A conjugate (coupled) forced convection heat transfer from a heated conducting plate under turbulent boundary layer flow is considered. A heated plate of finite thickness is cooled under turbulent forced convection boundary layer flow. Because the conduction and convection boundary layer flow is coupled (conjugated) in the problem, a semi-analytical solution based on Differential Transform Method (DTM) is presented for solving the non-linear integro-differential equation occurring in the problem. The main conclusion is that in the conjugate heat transfer case the temperature distribution of the plate is flatter than the one in the non-conjugate case. This feature is more pronounced under turbulent flow when compared with the laminar flow.

Keyword: Analytical solution, DTM, Conjugate, Coupled heat transfer

1. Introduction

When the temperature distribution of the solid body and that of the fluid flow are coupled, the resulting problem is known ‘conjugate heat transfer’. This problem, for the first time, was studied by Perelman [1] and Luikov et al. [2]. Perelman [1] used the method of the asymptotic solution to solve the integral equation occurring in the conjugate heat-transfer problem. A generalized Fourier sine transform was presented by Luikov et al. [2] for the semi-infinite plate. Trevino et al. [3] modeled the external heating of a flat plate cooled under a convective laminar flow with and accounted for the axial heat conduction in the plate by solving the integro-differential equation using perturbation. Determination of plate temperature in case of combined conduction, convection and radiation heat exchange is carried out by Sohal and Howel [4]. Later, Karvinen [5] presented an approximate method is presented by for calculating heat transfer from a flat plate in forced flow and compared the results with experimental data and previous results obtained in [4] for the case of combined convective heat exchange with the environment, conduction in the plate and internal heat sources. Forced convection conjugate heat transfer in a laminar plane wall jet was considered by Kanna and Das [6]. A problem of conduction-convection in fins [7, 8] and in cavities [9, 10] and the combined effect of conduction and radiation in a T-Y shaped fin [11] are carried out in recent years. Applying the superposition principle and Green’s function, Nagasue [12] carried out a solution for conjugate heat transfer in a circular tube. Chiu et al. [13] considered the conjugate heat transfer in a horizontal heated channel experimentally and numerically. Transient conjugated heat transfer to laminar flow in a tube or channel was studied by Karvinen [14]. Numerical and asymptotic solution was conducted by Escandon et al. [15] to solve a steady-state problem of a conjugate heat transfer in
an electro-Osmotic fully developed laminar flow. Optimization under the platform of conjugate heat transfer has also been studied [16-18]. Recently, Lindstedt and Karvinen [19] carried out a detailed analysis on conjugate heat transfer in a plate that one surface was kept at constant temperature and the other cooled by forced or natural convection. Interested readers can refer to the perfect review of Dorfman and Renner [20] and other papers [21-27] for acquiring more details about the history and characteristics of conjugate heat transfer problems.

In the recent years, semi-analytical treatments are used to solve the complex equations occurring in heat transfer and fluid mechanic studied. The most common treatments are the homotopy analysis method (HAM) [28,29], homotopy perturbation method (HPM) [30, 31], variational iteration method (VIM) [32] and the differential transform method (DTM) [33-37]. DTM was first proposed by Zhou [33]. DTM requires no symbolic computations of the necessary derivatives of the data functions, unlike traditional Taylor series method. The purpose of the present work is present a new algorithm for solving the integro-differential equation occurring in a turbulent conjugate heat transfer problems. To do this, a turbulent forced convection conjugate heat transfer from a good conducting plate is modeled. Because turbulent flow usually occurs in practical cooling systems cooled by forced convection flow, the present work may be of sufficient interests as well as the interesting research recently carried out by Hajmohammadi et al. [35], who assumed laminar forced convection flow for the studied problem.

2. Mathematical description of the problem

Shown in Fig. 1 is a schematic of a conjugate force convection flow over a conducting solid plate. An incompressible fluid is flowing over the upper surface of the plate. The flow is assumed to be turbulent, steady and two-dimensional in x-y plane with the free stream of velocity, \( U_\infty \), and free stream temperature, \( T_\infty \). The lower surface of the plate is heated with uniform and constant heat flux, \( q'' \). The left and the right sides of the rectangular plate are insulated whereas a constant heat flux is traveling through the bottom side toward the top of the plate. The upper side of the solid is in direct contact with the flowing fluid and heat transfer to the fluid occurs via this surface. For the sake of simplicity, the effects of buoyancy and viscous dissipation are neglected. The physical properties of the fluid and the solid are assumed to be uniform, isotropic and constant. Given the axial heat conduction occurring in a plate of finite thickness, \( b \), the heat flux distribution at the interface surface (upper surface of the plate), \( q''_s \), differs with that at the bottom, \( q'' \) and may be correlated to the temperature distribution of the interface surface, \( T_s \), by the following classical expression [38].

