MULTI-WAVE SOLUTIONS FOR A NON-ISOSPECTRAL KDV-TYPE EQUATION WITH VARIABLE COEFFICIENTS

by

Sheng ZHANG*, Qun GAO, Qian-An ZONG, and Dong LIU

Department of Mathematics, Bohai University, Jinzhou, China

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As a typical mathematical model in fluids and plasmas, Korteweg-de Vries equation is famous. In this paper, the Exp-function method is extended to a nonisospectral Korteweg-de Vries type equation with three variable coefficients, and multi-wave solutions are obtained. It is shown that the Exp-function method combined with appropriate ansatz may provide with a straightforward, effective and alternative method for constructing multi-wave solutions of variable-coefficient non-linear evolution equations.

Key words: Korteweg-de Vries type equation, exp-function method, multi-wave solution

Introduction

With the development of soliton theory, finding exact solutions of non-linear evolution equations (NLEE) has attached much attention and developed into a significant direction in non-linear science. Since proposed by He and Wu in 2006, the Exp-function method [1] has been applied to many equations, such as the double sine-Gordon equation [2], Maccari’s system [3] and variable-coefficient Korteweg-de Vries (KdV) equation [4]. In addition, this method can be generalized for solving differential-difference equation [5], stochastic equation [6], and fractional differential equation [7].


In this paper, we extend the Exp-function method for constructing multi-wave solutions of a non-isospectral KdV-type equation with variable coefficients \( K_0(t), K_1(t) \) and \( h(t) \) of time \( t \):

\[
 u_t + K_0(t)(u_{xxx} + 6uu_x) + 4K_1(t)u_x - h(t)(xu_x + 2u) = 0
\]  

(1)

which was derived by Chan et al. from a non-isospectral Lax pair [12].

* Corresponding author; e-mail: zhshaeng@yahoo.com.cn
Methodology

We describe the basic idea of the Exp-function method combined with a new and more general ansatz for multi-wave solutions of the given NLEE with variable coefficients, say, in three variables \( x, y, \) and \( t \):

\[
P(x, y, t, u, u_x, u_y, u_{xt}, u_{yt}, u_{xx}, u_{yy}, u_{tt}, \cdots) = 0
\]  

(2)

The Exp-function method for 1-wave solution is based on the assumption that eq. (2) has a solution:

\[
u(x, y, t) = \sum_{i=0}^{p_1} a_{i_1} (x, y, t) e^{i \xi_1(x, y, t)}
\]

(3)

where \( a_{i_1} (x, y, t), \ b_{j_1} (x, y, t), \) and \( \xi_1(x, y, t) \) are unknown functions of the indicated variables, the values of \( p_1 \) and \( q_1 \) can be determined by balancing the linear term of highest order in eq. (2) with the highest order non-linear term.

To seek \( N \)-soliton solutions for integer \( N > 1 \), we generalize eq. (3) to the form:

\[
u(x, y, t) = \sum_{i_x=0}^{p_2} \sum_{i_y=0}^{q_2} \cdots \sum_{i_y=0}^{q_2} a_{i_1, \ldots, i_y} (x, y, t) e^{i \xi_1(x, y, t)}
\]

(4)

Substituting eq. (4) with \( N = 2 \) into eq. (2) and equating to zero each coefficient of the same order power of the exponential functions yields a set of equations. Solving the set of equations, we can determine the 2-wave solution, and the following 3-wave solution by eq. (4) with \( N = 3 \), provided they exist. If possible, we may conclude with the uniform formula of \( N \)-wave solution for any \( N > 1 \).

Multi-wave solutions

Let us apply the method described in the section Methodology to solve the non-isospectral KdV-type equation. To begin with, we suppose that eq. (1) admits 1-wave solution in the form:

\[
u(x, t) = \frac{a_1(t) \exp(\xi_1)}{[1 + b_1 \exp(\xi_1)]^2}
\]

(5)

where \( \xi_1 = k_1(t)x + s_1(t) + w_1, \ k_1(t), \ s_1(t), \) and \( a_1(t) \) are undetermined functions of \( t, w_1, \) and \( b_1 \) are constants to be determined.

Substituting eq. (5) into eq. (1), and using Mathematica, then equating to zero each coefficient of the same order power of \( x^j e^{\theta \xi_1} (\theta = 0, 1; \ 0 = 2, 3, 4) \) yields a set of equations for \( k_1(t), s_1(t), a_1(t) \) and \( b_1 \). Solving the set of equations, we have:

\[
\begin{align*}
a_1(t) & = 2b_1k_0^2 e^{2i[h(t)]dt}, \ k_1(t) = k_0 e^{i[h(t)]dt}, \ s_1(t) = - \int[k_0^2 K_0(t) e^{3i[h(t)]dt} + 4k_1(t) e^{i[h(t)]dt}]dt
\end{align*}
\]
from which we obtain the 1-wave solution of eq. (1):

\[ u(x,t) = \frac{2b_1k_{10}^2e^{2\int h(t)dt}}{(1 + b_1e^{\xi_1})^2} = 2[\ln(1 + b_1e^{\xi_1})]_{xx} \]  \hspace{1cm} \text{(6)}

where \( \xi_1 = xk_{10}e^{\int h(t)dt} - \int[k_{10}^3K_0(t)e^{3\int h(t)dt} + 4k_{10}K_1(t)e^{\int h(t)dt}]dt + w_1, \) \( b_1, k_{10}, \) and \( w_1 \) are arbitrary constants.

