OSCILLATION CRITERIA FOR HALF-LINEAR FUNCTION DIFFERENTIAL EQUATIONS WITH Damping

by

Jun LIU*, Hong-Ying LUO*, and Xi LIUb

a College of Mathematics and Information Science, Qujing Normal University, Qujing, China
b College of Information Science and Engineering, Yunnan University, Kunming, China

Original scientific paper
DOI: 10.2298/TSCI1405537L

In this paper, a class of half-linear functional differential equations with damping are studied. By using the generalized Riccati transformation and integral average skills, new oscillation criteria are obtained which generalize and improve some known results.

Key words: differential equation, oscillation criteria, half-linear

Introduction

A functional differential equation can describe a generalized heat problem [1]:

\[ \frac{d}{dt} \{ r(t) \phi[x^{(n-1)}(t)] \} + p(t) \phi[x^{(n-1)}(t)] + q(t) \phi[g(t)] = 0, \quad t \geq t_0 \]  

(1)

In this paper, we mainly study the oscillation criteria of a half-linear functional differential equation with damping:

\[ \{ r(t) \phi_{\alpha}[z^{(n-1)}(t)] \} + p(t) \phi_{\alpha}[z^{(n-1)}(t)] + q(t) \phi_{\beta}[x[\sigma(t)]] = 0, \quad t \geq t_0 \]  

(2)

where \( \phi_{\alpha}(s) = |s|^{-\alpha}, \quad \phi_{\beta}(s) = |s|^{-\beta}, \) \( \alpha \) and \( \beta \) are positive constants, and \( n \) is even number,

\[ z(t) = x(t) + f(t)x[\tau(t)] \]  

(3)

and the following conditions are assumed to hold:

\begin{align*}
(H_1) & \quad r(t) \in C^1(I, R), I = [t_0, \infty], \quad r(t) > 0, \quad r'(t) \geq 0 \\
(H_2) & \quad p(t), q(t), f(t) \in C(I, R), \quad p(t) \geq 0, \quad q(t) > 0, \quad 0 \leq f(t) \leq 1 \\
(H_3) & \quad \sigma(t) \in C^1(I, R^+), \quad 0 < \sigma(t) \leq t, \quad \lim_{t \to \infty} g(t) = \infty, \quad \sigma'(t) > 0 \\
(H_4) & \quad \tau(t) \in C^1(I, R), \quad \tau(t) \leq t, \quad \lim_{t \to \infty} \tau(t) = \infty
\end{align*}

Obviously, eq. (1) is a special case of the eq. (2). When \( f(t) = 0 \), eq. (2) is \( n \)-order half-linear functional differential equation with damping:
\[ \{r(t)\varphi(x^{(n-1)}(t))\} + p(t)\varphi(x^{(n-1)}(t)) + q(t)\varphi_x(x\{\sigma(t)\}) = 0, \quad t \geq t_0 \] 

(4)

when \( p(t) = 0 \), eq. (4) is non-damping differential equation.

Using the generalized Riccati transformation and integral average skills, we obtain new oscillation criteria of eq. (2) which generalize and improve some known results.

**Oscillation criteria**

Consider the following equation:

\[ x^{(n)}(t) + q(t)x(t) = 0, \quad q(t) > 0 \] 

(5)

**Theorem 1.** Let \( w(t) = x'(t)/x(t) > 0 \), then \( w'(t) = -q(t) - w^2(t) \), and eq. (5) is oscillatory.

**Proof.** Suppose that eq. (5) has a non-oscillatory positive solution \( x(t) > 0 \), then \( x''(t) < 0 \), that is \( x'(t) \) is decreasing function, therefore \( x'(t) > 0 \) or \( x'(t) < 0 \). Suppose \( x'(t) < 0 \), there exists a \( t_1 \geq t_0 \) such that \( x'(t) < 0 \), we have \( x'(t) \leq -\beta \) for \( t > t_1 \), 

\[
\int_{t_1}^{t} \frac{1}{r(s)} [\rho(s)Q(s) - r(s)] ds = \infty
\]

(6)

where \( Q(t) = q(t) [1 - p(\sigma(t))]^{\beta} \), then eq. (2) is oscillatory.