\[
q''_s(x) = 0.0287 \frac{k_f}{x} Pr^{0.0} Re_x^{0.85} \left\{ \frac{d}{dx} \left[ \int_{0}^{x} \left( 1 - \left( \frac{\xi}{x} \right)^{9/10} \right)^{9/5} \frac{dT_s}{d\xi} d\xi \right] + \Delta T_0 \right\} \tag{1}
\]

The first term in Eq. (1) is related to the integration of the temperature variation along the plate and the second term corresponds to the temperature jump in the leading edge of the plate, \( \Delta T_0 \). \( x \) is the axial location measured from the leading edge of the plate and \( k_f \) is the thermal conductivity of fluid. Besides, \( Pr \) is the Prandtl number and the local Reynolds number, \( Re_x \) is defined as
To arrive at Eq. (1), it is assumed that the velocity profiles are polynomial, there is no gravity and no external force, and all thermo-physical properties are constant. However, it must be noted that in real practical systems, the properties may vary with temperature. Besides, Eq. (1) is valid for high Reynolds number and not for the transient situation. It is further assumed that the temperature of the plate is uniform in the transversal coordinate, \( y \), that is, \( q^* = \text{const} \).

Figure 1. Conjugate forced convection flow over a solid conducting plate.

\[
\frac{1}{Bi_b} = \frac{q_{\text{transverse conduction}}}{q_{\text{convection}}} \approx \frac{k_s l (\Delta T)}{hb} \approx \frac{k_s}{hb} \gg 1. \tag{3}
\]

where, Biot number, \( Bi_b \), is defined based on the height of the plate, \( b \).

Substituting the convective heat transfer coefficient, \( h \), from the classical relations [38], Eq. (3) is rewritten as,

\[
0.0287 \left( \frac{k_s}{k_f} \right) Re_i^{-0.8} Pr^{-0.6} \gg 1 \tag{4}
\]

where, \( l \) is the length of the plate and \( \lambda \) is the aspect ratio of the plate, the ratio of the height to the length of the plate, \( b/l \). Applying the energy balance in a control volume of width, \( dx \) and height, \( b \) (with uniform temperature along the transversal coordinate), the energy equation is obtained as

\[
q^* + k_{s,x} b \frac{\partial^2 T}{\partial x^2} = q_s^* (x) \tag{5}
\]

Where \( q_s^* \) is calculated from Eq. (1). To formulate the problem under study, it is natural to define the following dimensionless variables,

\[
(\tilde{\xi}, \tilde{x}) = \frac{(\xi, x)}{l}; \quad \theta = \frac{Pr^{0.6} Re_i^{0.8} (T_s - T_w)}{0.0287 q^* l/k_f} \tag{6}
\]
and also
\[ \sigma = 0.0287 \left( \frac{k_{s,n}}{k_f} \right) Pr^{-0.6} Re_{\text{f}}^{-0.8} \]
\[ \sigma_\lambda = \sigma \lambda \]  
(7)

In this study, the parameter, \( \sigma_\lambda \) is called the “axial conduction parameter” which represents the ratio of the ability of the plate to carry heat in the stream-wise direction to the ability of the fluid to carry heat from the plate. We also call a plate as “good conducting plate”, if in addition to meet the condition described in Eq. (4), the magnitude of the parameter \( \sigma_\lambda \) is sufficiently large and is defined as,
\[ \sigma_\lambda = 0.0287 \left( \frac{k_{s,n}}{k_f} \right) Pr^{-0.6} Re_{\text{f}}^{-0.8} \lambda \left( \frac{k_i h_l}{h l (\Delta T)} \right) \approx \frac{Q_{\text{axial conduction}}}{Q_{\text{convection}}} = \frac{k_i}{h l} \lambda = \frac{1}{Bi_i} \lambda \gg 0. \]
(8)

Where the Biot number, \( Bi_i \), is defined based on the length of the plate, \( l \). We will later prove that the aforementioned power series solution derived for the plate temperature, is quickly convergent when the plate under consideration is a “good conducting” plate, \( \sigma_\lambda > 0 \). Utilizing Eqs. (1)-(9), the governing equation for dimensionless temperature distribution of the good conducting plate can be obtained as:
\[ \sigma_\lambda \frac{d^2 \theta}{dx^2} = -1 + \frac{\theta^0}{x^{0.2}} + \frac{1}{x^{0.2}} \int_0^x \left[ 1 - \left( \frac{\xi}{x} \right)^{10} \right]^{1/9} \frac{\partial \theta}{\partial \xi} d\xi \]  
(9)

with the boundary conditions,
\[ \frac{\partial \theta}{\partial \xi} \bigg|_{\xi=0} = 0 \]  
(11)
\[ \frac{\partial \theta}{\partial \xi} \bigg|_{\xi=1} = 0 \]  
(12)

