PERIODIC SOLUTION TO GENERAL CONDUCTION PROBLEMS

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In this paper, we present a modified exp-function method, where hyperbolic cosine and cosine functions are used. The hyperbolic cosine functions are responsible for energy localization while cosine functions reveal the periodic effect. A general conduction problem is used as an example to illustrate the solution process.

Key words: non-linear equation, exp-function method, solitary solution

Introduction

Many kinds of soliton equations have been discovered up to now, for examples, non-linear Schrodinger equation, KdV equation, Sine-Gordon equation, and others. All of these equations can be transformed into bilinear forms by some special transformations [1] including the rational transformation, the logarithmic transformation, and the bi-logarithmic transformation. Once we get the bilinear forms of these equations, we can directly construct N-soliton solutions following the Hirota’s basic assumptions. Furthermore, bilinear forms have some special intrinsic properties, which can bring us some free considerations. Own to these bilinear forms, Lou [2, 3] constructed many localized structure by the variable separation method, Hirota [1] obtained determinants and pfaffians solutions. Recently, Dai et al. [4] proposed the three-wave method for non-linear evolution equations (NEE), and He and Wu suggested the exp-function method for solitary solutions [5, 6]. Review on various methods is available in [7-9]. In this paper, we will suggest a modification of the exp-function method.

Consider a (2+1) dimensional non-linear evolution equation of the general form:

\[ F(u, u_x, u_y, u_{xx}, u_{yy}) = 0 \]  \hspace{1cm} (1)

where \( F \) is a polynomial of \( u(x, y, t) \) and its derivatives.

We consider a bilinear equation of the form:

\[ G(D_t, D_x, D_y, \cdots) f \cdot f = 0 \]  \hspace{1cm} (2)

where \( G \) is a general polynomial in \( D_t, D_x, \) and \( D_y \), where the \( D \)-operator is defined by:

\[
D^n_t D^m_x F(x, y, t) \cdot G(x, y, t) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n F(x, y, t) G(x', y', t') \bigg|_{x'=x, y'=y, t'=t} \]

Traditionally, one obtains \( N \) soliton solutions using the assumption:

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According to the exp-function method [5-9], we assume that:

$$f = \sum_{i=1}^{m} a_i \left[ \exp(\xi_i) + \exp(-\xi_i) \right] + \sum_{j=1}^{n} b_j \left[ \exp(\eta_j) + \exp(-\eta_j) \right]$$

or equivalently:

$$f = 2 \sum_{i=1}^{m} a_i \cosh(\xi_i) + 2 \sum_{j=1}^{n} b_j \cos(\eta_j)$$

where $\xi_i = k_i x + l_i y + c_i t$ and $\eta_j = d_j x + e_j y + f_j t$.

In eq. (5), cosh functions are responsible for energy localization but trigonometric cos functions reveal periodic effect in real physical background.

**Application to (2+1) dimensional NLEE equation**

In this work, we study the following general conduction problem arising in fluid mechanics [5]:

$$u_{xxx} + 3u_x u_{xx} + 3u_x u_{xy} + 2u_{xt} = 0$$

Bekir [10] has studied its Painleve property. By the dependent variable transformation $u = 2(\ln \phi)_x$, then, eq. (6) is reduced to Hirota bilinear form:

$$(D_x D_t + D_x^3) \phi \cdot \phi = 0$$

One soliton solution is assumed to have the form:

$$\phi = 1 + e^{k x + l y + c t}$$

Inserting eq. (8) into eq. (7), and after simple calculation, we obtain:

$$u(x,t) = \frac{2k_1 e^{k x + l y - k t}}{1 + e^{k x + l y - k t}}$$

Two soliton solutions can be constructed by substituting:

$$\phi = 1 + e^{k_1 x + l_1 y + c_1 t} + e^{k_2 x + l_2 y + c_2 t} + a_{12} e^{k_1 x + l_1 y + c_1 t + k_2 x + l_2 y + c_2 t}$$

into eq. (7) and solving for the phase shift $a_{12}$, we find two-soliton solution in the form:

$$u(x,t) = \frac{2 \left( k_1 e^{k_1 x + l_1 y - k_1 t} + k_2 e^{k_2 x + l_2 y - k_2 t} + a_{12} e^{k_1 x + l_1 y - k_1 t + k_2 x + l_2 y - k_2 t} \right)}{1 + e^{k_1 x + l_1 y - k_1 t} + e^{k_2 x + l_2 y - k_2 t} + a_{12} e^{k_1 x + l_1 y - k_1 t + k_2 x + l_2 y - k_2 t}}$$

Alternatively, we assume that:

$$\phi = \cosh(k_1 x + l_1 y + c_1 t) + \cos(k_2 x + l_2 y + c_2 t) + a_3 \cosh(k_3 x + l_3 y + c_3 t)$$

(10)
Substituting eq. (10) into eq. (7), we have the following relations:

\[ c_1 = -k_3^2(-1 + 3l_3^2 - 6l_3^2a_3^2 + 3l_3^4a_3^4), \quad c_2 = -k_3l_3^2(1 - a_3^4)(l_3^2 - 2l_3^2a_3^2 + a_3^4l_3^2 - 3), \]

\[ c_3 = k_3^2(-1 + 3l_3^2 - 6l_3^2a_3^2 + 3l_3^4a_3^4), \quad k_1 = -k_3, \quad k_2 = l_3k_3(1 - a_3^4), \quad l_2 = 1, l_1 = l_3 \]

where \( l_3 \), \( a_3 \), and \( k_3 \) are free parameters. This case leads to a breath-kink solitary solution:

\[
\begin{align*}
    u(x,t) &= \frac{2[k_1 \sin(k_1x + l_3y + c_1t) - k_3 \sin(k_3x + l_3y + c_2t) + a_3k_3 \sin(k_3x + l_3y + c_2t)]}{\cosh(k_1x + l_3y + c_1t) + \cos(k_2x + l_3y + c_2t) + a_3 \cosh(k_3x + l_3y + c_2t)}
\end{align*}
\]

This solution shows periodic breathing resulting from cosine function in above expressions.

**Conclusions**

Generally, \( N \)-soliton solution of non-linear evolution equation can be obtained by a similar manner illustrated. In this article, by the modified exp-function method, we obtain various solutions including the multiple kink solution and the breath-kink solitary solution. The method is proved to be an effective method to construct new exact solutions of non-linear evolution equation.

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**References**


