FRACTIONAL MODEL FOR HEAT CONDUCTION IN POLAR BEAR HAIRS

by

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Time-fractional differential equations can accurately describe heat conduction in fractal media, such as wool fibers, goose down and polar bear hair. The fractional complex transform is used to convert time-fractional heat conduction equations with the modified Riemann-Liouville derivative into ordinary differential equations, and exact solutions can be easily obtained. The solution process is straightforward and concise.

Key words: *fractional complex transform, modified Riemann-Liouville derivative, time-fractional heat conduction equation, anomalous heat conduction*

Introduction

Fractional differential equations can describe many real-life problems in a more accurate way than partial/ordinary differential equations, whereas numerous definitions of the fractional derivatives make an engineer elusive to how to choose a suitable fractional derivative, furthermore, the fractional differential equations are very difficult to be solved if not impossible. Recently much progress has been made in both issues. Firstly some effective analytical approaches to fractional differential equations are available, for examples, the heat-balance integral method [1-3],the fractional Lie Group method [4], the variational iteration method [5], the homotopy perturbation method [6-8], and the exp-function method [9].

Secondly a simple fractal derivative [10, 11] is introduced for any discontinuous media, and the fractal dimensions are equivalent to the order of the fractal derivative.

Consider heat conduction in a fractal media such as wool fibers, goose down and polar bear hair, and assume the smallest measure is L_0, any discontinuity less than L_0 is ignored, then the distance between two points of A and B can be expressed using fractal geometry. The fractal derivative is defined as

$$\frac{\mathrm{D}u(x)}{\mathrm{D}x^{a}}=\lim_{\Delta x\to L_0}\frac{u(A)-u(B)}{\text{The distance between two points } A \text{ and } B}=\frac{\mathrm{d}u}{\mathrm{d}s}=\lim_{\Delta x\to L_0}\frac{u(A)-u(B)}{kL_0^{\alpha}} \quad (1)$$

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where k is a constant, α – the fractal dimension, and the distance between two points in a discontinuous space can be expressed as:

$$ds = kL_0^{\alpha} \tag{2}$$

Thirdly the fractional complex transform proposed by Li *et al.* [12, 13] can convert fractional differential equations to ordinary differential equations, so that non-mathematicians can easily deal with fractional calculus. In this paper we will extend the fractional complex transform to a time-fractional model for heat conduction in animal's hairs.

Fractional model for heat conduction in polar bear Hairs

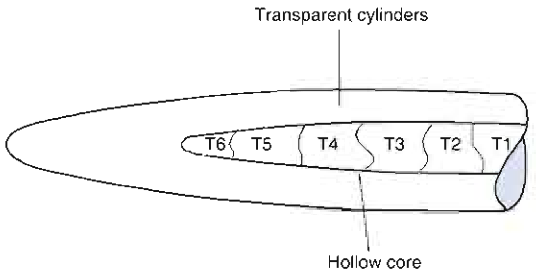

Figure 1. Hollow and membrane structure of a polar bear hair, T1 is the body temperature; and T6 is the environment temperature

The polar bear (*Ursus Maritimus*) has superior ability to survive in the harsh Arctic regions, its hollow and membrane structure of the polar bear hair, fig. 1, plays an important role in body temperature lose and energy absorb from environment [14]. The transparent part of the polar bear hair enables its excellent optical character to absorb solar energy and relatively poor heat conduction to prevent from body temperature lose and to absorb environment temperature from ice water or air; while the hollow and membrane structure behaves as a good thermal insulation.

If the hair is considered as a continuous media, we can write down the following 1-D heat conduction equation:

$$\frac{\partial \tilde{T}}{\partial t} + \tilde{D}\frac{\partial^2 \tilde{T}}{\partial x^2} = 0, x \in (0, L), t \geq 0 \tag{3}$$

where $\tilde{T}(x, t)$ is the temperature at the point x and time t and $\tilde{D}$ – the thermal diffusivity.

The polar bear's body temperature is about 37 °C and the environment temperature can be as low as –50 °C, we, therefore, have the following boundary conditions:

$$\tilde{T}(0, t) = 37\ °\text{C and } \tilde{T}(L, t) = -50\ °\text{C} \tag{4}$$

where L is the length of the hair.

Equation (3) is valid only for continuous media. For discontinuous polar bear hairs, eq. (3) can be modified as:

$$\frac{\partial \tilde{T}}{\partial t} + \frac{\partial^a}{\partial x^a}\left(\tilde{D}\frac{\partial^b \tilde{T}}{\partial x^2}\right) = 0 \tag{5}$$

where $\partial^a \tilde{T}/\partial x^a$ is the modified Riemann-Liouville fractional derivative [15-17] of order α with respect to x.

By a suitable transform, eq. (8) can be converted into the following dimensionless time-fractional differential equation:

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} + \mathrm{D}\frac{\partial^2 T}{\partial x^2} = 0, \quad x \in (0,1), \quad t > 0 \tag{6}$$

Assume the temperature decreases along the hair exponentially, we have the following initial condition:

$$T(x,0) = a - \frac{a-b}{1-\exp\left(\frac{-1}{k\mathrm{D}}\right)} + \frac{a-b}{1-\exp\left(\frac{-1}{k\mathrm{D}}\right)}\exp\left(\frac{-x}{k\mathrm{D}}\right) \tag{7}$$

where a is the body temperature, b – the environment temperature, and k – a constant.

Fractional complex transform

The fractional complex transform is to introduce a transform in the form [12, 13]:

$$\xi = \frac{qT^{\alpha}}{\Gamma(1+\alpha)} + px \tag{8}$$

Equation (6) becomes:

$$qT_{\xi} + \mathrm{D}p^2 T_{\xi\xi} = 0 \tag{9}$$

Solving eq. (9) results in:

$$T(\xi) = c_1 + c_2 \exp\left(-\frac{q\xi}{\mathrm{D}p^2}\right) \tag{10}$$

where c_1 and c_2 are integral constants.

We, therefore, obtain the following general exact solution:

$$T(x,t) = c_1 + c_2 \exp\left(-\frac{qx}{\mathrm{D}p} - \frac{q^2 t^{\alpha}}{\mathrm{D}p^2\Gamma(1+\alpha)}\right) \tag{11}$$

Incorporating the initial condition, eq. (7), we finally have:

$$T(x,t) = a - \frac{a-b}{1-\exp\left(\frac{-1}{k\mathrm{D}}\right)} + \frac{a-b}{1-\exp\left(\frac{-1}{k\mathrm{D}}\right)}\exp\left(-\frac{1}{k\mathrm{D}}x - \frac{t^{\alpha}}{\mathrm{D}k^2\Gamma(1+\alpha)}\right) \tag{12}$$

Equation (12) implies that the body temperature will decrease to $\lim_{t\to\infty} T(x,t) = a - (a-b)/$ $/1-\exp(-1/k\mathrm{D})$. Actually this is not the case, because the hair with hollow and membrane structure can also absorb energy from water, ice, wind and solar light as well, see a detailed description in ref. [18].

Conclusion

In this paper, the fractional complex transform has been successfully used to obtain explicit and exact solutions of homogeneous heat conduction equation arising in polar bear hairs.

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