Due to obvious reasons, the solution of non-linear, integro-differential equation (10) is not easily tractable by pure analytical means and, therefore, has to be obtained numerically or via a recursive semi-analytical method. In what follows, it is shown that the DTM is capable of providing a quick and highly accurate power series solution of Eq. (10) satisfying the boundary conditions (11) and (12).

### 3. Solution of the problem with Differential Transform Method

The basic definitions and the fundamental theorems of DTM and its applicability for various types of equations are given in [33-37]. For the convenience of the reader, in what follows, a brief review of the DTM is presented. The differential transform of a function \( f(x) \) is defined as:
\[ \tilde{f}(k) = \left. \left( \frac{1}{k!} \right) \frac{d^k f(x)}{dx^k} \right|_{x=x_0} = 0 \]  
(13)
where \( f(x) \) stands for the original function and \( \tilde{f}(k) \) is the differential transform function. The inverse differential transform of a sequence, \( \tilde{f}(k) \), is approximated as,

\[
f(x) \approx \sum_{k=0}^{N} \tilde{f}(k)(x-x_0)^k
\]

(14)

where, \( N \), is the highest order of the power series solution, namely, Eq. (14). The fundamental operations performed by DTM are listed in Table 1. Applying these fundamental operations to the given problem, a recurrence relation is obtained. Then solving this recurrence relation and using the differential inverse transform, we can obtain the solution of the problem.

As an initial step, we substitute Eq. (14) into the integral of Eq. (10). Owing that here \( x_0 = 0 \) (leading edge of the plate) and \( \theta^0 = \tilde{\theta}(0) \), we obtain,

\[
\sigma \frac{\partial^2 \theta}{\partial \hat{x}^2} = -1 + \tilde{\theta}(0) + \frac{1}{\hat{x}^{0.2}} \sum_{j=1}^{N} j \tilde{\theta}(j) \left[ 1 - \left( \frac{\xi}{\hat{x}} \right)^{9/10} \right] \hat{x}^{-1} d\xi
\]

(15)

Where \( \tilde{\theta} \) stands for the differential transform of function, \( \theta \). Using the variable transformation, \( u = 1 - \left( \frac{\tilde{\xi}}{\hat{x}} \right)^{9/10} \), Eq. (15) is reduced to the following expression,

\[
\sigma \frac{\partial^2 \theta}{\partial \hat{x}^2} = -1 + \frac{\tilde{\theta}(0)}{\hat{x}^{0.2}} + \frac{10}{9} \sum_{j=1}^{N} j \tilde{\theta}(j) B \left( \frac{8}{9}, \frac{10 j}{9} \right) \hat{x}^{-0.2}
\]

(16)

where \( B(m,n) \) is the numerical beta function defined as:

\[
B(m,n) = \int_0^1 u^{m-1} (1-u)^{n-1} \, du
\]

(17)

Applying the fundamental operations performed by DTM (listed in Table 1) to Eq. (16), the following relation is obtained,

\[
\text{Table 1. Fundamental operations of DTM.}
\]

<table>
<thead>
<tr>
<th>Original function</th>
<th>Transformed function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) = au(t) \pm bv(t) )</td>
<td>( \tilde{f}(k) = a\tilde{u}(k) \pm b\tilde{v}(k) )</td>
</tr>
<tr>
<td>( f(t) = u(t)v(t) )</td>
<td>( \tilde{f}(k) = \sum_{i} \tilde{u}(k_i)\tilde{v}(k - k_i) )</td>
</tr>
<tr>
<td>( f(x) = \frac{d^k u(x)}{dx^k} )</td>
<td>( \tilde{f}(k) = (k+1)(k+2)...(k+m)\tilde{u}(k+m) )</td>
</tr>
<tr>
<td>( f(x) = \int_{x_0}^{x} u(x) , dx )</td>
<td>( \tilde{f}(k) = \frac{\tilde{u}(k-1)}{k}, \quad k \geq 1 )</td>
</tr>
<tr>
<td>( f(x) = x^m )</td>
<td>( \tilde{f}(k) = \delta(k-m), \quad \delta(k-m) = \begin{cases} 1 &amp; k = m \ 0 &amp; k \neq m \end{cases} )</td>
</tr>
<tr>
<td>( f(x) = \exp(x) )</td>
<td>( \tilde{f}(k) = \frac{\lambda^k}{k!} )</td>
</tr>
</tbody>
</table>
\[
\sigma_\lambda (k+1)(k+2)\tilde{\theta}(k+2) = -\delta(k) + \\
\tilde{\theta}(0)\delta(k+1/5) + \frac{10}{9} \sum_{j=1}^{N} j \tilde{\theta}(j) B \left( \frac{8}{9} \cdot \frac{10j}{9} \right) \delta(k-j+1/5)
\]

