The purpose of this paper is to extend the homotopy perturbation method to fractional heat transfer and porous media equations with the help of the Laplace transform. The fractional derivatives described in this paper are in the Caputo sense. The algorithm is demonstrated to be direct and straightforward, and can be used for many other non-linear fractional differential equations.

Key words: homotopy perturbation method, Laplace transform, Caputo derivative, heat transfer equation, porous media equation

Introduction

As the generalization of ordinary calculus, fractional calculus was firstly proposed by Leibniz [1]. But only in recent years have fractional differential equations been gained much attention due to their exact description and extensive applications in various of scientific fields from physics to biology, chemistry, and engineering etc. The investigation of solutions to fractional differential equations plays an important role in the study of mathematical physical phenomena. But we always encounter some difficulty in finding the exact analytical solutions of these problems. So some numerical methods have been developed to handle these equations such as the homotopy perturbation method [2-4], the homotopy analysis method [5-7], the variational iteration method [8-10], the Laplace decomposition method [11, 12] and so on. As Liu [13] pointed out that these methods have their deficiencies like the calculation of Adomian polynomials, the Lagrange multiplier, and huge computation work.

The purpose of this paper is to use the modified homotopy perturbation method [13-15], which unifies homotopy perturbation method and Laplace transform method, to solve the heat transfer and porous media equations. The method provides the solutions in a rapid convergent series which may lead to the solution in a closed form.

Caputo derivative and the modified homotopy perturbation method

The Caputo derivative of order $\alpha$ is defined by the formula [16]:

$$D_0^\alpha u(x,t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{u^{(n)}(x,t)}{(t - \tau)^{1+\alpha-n}} \, \mathrm{d} \tau, \quad (n - 1 < \alpha \leq n, n \in N)$$

(1)

where $\Gamma(\cdot)$ denotes the Gamma function.

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The important properties of the Caputo derivative that will be used in this paper are:

\[ D^\alpha_t t^\beta = \frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta - \alpha)} t^{\beta-\alpha} \]  
(2)

\[ D^\alpha_t e^0 = 0 \]  
(3)

The Laplace transform of the Caputo derivative is:

\[ L[D^\alpha_t u(x,t)] = s^\alpha u(x,s) - \sum_{i=0}^{n-1} u^{(i)}(x,0^+) s^{\alpha-1-i}, \quad (n - 1 < \alpha \leq n) \]  
(4)

where \( L[\cdot] \) is the Laplace transform and is defined by:

\[ L[u(t)] = \int_0^\infty u(t) e^{-st} dt \]  
(5)

We will outline the main steps of the modified homotopy perturbation method [13-15]. For a given time-fractional non-linear non-homogeneous partial differential equation of the form:

\[ D^\alpha_t u(x,t) = Ru(x,t) + Nu(x,t) + g(x,t) \]  
(6)

with initial conditions:

\[ u(x,0) = h(x) \]  
(7)

where \( g(x,t) \) is the non-homogeneous term, \( N \) – the non-linear differential operator, \( R \) – the linear differential operator, and \( D^\alpha_t u(x,t) \) – the Caputo fractional derivative.

Assuming that the solution of eq. (6) can be written as a power series in \( p \):

\[ u(x,t) = \sum_{n=0}^\infty p^n u_n(x,t) \]  
(8)

and the non-linear term \( Nu(x,t) \) can be decomposed:

\[ Nu(x,t) = \sum_{n=0}^\infty p^n H_n(u_0, u_1, \cdots, u_n) \]  
(9)

where \( H_n(u_0, u_1, \cdots, u_n) \) is He's polynomials [17] and can be computed by:

\[ H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ N \left( \sum_{i=0}^n p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \cdots \]  
(10)

To determine \( u_i \) explicitly, we take the following several steps.

**Step 1.** Taking Laplace transform on both sides of eq. (6), and by the property of the Laplace transform, we get:

\[ L[u(x,t)] = \frac{h(x)}{s} + \frac{1}{s^\alpha} L[Ru(x,t) + Nu(x,t)] + \frac{1}{s^\alpha} L[g(x,t)] \]  
(11)

