p}{\partial x} (r^2 - h^2) + \frac{Gr \beta}{64} \left( r^4 + 3h^4 - 4h^2 r^2 \right).
\]

(13)

The volume flow rate is defined as

\[Q = \int_0^h 2 \pi u dr,\]

which by virtue of Eq. (13), reduces to
\[
\left(1 + \lambda^\prime \left[ \frac{\partial^\prime}{\partial t^\prime} \right] \right) Q = -\frac{h^4}{8} \left( \frac{\partial p}{\partial x} \right) - Gr \beta \left( \frac{h^2}{8} - \frac{h}{20} \right) \]  
\]  
(14)

The transformation between wave and laboratory frames, in dimensionless form is given by
\[
X = x - 1, \quad R = r, \quad U = u - 1, \quad V = \nu, \quad q = Q - h^2 
\]  
(15)

Further the wall under goes contraction and relaxation is mathematically formulated as
\[
h = 1 - \phi \cos^2(\pi X). 
\]  
(16)

The following are the existing relations between the averaged flow rate, the flow rate in the wave frame and that in the laboratory frame:
\[
\bar{Q} = q + 1 - \phi + \frac{3\phi^2}{8} = Q - h^2 + 1 - \phi + \frac{3\phi^2}{8} 
\]  
(17)

Eq. (14), in view of Eq. (17), we get
\[
\frac{\partial p}{\partial x} = \frac{Gr \beta}{8} \left( 1 + \frac{8h}{20} \right) - \frac{8}{h^4} \left( \frac{\bar{Q} + h^2 - 1 - \phi + \frac{3\phi^2}{8}}{1 + \lambda^\prime \left[ \frac{\partial^\prime}{\partial t^\prime} \right]} \right) 
\]  
(18)

It is observed that as \( \beta \to 0 \), equation (13), (14) and (18) reduces to corresponding results of Tripathi [23].

The dimensionless pressure rise \( \Delta p \) and friction force \( F_\alpha \) per one wave length are defined by
\[
\Delta \rho = \int_0^1 \frac{\partial p}{\partial x} \, dx 
\]  
(19)

\[
F_\alpha = \int_0^1 \left( -h^2 \frac{\partial p}{\partial x} \right) \, dx 
\]  
(20)

The heat transfer coefficient at the outer wall is given by Ref. [13]
\[
Z = \left[ \frac{\partial \theta}{\partial r} \frac{\partial r}{\partial x} \right]_{r=h} = -\frac{\phi \beta h \pi}{2} \left( \sin(2\pi x) \right) 
\]  
(21)

3. Results and Discussion

In order to see quantitative effects of various emerging parameters involved in results on pumping characteristics and heat transfer coefficient the MATHEMATICA package is used. The salient features of temperature \( \theta \), heat transfer coefficient \( Z \) and pressure rise and friction forces are analyzed through the Figs. 2-15. Figs 2-5 demonstrate that there is a linear relation between pressure and time-averaged flow rate. It is worth mentioning that an increase in averaged flow rate makes the pressure fall and thus maximum flow rate is achieved at zero pressure and maximum pressure occurs at zero time-averaged flow rate. The variation of pressure rise \( \Delta p \) against flow rate \( \bar{Q} \) for various values of \( a_1 \) at \( \phi = 0.4, t = 0.5, \lambda_1 = 1, Gr = 3, \beta = 5 \) is depicted in Fig.2. It is observed that the pumping rate decreases with increase of \( a_1 \) for pumping \( (\Delta p > 0) \) and as well as for free pumping \( (\Delta p = 0) \). Also, it can be noted that the fractional behavior of second grade fluids increases, the pressure for flow diminishes. Fig.3 shows the variation of pressure rise \( \Delta p \) against flow rate \( \bar{Q} \) for various values of \( \phi \) at \( \alpha = 0.2, \lambda_1 = 1, Gr = 3, \beta = 5 \) is depicted in Fig.2. It is observed that the pressure increases with increasing amplitude ratio \( \phi \). Fig.4 shows that the graph between \( \Delta p \) and \( \bar{Q} \) for various values of \( t \) at \( a_1 = 0.2, \phi = 0.4, \lambda_1 = 1, \beta = 5 \). It is found that pressure increases with an increase in the magnitude of the
parameter \( t \). Fig. 5 depicts the variation of pressure rise \( \Delta p \) with time averaged flow rate \( \bar{Q} \) for different values of \( \lambda_i \) at \( \alpha = 0.2, \phi = 0.4, t = 0.5, Gr = 3, \beta = 5 \). It is revealed that the pressure increase with increasing \( \lambda_i \). This means that the viscoelastic behavior of fluids increases, the pressure for flow of fluids decreases, i.e., the flow for second grade fluid requires more pressure than that of Newtonian fluids \( \lambda_i \to 0 \). The variation of pressure rise \( \Delta p \) against flow rate \( \bar{Q} \) for various values of Grashof number \( Gr \) at \( a_i = 0.2, \phi = 0.4, t = 0.5, \lambda_i = 1, \beta = 5 \) is presented in Fig. 6. It is evident that an increase in flow rate \( \bar{Q} \) reduces the pressure. Volume flow rate increases with increasing magnitude of Grashof number.