Rearranging and simplifying Eq. (18), gives the following recurrence relations,

\[
\tilde{\theta}(9/5) = \frac{25}{36\sigma_\lambda} \tilde{\theta}(0)
\]

\[
\tilde{\theta}(2) = -\frac{1}{2\sigma_\lambda}
\]

\[
\tilde{\theta}(k) = \frac{10}{9\sigma_\lambda k(k-1)} B \left( \frac{8}{9} \cdot \frac{10(k-9/5)}{9} \right) \left( k - \frac{9}{5} \right) \tilde{\theta} \left( k - \frac{9}{5} \right) \quad k \geq 18/5
\]

The differential transform of the boundary conditions, Eqs. (11) and (12), can easily be obtained as,

\[
\tilde{\theta}(1) = 0
\]

\[
\sum_{j=1}^{N} j \tilde{\theta}(j) = 1
\]

The numerical value of constant \( \tilde{\theta}(0) \) is determined by substituting the given number of series solution components up to \( N \) into relation (23) and solving the resulting equation for \( \tilde{\theta}(0) \) when the value of \( \sigma_\lambda \) is known. Substituting the numerical value of \( \tilde{\theta}(0) \) in power series solution coefficients, and utilizing the inverse differential transform, Eq. (14), we may construct the following power series solution up to \( N = 8 \) for the plate non-dimensional temperature, when \( \sigma_\lambda = 1 \) and \( \lambda = 0.1 \), as followings,

\[
\theta(\hat{x}) = 0.7719409502 + 0.5360701043 \hat{x}^{9/5} - 0.5\hat{x}^2 + 0.06822159130 \hat{x}^{18/5}
\]

\[
- 0.0565135628 \hat{x}^{19/5} + 0.00365322472 \hat{x}^{27/5} - 0.002806422020 \hat{x}^{28/5}
\]

\[
+ 0.000108500 \hat{x}^{36/5} - 0.00007859278 \hat{x}^{37/5} + 0.000002058733474 \hat{x}^9
\]

The local Nusselt number of the flow can be determined from,

\[
Nu(\hat{x}) = \frac{h \hat{x}}{k_f} = \frac{1}{0.0287 Pr_{Re}^{0.6} \lambda^{0.8} \hat{x}^{0.8}} \tilde{\theta}(0) + \frac{10}{9} \sum_{k=9/5}^{N} k \tilde{\theta}(k) B(8/9,10k/9) \hat{x}^k
\]

Substituting Eq. (31) into Eq. (32), gives the local Nusselt number as:

\[
Nu(\hat{x}) = \frac{1}{0.0287 Pr_{Re}^{0.6} \lambda^{0.8} \hat{x}^{0.8}} \tilde{\theta}(0) + \frac{10}{9} \sum_{k=9/5}^{N} k \tilde{\theta}(k) B(8/9,10k/9) \hat{x}^k
\]

To characterize the convergence trend of the present series solution it is clear that each term in the series solution should be less than the next subsequent term, meaning that;
\( \tilde{\theta}(k) < \tilde{\theta}(k - 3/2) \)  \hspace{1cm} (25)

which in the view of Eq. (23) it gives the following relation,

\[
\frac{4}{3\sigma_z k(k-1)} B\left(\frac{2}{3}, \frac{4k-3}{3}; \frac{k-3}{2}\right) < 1
\]

Equation (26) demonstrates that; in order to have a reasonable and fast convergence of the presented series solution, the magnitude of \( \sigma_z \) must be sufficiently large, depending on the number of the terms contributing in the series solution.

4. Results and discussion

The effect of \( \sigma_z \) on the one-dimensional temperature distribution of the plate, \( \theta(\hat{x}) \), is shown in Fig. (2) when \( \lambda = 0.1 \). Figure (2) reveals that, increasing \( \sigma_z \) results in flattening the plate temperature distribution. As a consequence of high axial heat conduction in the plate, when \( \sigma_z = 50 \), the plate temperature is nearly uniform along the \( x \) coordinate. The variations of Nusselt number of the flow in the axial coordinate is plotted in Fig. (3) for several values of \( \sigma_z \). Based on this figure, the variations of Nusselt number of the flow is not sensitive with the variations of \( \sigma_z \), however, the Nusselt number in the conjugate laminar flow is affected by the variation of \( \sigma_z \) [15]. This feature is justified as: quite different in the laminar flow, the thermal boundary conditions of the plate has a negligible effect on the heat transfer coefficient and the Nusselt number, in the turbulent flow.