**Step 2.** Operating the Laplace inverse on both sides of eq. (11) results:

\[ u(x,t) = \left\{ \frac{h(x)}{s} + \left[ \frac{1}{s^\alpha} L[g(x,t)] \right] \right\} + \left[ \frac{1}{s^\alpha} L[Ru(x,t) + Nu(x,t)] \right] \]  
(12)
Step 3. Substituting eqs. (8) and (9) into eq. (12), and by the modified homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n = \left\{ h(x) + L^{-1} \left[ \frac{1}{s^{\alpha}} L\left[ g(x,t) \right] \right] \right\} + p \left\{ L^{-1} \left[ \frac{1}{s^{\alpha}} L\left[ R \left( \sum_{n=0}^{\infty} p^n u_n \right) + \sum_{n=0}^{\infty} p^n H_n \right] \right] \right\} \quad (13)$$

Step 4. Equating the terms with identical powers of $p$, we get the approximations of $u_n$, $i = 0, 1, 2, \cdots$:

$$p^0 : u_0(x,t) = h(x) + L^{-1} \left[ \frac{1}{s^{\alpha}} L\left[ g(x,t) \right] \right], \quad p^1 : u_1(x,t) = L^{-1} \left[ \frac{1}{s^{\alpha}} L[Ru_0(x,t) + H_0] \right], \quad p^2 : u_2(x,t) = L^{-1} \left[ \frac{1}{s^{\alpha}} L[Ru_1(x,t) + H_1] \right], \cdots \quad (14)$$

Therefore, the numerical solution of eq. (6) is:

$$u = u_0 + u_1 + u_2 + \cdots \quad (15)$$

Solutions of fractional heat transfer and porous media equations

Example 1. Consider the following non-linear time-fractional heat transfer equation with cubic non-linearity [18, 19]:

$$D_t^\alpha u = u_{xx} - 2u^3 \quad (16)$$

with initial condition:

$$u(x,0) = \frac{1 + 2x}{x^3 + x + 1} \quad (17)$$

Taking Laplace transfer on both sides of eq. (16), we have:

$$L[u(x,t)] = \frac{u(x,0)}{s} + \frac{1}{s^{\alpha}} L[u_{xx}(x,t) - 2u^3(x,t)] \quad (18)$$

Operating with the Laplace inverse on both sides of eq. (18) results:

$$u(x,t) = u(x,0) + L^{-1} \left[ \frac{1}{s^{\alpha}} L[u_{xx}(x,t) - 2u^3(x,t)] \right] \quad (19)$$

Substituting eqs. (8) and (9) into eq. (19) and applying the modified homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n = u(x,0) + p \left\{ L^{-1} \left[ \frac{1}{s^{\alpha}} L\left( \sum_{n=0}^{\infty} p^n u_{xx} - \sum_{n=0}^{\infty} p^n H_n \right) \right] \right\} \quad (20)$$

where $H_n$ is He's polynomial that represents the non-linear term $2u^3$, and the first few terms of $H_n$ are given by:

$$H_0 = 2u_0^3, \quad H_1 = 6u_0^2u_1, \quad H_2 = 6u_0u_1^2 + 6u_0^2u_2, \cdots \quad (21)$$

Comparing the coefficient of powers of $p$ in eq. (20), we get recursive relations:
\begin{align*}
\nonumber u_0 &= u(x,0), \quad u_n = L^{-1}\left\{ \frac{1}{s^{\alpha}}L[u_{(n-1)xx} - H_n] \right\}, \quad n \geq 1. \\
\text{(22)}
\end{align*}

Therefore, we have:

\begin{align*}
\nonumber u_0 &= \frac{1 + 2x}{x^2 + x + 1} \\
\nonumber u_1 &= L^{-1}\left[ \frac{1}{s^{\alpha}}L(u_{0xx} - 2u_0^3) \right] = \frac{-6(1 + 2x)}{(x^2 + x + 1)^2} \frac{t^\alpha}{\Gamma(1 + \alpha)} \\
\nonumber u_2 &= L^{-1}\left[ \frac{1}{s^{\alpha}}L(u_{1xx} - 6u_0^2u_1) \right] = \frac{72(1 + 2x)}{(x^2 + x + 1)^3} \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} \\
\nonumber u_3 &= L^{-1}\left[ \frac{1}{s^{\alpha}}L(u_{2xx} - 6u_0u_1^2 - 6u_0^2u_2) \right] = \\
\nonumber &= \left[ -1296(1 + 2x) + 432(1 + 2x)^3 \right] \frac{t^{3\alpha}}{(x^2 + x + 1)^4 (x^2 + x + 1)^5} - 216(1 + 2x)^3 \Gamma(1 + 2\alpha) \Gamma(1 + 3\alpha) \\
\text{(23)}
\end{align*}