**Lemma [2].** Assume that:

\[ \theta > 0, \quad A > 0, \quad B \in R, \quad Bu - Au^\theta \leq \frac{\theta \theta^{\theta+1}}{(\theta+1)^{\theta+1}} A^\theta \]

**Theorem 2.** Assume that \((H_1) \sim (H_4)\) and \( \beta \geq \alpha > 0 \) hold, if for each \( t \geq t_0 \), there exist a function \( \rho(t) \in C^1[I, (0, \infty)] \), \( \rho(t) \geq 0 \) such that:

\[
\int_{t_0}^{t} \frac{1}{r(s)} [\rho(s)Q(s) - r(s)] ds = \infty
\]

\[
\int_{t_0}^{t} \frac{1}{r(s)} [\rho(s)Q(s) - r(s)] ds = \infty
\]

(7)

Therefore \( r(t)z^{(n-1)}(t) \) is a non-increasing function and \( z'(t) \) is eventually of one sign. That is \( z'(t) > 0 \) or \( z'(t) \leq 0 \). Otherwise, if there exists a \( t_1 \geq t_0 \) such that \( z'(t) < 0 \) for \( t \geq t_1 \), 

\[
r(t)z^{(n-1)}(t) \leq -r(t)z^{(n-1)}(t) \leq K, \quad t \geq t_1
\]

that is:

\[
z^{(n-1)}(t) \leq -\frac{K}{r(t)} \sigma, \quad t \geq t_1
\]
Integrating the inequality from $t_1$ to $t$:
\[
z^{(n-2)}(t) \leq z^{(n-2)}(t_1) - K^2 \int_{t_1}^{t} \frac{1}{r(t)} \, ds
\]

Letting $t \to \infty$, we get $\lim_{t \to \infty} z^{(n-2)}(t) = -\infty$. Thus, we have $z^{(n-1)}(t) > 0$.

\[
x(t) = z(t) - f(t)x[r(t)] \geq [1 - f(t)]z(t)
\]

\[
p(t)\phi_\alpha[z^{(n-1)}(t)] = p(t)\left|z^{(n-1)}(t)\right|^\alpha - z^{(n-1)}(t) \geq 0
\]

\[
q(t)\phi_\beta[x[\sigma(t)]] = q(t)x[\sigma(t)]^{\beta-1} x[\sigma(t)] \geq q(t)\{1 - p[\sigma(t)]\}^\beta z^\beta[\sigma(t)] = Q(t)z^\beta[\sigma(t)]
\]

Therefore, we have:
\[
[r(t)z^{(n-1)}(t)^\alpha]^\prime + Q(t)z^\beta[\sigma(t)] \leq 0, \quad Q(t) \leq -\frac{r(t)[z^{(n-1)}(t)^\alpha]^\prime}{z^\beta[\sigma(t)]}
\]

Define:
\[
w(t) = p(t)\frac{r(t)[z^{(n-1)}(t)]^\alpha}{z^\beta[\sigma(t)]}
\]

Suppose $w(t) > 0$, we have:
\[
w'(t) = \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)\frac{\{r(t)[z^{(n-1)}(t)]^\alpha\}'}{z^\beta[\sigma(t)]} - \rho(t)r(t)[z^{(n-1)}(t)]^\alpha \frac{\beta z^{\beta-1}[\sigma(t)]z'[\sigma(t)]\sigma'(t)}{z^\beta[\sigma(t)]} \leq
\]
\[
\leq -\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha\sigma'(t)}{[\rho(t)r(t)]^{\beta}}[w(t)]^{\alpha+1} \quad (9)
\]