Fig. (4), show the comparison of the temperature distribution of plate for the case of conjugate convection heat transfer (\( \sigma_z = 1 \), flow over a plate of finite thickness) with those for the non-conjugate convection heat transfer (\( \sigma_z = 0 \), flow over a very thin plate). Figure (4) reveals that the temperature distribution in the conjugate case is flatter than temperature distribution at the non-conjugate case. The flattered temperature distribution in the conjugate case are compared in two cases; laminar forced convection and turbulent forced convection. It is observed that the effect of \( \sigma_z \), or in other words, the plate thickness, on flattering the temperature distribution is more pronounced under in the conjugate turbulent flow when compared with the temperature distribution in the conjugate laminar flow. This feature is extremely significant, because in the cooling systems, the objective is to spread heat as much as possible, such that the temperature distribution tends to be uniform at the entire devices.

5. Conclusion

Owing that turbulence often occurs in real practical systems, consideration is given to the conjugate heat transfer from a heated plate of finite thickness when the flow is supposed to be turbulent. The mathematical formulation of the problem derived in this study deals mainly with the one-dimensional variation of the plate temperature and Nusselt number of the flow. The integro-differential equation occurring in problem is solved by DTM.

As a significant physical conclusion of this study we conclude that the use of the conjugate heat transfer over a flat plate is higher than the non-conjugate case, because the temperature distribution is more uniform in the conjugate case. It is further concluded that the use of flat plate in reducing the hot spot
temperature of the plate is of greater advantage in turbulent case when compared with the laminar flow situation.

Acknowledgements
The author wishes to thank the reviewers for their careful, unbiased and constructive suggestions, which led to this revised manuscript.

![Figure 2](attachment:image.png)

Figure 2. Effect of \( \sigma_\lambda \) on the one-dimensional temperature distribution of the plate, \( \theta(\hat{x}) \) when \( \lambda = 0.1 \).

![Figure 3](attachment:image.png)

Figure 3. The variation of Nusselt number of the flow in the axial coordinate for several values of \( \sigma_\lambda \).
Figure 4. The comparison between the plate temperature distribution of the conjugate and non-conjugate convection heat transfer, $\sigma_\lambda = 1$ and $\sigma_\lambda = 0$ respectively.

Figure 5. The comparison between the plate temperature distribution of the plate cooled under laminar forced convection flow with the ones in the case of turbulent flow.
Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>Beta function</td>
</tr>
<tr>
<td>( b )</td>
<td>plate thickness</td>
</tr>
<tr>
<td>( h )</td>
<td>convective heat transfer coefficient</td>
</tr>
<tr>
<td>( k )</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>( l )</td>
<td>length</td>
</tr>
<tr>
<td>( N )</td>
<td>highest order of the series solution</td>
</tr>
<tr>
<td>( Pr )</td>
<td>Prandtl number</td>
</tr>
<tr>
<td>( q^* )</td>
<td>heat flux</td>
</tr>
<tr>
<td>( q^*_{uw} )</td>
<td>heat flux on the upper surface of the plate</td>
</tr>
<tr>
<td>( Re_x )</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>( T_s )</td>
<td>temperature distribution of the plate</td>
</tr>
<tr>
<td>( T )</td>
<td>local temperature</td>
</tr>
<tr>
<td>( T_\infty )</td>
<td>free stream temperature</td>
</tr>
<tr>
<td>( U_\infty )</td>
<td>Free stream velocity</td>
</tr>
</tbody>
</table>

\( x \) Cartesian coordinate

Greek symbols

\( \theta \) Dimensionless temperature, defined in Eq. (6)

\( \sigma \) Dimensionless parameter, defined in Eq. (7)

\( \sigma_\lambda \) dimensionless parameter, defined in Eq. (8)

\( \lambda \) plate aspect ratio \( (b/l) \)

\( \delta \) defined in table 1.

\( \nu \) kinematic viscosity

\( \nu \) fluid

\( \alpha \) scripts

\( \bar{u} \) heat flux on the upper surface of the plate

\( \bar{T} \) temperature distribution of the plate

(\( \wedge \)) Dimensionless variables defined in Eq. (6)

(\( \sim \)) Differential Transformed variables

References