and so on. In the same manner the rest of components of the solution can be obtained. The solutions of eqs. (16) and (17) are:

\begin{align*}
\nonumber u(x,t) &= \frac{1 + 2x}{x^2 + x + 1} - \frac{6(1 + 2x)}{(x^2 + x + 1)^2} \frac{t^\alpha}{\Gamma(1 + \alpha)} + \frac{72(1 + 2x)}{(x^2 + x + 1)^3} \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \\
&\quad \left[ -1296(1 + 2x) + 432(1 + 2x)^3 \right] \frac{t^{3\alpha}}{(x^2 + x + 1)^4 (x^2 + x + 1)^5} - 216(1 + 2x)^3 \Gamma(1 + 2\alpha) \Gamma(1 + 3\alpha) + \cdots \\
\text{(24)}
\end{align*}

If \( \alpha = 1 \), then eq. (24) can be reduced to the formula:

\begin{align*}
\nonumber u(x,t) &= \frac{1 + 2x}{x^2 + x + 1} - \frac{6(1 + 2x)}{(x^2 + x + 1)^2} t + \frac{36(1 + 2x)}{(x^2 + x + 1)^3} t^2 - \frac{216(1 + 2x)}{(x^2 + x + 1)^4} t^3 + \cdots \\
\text{(25)}
\end{align*}

which is exactly the same as the result given by [18].

Example 2. Consider the following non-linear porous media equation [20]:

\begin{align*}
\nonumber D_\alpha^\alpha u = D_x(uD_xu) \\
\text{(26)}
\end{align*}

with initial condition:

\begin{align*}
\nonumber u(x,0) = x \\
\text{(27)}
\end{align*}

Taking Laplace transfer on both sides of eq. (26), we have:

\begin{align*}
\nonumber L[u(x,t)] = \frac{u(x,0)}{s} + \frac{1}{s^{\alpha}}L\left[ D_x[u(x,t)D_xu(x,t)] \right] \\
\text{(28)}
\end{align*}

Operating with the Laplace inverse on both sides of eq. (28) results:

\begin{align*}
\nonumber u(x,t) &= u(x,0) + L^{-1}\left\{ \frac{1}{s^{\alpha}}L\left[ D_x[u(x,t)D_xu(x,t)] \right] \right\} \\
\text{(29)}
\end{align*}
Substituting eqs. (8) and (9) into eq. (29) and applying the modified homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n = u(x, 0) + p \left\{ L^{-1} \left[ \frac{1}{s^\alpha} L \left[ D_\alpha (\sum_{n=0}^{\infty} p^n H_n) \right] \right] \right\}$$

(30)

where $H_n$ is He's polynomial that displays the non-linear term $u(x,t)D_\alpha u(x,t)$, and the first few terms of $H_n$ are given by:

$$H_0 = u_0 u_{0x}, \quad H_1 = u_0 u_{1x} + u_1 u_{0x}, \quad H_2 = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} \ldots$$

(31)

Comparing the coefficient of powers of $p$, we get recursive relations:

$$u_0 = u(x, 0)$$

$$u_n = L^{-1} \left\{ \frac{1}{s^\alpha} L \left[ D_\alpha (H_{n-1}) \right] \right\}, \quad n \geq 1$$

(32)

Therefore

$$u_0 = x, \quad u_1 = L^{-1} \left\{ \frac{1}{s^\alpha} L \left[ D_\alpha (H_0) \right] \right\} = \frac{t^\alpha}{\Gamma(1 + \alpha)}$$

$$u_2 = L^{-1} \left\{ \frac{1}{s^\alpha} L \left[ D_\alpha (H_1) \right] \right\} = 0, \quad u_3 = L^{-1} \left\{ \frac{1}{s^\alpha} L \left[ D_\alpha (H_2) \right] \right\} = 0 \ldots$$

(33)

and so on. In the same manner the rest of components of the solution can be obtained. So the solution of eqs. (26) and (27) are:

$$u = x + \frac{t^\alpha}{\Gamma(1 + \alpha)}$$

(34)

If $\alpha = 1$, then eq. (34) can be reduced to the formula:

$$u(x, t) = x + t$$

(35)

which is exactly the same as the result given by [20].

**Conclusions**

In this paper, by applying the modified homotopy perturbation method, the approximate solutions of non-linear fractional heat transfer and porous media equations with initial conditions have successfully been obtained. The examples can demonstrate that the method provides an efficient, direct, and straightforward way for us in solving non-linear fractional differential equations.

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**References**