The variation of pressure rise \( \Delta p \) against flow rate \( \bar{Q} \) for various values of heat Source/Sink parameter \( \beta \) at \( a_i = 0.2, \phi = 0.4, t = 0.5, \lambda_i = 1, Gr = 4 \) is presented in Fig. 7. It is observed that an increase in \( \beta \) increases the time averaged flow rate \( \bar{Q} \) in all the three (pumping, free-pumping and co-pumping) regions.

The variation of fractional force against flow rate is shown in Figs. 8-13. It can be seen that the effect of all parameters on friction force have opposite behavior as compared to the pressure rise.

Figs. 14-15 depict the behavior of heat transfer coefficient at the wall and have an oscillatory behavior due to peristalsis. Fig. 14 shows the variation of heat transfer coefficient \( Z \) with \( \beta \). Absolute value of heat transfer coefficient increases with an increase in \( \beta \). Fig. 15 illustrates that with increasing \( \phi \) the magnitude of heat transfer coefficient increases.

4. Conclusions

The study examines the interaction of heat transfer and peristaltic pumping of a fractional second grade fluid through cylindrical tube under low Reynolds number and long wavelength approximation. The Caputo’s definition is used for differentiating the fractional derivatives. Closed form solutions are derived for velocity and temperature. The main points of the performed analysis are as follows.

- Pressure rise decrease with an increase in fractional parameter \( \alpha \).
- The qualitative behaviors of \( \phi, t, \lambda_i, Gr \) and \( \beta \) on the pressure are similar.
- It is observed that frictional forces have an opposite behavior to that of pressure rise.
- The absolute value of heat coefficient increases with increase of \( \beta \) and \( \phi \).

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Fig. 2. Pressure vs. averaged flow rate for various values of $\alpha_i$ at $\phi = 0.4$, $t = 0.5$, $\lambda = 1$, $Gr = 3$, $\beta = 5$.

Fig. 3. Pressure vs. averaged flow rate for various values of $\phi$ at $\alpha_i = 0.2$, $t = 0.5$, $\lambda = 1$, $Gr = 3$, $\beta = 5$.

Fig. 4. Pressure vs. averaged flow rate for various values of $t$ at $\alpha_i = 0.2$, $\phi = 0.4$, $\lambda_i = 1$, $Gr = 3$, $\beta = 5$. 


Fig. 5. Pressure vs. averaged flow rate for various values of $\lambda_i$ at 
$\alpha_i = 0.2, \phi = 0.4, t = 0.5, Gr = 3, \beta = 5$

Fig. 6. Pressure vs. averaged flow rate for various values of $Gr$ at 
$\alpha_i = 0.2, \phi = 0.4, t = 0.5, \lambda_i = 1, \beta = 5$

Fig. 7. Pressure vs. averaged flow rate for various values of $\beta$ at 
$\alpha_i = 0.2, \phi = 0.4, t = 0.5, \lambda_i = 1, Gr = 3$
Fig. 8. Friction force vs. averaged flow rate for various values of $\alpha_i$ at $\phi=0.4, t=0.5, \lambda_i=1, Gr=3, \beta=5$

Fig. 9. Friction force vs. averaged flow rate for various values of $\phi$ at $\alpha_i=0.2, t=0.5, \lambda_i=1, Gr=3, \beta=5$

Fig. 10. Friction force vs. averaged flow rate for various values of $t$ at $\phi=0.4, \alpha_i=0.2, \lambda_i=1, Gr=3, \beta=5$
Fig. 11. Friction force vs. averaged flow rate for various values of $\lambda_1$ at $\alpha_i=0.2$, $\phi=0.4$, $t=0.5$, $Gr=3$, $\beta=5$.

Fig. 12. Friction force vs. averaged flow rate for various values of $\beta$ at $\alpha_i=0.2$, $\phi=0.4$, $t=0.5$, $\lambda_1=1$, $Gr=3$.

Fig. 13. Friction force vs. averaged flow rate for various values of $Gr$ at $\alpha_i=0.2$, $\phi=0.4$, $t=0.5$, $\lambda_1=1$, $\beta=5$. 
Fig. 14. Coefficient of heat transfer for various values of $\beta$ at $\phi = 0.4$

Fig. 15. Coefficient of heat transfer for various values of $\phi$ at $\beta = 5$

References


