Initial boundary value problem for fractal heat equation in the semi-infinite region by Yang-Laplace transform

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Abstract
Analytical solution of transient heat conduction through a semi-infinite fractal medium is developed. The solution focuses on application of a local fractional derivative operator to model the heat transfer process and a solution through the Yang-Laplace transform.

Keywords:
Initial boundary value problem, heat equation, fractal media, Yang-Laplace transform.

1 Introduction

Boundary value problems in heat conduction [1-4] are ever attractive due to their significant practical and academic aspects and commonly solved by numerical [5] or analytical [6] techniques such as variational iteration methods [7], Adomian's decomposition method [8] and the homotopy analysis method [9].

In case of fractional difference models of heat conduction [10-14] describing anomalous transport of thermal energy [15] in fractal media, the boundary value problems are described by fractional diffusion equations [16,17] solved numerically or analytically [18-21]. Moreover, especially in the case of heat conduction, this leads to non-differentiable transport problems [22-27] solved in a variety ways, among them: local fractional variation iteration method [22, 23], local fractional Fourier series method [26], local fractional Laplace variational iteration method [27], etc.

This communication addresses a solution of transient heat conduction problem through a semi-infinite fractal medium [26] and developed by the Yang-Laplace transform. The Yang-Laplace transform [27-29] was conceived to solve some differential equations expressed through local fractional derivatives.

2 The mathematical method

For seek of clarity of the explanation, the properties for Yang-Laplace transform are briefly outlined. The Yang-Laplace transform is defined as [27-29]

\[ L_\alpha \{ f(x) \} = f_\alpha^L (s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha \left( -s^\alpha x^\alpha \right) f(x)(dx)^\alpha, \ 0 < \alpha \leq 1 \]  (1)

and its inverse formula is defined as [27-29]
\[ f(x) = L^{-1}_α \{ f^{L,α}(s) \} = \frac{1}{(2\pi)^α} \int_{β→∞}^{β→α} E_α \left( s^α x^α \right) f^L_α(s)(ds)^α, \]  
(2)

where \( f(x) \) is local fractional continuous, \( s^α = β^α + i^α x^α \) and \( \text{Re}(s^α) = β^α \).

The following properties for Yang-Laplace transform are valid [28]:
\[ L_α \{ f^{(α)}(x) \} = s^α L_α \{ f(x) \} - f(0), \quad L_α \{ x^{α} \} = \frac{Γ(1+ka)}{s^{(k+1)a}}, \quad L_α \{ f(ax) \} = \frac{1}{a^α} f^{L,α}_α\left( \frac{s}{a} \right), a > 0. \]
(3a,b,c)

For the more details of local fractional derivatives and integrals, see [26, 28].

### 3 Fractal heat equation in the semi-infinite region and its solution

The first law of thermodynamic states in fractal media reads as [26]
\[ \frac{1}{Γ(1+α)} \int_0^T \left[ \int_{v(0)}^{v(T)} \int_{t(0)}^{t(T)} \left( K^{2α} \frac{∂^α u}{∂t^α} + g - ρ_α c_α \frac{∂^α u}{∂t^α} \right) dV^{(r)} \right] dt^{α} = 0, \]
(4a)
which leads to the following equation [25, 26]
\[ K^{2α} \frac{∂^α u}{∂t^α} + g - ρ_α c_α \frac{∂^α u}{∂t^α} = 0, \]
(4b)
where the volume integral is the local fractional volume integral [26]. When the fractal dimension is equal to 1, eq. (4b) becomes the classical Fourier equation.

Making use of \( g = 0 \) and \( ρ_α c_α = K^{2α} \), the one-dimensional heat conduction through a semi-infinite fractal medium is modelled by [26, 27, 30]
\[ \frac{∂^α u(x,t)}{∂t^α} - \frac{∂^{2α} u(x,t)}{∂x^{2α}} = 0, \quad x > 0, \quad t > 0. \]
(5a)
\[ u(x, 0) = 0, u(0,t) = u_0. \quad (5b,c) \]

With the Yang-Laplace transform, the model (5a,b,c) can be transformed into
\[ s^α u(x) - (u, 0) - \frac{∂^{2α} u(x)}{∂x^{2α}} = 0, \quad u(0,s) = \frac{u_0}{s^α}. \]
(6a,b)
Then, from Eq.(6a), we get
\[ s^α u(x,s) - \frac{∂^{2α} u(x,s)}{∂x^{2α}} = 0, \]
(7)
where the initial value condition is Eq.(6b).