We next suppose that eq. (1) has 2-wave solution in the form:

\[ u(x,t) = \frac{a_{10}(t)e^{\xi_1} + a_{01}(t)e^{\xi_2} + a_{11}(t)e^{\xi_1 + \xi_2} + a_{21}(t)e^{2\xi_1 + \xi_2} + a_{12}(t)e^{\xi_1 + 2\xi_2}}{(1 + b_1e^{\xi_1} + b_2e^{\xi_2} + b_3e^{\xi_1 + \xi_2})^2} \]  \hspace{1cm} \text{(7)}

where \( \xi_1 = k_1(t)x + s_1(t) + w_1, \) \( \xi_2 = k_2(t)x + s_2(t) + w_2, \) \( k_1(t), k_2(t), s_1(t), s_2(t), a_{10}(t), a_{01}(t), a_{11}(t), a_{21}(t), a_{12}(t) \) are undetermined functions of \( t, \) and \( w_1, w_2, b_1, b_2, \) and \( b_3 \) are constants to be determined.

Substituting eq. (7) into eq. (1) and using Mathematica yields:

\[ a_{10}(t) = 2b_1k_{10}^2e^{2\int h(t)dt}, \quad a_{01}(t) = 2b_2k_{20}^2e^{2\int h(t)dt}, \quad a_{11}(t) = 2b_1b_2k_{10}^2e^{2\int h(t)dt}(k_{10} - k_{20})^2, \]

\[ a_{21}(t) = \frac{2b_1b_2k_{20}^2e^{2\int h(t)dt}(k_{10} - k_{20})^2}{(k_{10} + k_{20})^2}, \quad a_{12}(t) = \frac{2b_1b_2k_{10}^2e^{2\int h(t)dt}(k_{10} - k_{20})^2}{(k_{10} + k_{20})^2}, \quad b_3 = \frac{b_2(b_1k_{10} - b_2k_{20})^2}{(k_{10} + k_{20})^2}, \]

\[ k_1(t) = k_{10}e^{\int h(t)dt}, \quad s_1(t) = \int k_{10}^3K_0(t)e^{3\int h(t)dt} + 4k_{10}K_1(t)e^{\int h(t)dt}]dt, \]

\[ k_2(t) = k_{20}e^{\int h(t)dt}, \quad s_2(t) = \int k_{20}^3K_0(t)e^{3\int h(t)dt} + 4k_{20}K_1(t)e^{\int h(t)dt}]dt, \]

from which we obtain 2-wave solution of eq. (1):

\[ u(x,t) = 2[\ln(1 + b_1e^{\xi_1} + b_2e^{\xi_2} + b_3e^{\xi_1 + \xi_2})]_{xx} \]  \hspace{1cm} \text{(8)}

Similarly, we can also determine the 3-wave solution of eq. (1):

\[ u(x,t) = 2[\ln(1 + b_1e^{\xi_1} + b_2e^{\xi_2} + b_3e^{\xi_1 + \xi_2} + b_4e^{\xi_1 + 2\xi_2} + b_5e^{\xi_1 + 3\xi_2})]_{xx} \]

\[ + b_3b_3e^{\xi_1 + 2\xi_2 + \xi_3} + b_4b_3e^{\xi_1 + \xi_2 + 2\xi_3} + b_5b_3e^{\xi_1 + 2\xi_2 + \xi_3} \]

\[ + b_3b_5e^{\xi_1 + 3\xi_2 + \xi_3} + b_4b_5e^{\xi_1 + 2\xi_2 + 3\xi_3} + b_5b_5e^{\xi_1 + 2\xi_2 + \xi_3}]_{xx} \]  \hspace{1cm} \text{(9)}

where \( \xi_i = xk_{i0}e^{\int h(t)dt} - \int[k_{i0}^3K_0(t)e^{3\int h(t)dt} + 4k_{i0}K_1(t)e^{\int h(t)dt}]dt + w_i, \) \( b_i, k_{i0}, \) and \( w_i \) are arbitrary constants, \( i = 1, 2, 3, \) and \( e^{\xi_i} = \frac{(k_{i0} - k_{i0})^2}{(k_{i0} + k_{i0})^2}(1 \leq i < j \leq 3). \)

By analyzing the obtained solutions (6), (8) and (9), we can conclude with a uniform formula of \( N \)-wave solution for any \( N > 1 \) of eq. (1) as:

\[ u(x,t) = 2[\ln(\sum_{\mu=0}^{N-1} \prod_{i=1}^{N} \left( b_i k_{i0} e^{\xi_i} + \sum_{i<j<\mu} b_i b_j \right))]_{xx} \]  \hspace{1cm} \text{(10)}
where \( \tilde{z}_i = x_{i0} \int h(t) dt - \int [k_{i0} K_0(t) e^{\int h(t) dt}]^2 + 4k_{i0} K_1(t) e^{\int h(t) dt}] dt + w_i \), \( b_i, k_{i0}, w_i \) are arbitrary constants, the sum \( \sum_{\mu=0,1} \) refers to all combinations of each \( \mu = 0.1 \) for \( i = 1, 2, ..., N \), and
\[
e^{B_i} = (k_{i0} - k_{j0})^2/(k_{i0} + k_{j0})^2 (1 \leq i < j \leq N).
\]

Conclusions

In this paper, multi-wave solutions of the nonisospectral KdV-type equation with variable coefficients have successfully been obtained, from which the uniform formula of \( N \)-wave solution is derived. It is due to the devised new and more general ansatz (4). The paper shows that the Exp-function method combined with appropriate ansatz may provide us with a straightforward, effective and alternative method for constructing multi-wave solutions or testing their existence and can be extended to other NLEE with variable coefficients.

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References


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