The general solution of Eq.(7) can be expressed in the form
\[ u(x,s) = AE_α \left( \frac{α}{s^2 x^α} \right) + BE_α \left( -\frac{α}{s^2 x^α} \right), \]
(8)
In the expression (8) the pre-factors $A$ and $B$ are constants. However, taking into account that the temperature function is bounded, we get $A = 0$. Hence, from Eq.(8) we have

$$u(x,s) = \frac{u_0}{s^\alpha} E_\alpha \left(-\frac{a}{s^2} x^\alpha \right).$$  \hspace{1cm} (10)

Now, taking into account the transforms

$$L_\alpha \left\{ t^{2\alpha} f(t) \right\} = \frac{\partial^{2\alpha} T(s)}{\partial s^{2\alpha}}, \quad L_\alpha \left\{ t^{-\alpha} f(t) \right\} = \frac{\partial^{\alpha} T(s)}{\partial s^\alpha} \quad \text{and} \quad L_\alpha \left\{ f(t) \right\} = T(s) = E_\alpha \left(-\frac{a}{s^2} \right).$$

We may obtain

$$\frac{\partial^{\alpha} f(t)}{\partial t^\alpha} + mf(t) = 0, \quad m = \frac{4^\alpha t^\alpha - 1}{4^\alpha t^{2\alpha}}, \quad \mu = \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} - \frac{\Gamma(1-\alpha/2)}{\Gamma(1-3\alpha/2)}. \quad (12a,b,c)$$

Following (12d) we have

$$f(t) = \frac{1}{2^\alpha \Gamma \left(1-\frac{\alpha}{2} \right) t^{\frac{3\alpha}{2}}} E_\alpha \left(-\frac{\Gamma \left(1-\frac{5\alpha}{2} \right)}{4^\alpha \mu \Gamma \left(1-\frac{3\alpha}{2} \right)} \times \frac{1}{t^\alpha} \right).$$ \hspace{1cm} (13)

Now, applying the Yang-Laplace transform we have

$$L_\alpha \left\{ E_\alpha \left(-\frac{a}{s^2} \right) \right\} = \frac{1}{2^\alpha \Gamma \left(1-\frac{\alpha}{2} \right) t^{\frac{3\alpha}{2}}} E_\alpha \left(-\frac{\Gamma \left(1-\frac{5\alpha}{2} \right)}{4^\alpha \mu \Gamma \left(1-\frac{3\alpha}{2} \right)} \times \frac{1}{t^\alpha} \right)$$

and

$$L_\alpha \left\{ E_\alpha \left(-\frac{a}{s^2} x^\alpha \right) \right\} = \frac{1}{2^\alpha \Gamma \left(1-\frac{\alpha}{2} \right) \Gamma(1+\alpha)} \int_0^t \frac{1}{\tau^{\frac{3\alpha}{2}}} E_\alpha \left(-\frac{\Gamma \left(1-\frac{5\alpha}{2} \right)}{4^\alpha \mu \Gamma \left(1-\frac{3\alpha}{2} \right)} \times \frac{1}{\tau^\alpha} \right) \left(\tau^\alpha \right) \alpha. \quad (14b)$$

From Eq.(14b) we obtain the non-differentiable solution of Eq.(5a) in the form

$$u(x,t) = \frac{u_0 x^{2\alpha}}{2^\alpha \Gamma \left(1-\frac{\alpha}{2} \right) \Gamma(1+\alpha)} \times \int_0^t \frac{1}{\left(\frac{\tau}{x^\alpha} \right)^{\frac{3\alpha}{2}}} E_\alpha \left(\frac{1}{\tau} \right)^\alpha \left( \Gamma \left(1+2\alpha \right) \Gamma \left(1+\alpha \right) \Gamma \left(1-\frac{3\alpha}{2} \right) \right) \alpha. \quad (15)$$

The solution (15) is a fractal function in accordance with the local fractional continuity concept [26].
4 Conclusions

The Yang-Laplace transform was successfully applied to solve an initial boundary value problem for fractal heat equation in the semi-infinite region, with local fractional derivatives and non-differentiable conditions. The result differs from those developed in [10-14] due to differences in the employed fractal operators. The solution allows, when the fractal dimension is \( \alpha = \frac{\ln 2}{\ln 3} \) a solution on Cantor sets to be developed.

Nomenclature

\( c_\alpha \) is the specific heat of fractal material, \( Jkg^{-1} \)
\( g \) is energy generation term,
\( K^{2\alpha} \) the thermal conductivity of the fractal material, \( Wm^{-\alpha}K^{-1} \)
\( L_\alpha (f(x)) \)-Yang-Laplace transform of \( f(x) \)
\( L_\alpha^{-1}(f_s^{L\alpha}(s)) \)-inverse version of Yang-Laplace transform of \( f_s^{L\alpha}(s) \)
\( u(x,t) \) the temperature function,
\( K \)
\( x \)-space co-ordinate,
\( m \)
\( t \)-time,
\( \alpha \) - fractal dimensional order (dimensionless)
\( \rho_\alpha \) is the density, \( kgm^{-3} \)

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