From Lemma and (9), we have:
\[
w'(t) \leq -\rho(t)Q(t) + \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha+1}} \left[\frac{\rho'(t)}{\rho(t)}\right]^{\alpha+1} \frac{\rho(t)r(t)}{\alpha^\alpha[\sigma'(t)]^\alpha}
\]

\[
w(t) \leq w(t_1) - \int_{t_1}^{t} \left[\rho(s)Q(s) - \frac{\rho'(s)}{\alpha + 1}\left[\frac{\rho'(s)}{[\rho(s)]^{\alpha+1}}\right]\int_{t_1}^{\infty} s^{\alpha+1} \frac{r(s)}{[\rho(s)]^{\alpha+1}} ds\right] \, ds
\]

Letting $t \to \infty$, we get $\lim_{t \to \infty} w(t) = -\infty$, which contradicts $w(t) > 0$, then eq. (2) has not a positive solution. Similarly, then eq. (2) has not a negative solution. Therefore, eq. (2) is oscillatory.

Corollary 1. Assume that:
\[
\beta \geq \alpha > 0, \quad \int_{t_1}^{\infty} \rho(s)Q(s) ds = \infty, \quad \int_{t_1}^{\infty} s^{\alpha+1} \frac{r(s)}{[\rho(s)]^{\alpha}} ds = \infty
\]

then eq. (2) is oscillatory.
Corollary 2. Assume that $\beta \geq \alpha > 0$, and the following conditions are hold:

1. \[ \lim_{t \to \infty} \int_{t_0}^{t} \frac{1}{r(s)} \int_{s}^{\infty} Q(u)du \left( \frac{1}{\alpha} \right) ds = \infty \]

2. \[ \int_{t_0}^{\infty} \left[ \frac{\rho'(s)}{[\rho(s)]^{\alpha+1}} r(s) \right] ds = \infty \]

3. \[ \lim_{t \to \infty} \frac{[\rho(t)]^{\alpha+1}[\sigma'(t)]^{\beta}}{[\rho'(t)]^{\alpha+1} r(t)} Q(t) > \left( \frac{1}{\alpha+1} \right)^{\alpha+1} \]

then eq. (2) is oscillatory.

Proof. From conditions (3), we have:

\[ \rho(t)Q(t) - \left( \frac{1}{\alpha+1} \right)^{\alpha+1} \left[ \frac{\rho'(t)^{\alpha+1} r(t)}{[\rho(t)]^{\alpha+1} \sigma'(t)} \right] > \varepsilon \]

\[ \int_{t_0}^{t} \left[ \rho(s)Q(s) - \left( \frac{1}{\alpha+1} \right)^{\alpha+1} \left[ \frac{\rho'(s)^{\alpha+1} r(s)}{[\rho(s)]^{\alpha+1} \sigma'(s)} \right] \right] ds > \varepsilon \int_{t_0}^{t} \left[ \frac{[\rho'(s)]^{\alpha+1} r(s)}{[\rho(s)]^{\alpha+1} \sigma'(s)} \right] ds, \ t > t_0 \]

Letting $t \to \infty$, we get eq. (2) is oscillatory.

Theorem 3. Assume that $(H_1) \sim (H_4)$ and $\alpha \geq \beta > 0$ hold, if for each $t \geq t_0$, there exist a function $\rho(t) \in C^1[I, (0, \infty)], \rho'(t) \geq 0$ such that:

\[ \lim_{t \to \infty} \int_{t_0}^{t} \frac{1}{r(s)} \int_{s}^{\infty} Q(u)du \left( \frac{1}{\alpha} \right) ds = \infty, \ \lim_{t \to \infty} \int_{t_0}^{t} \left( \rho(s)Q(s) - \left[ \frac{\rho'(s)^{\beta+1} r(s)}{[\rho(s)]^{\beta+1} \sigma'(s)} \right] \frac{1}{b^{\frac{\beta+1}{\alpha}}} \right) ds = \infty \]

where $Q(t) = q(t)\{1 - p[a(t)]\}^{\beta}$, $0 < b \leq z'(t)$, then eq. (2) is oscillatory.

Proof. Suppose that eq. (2) has a positive solution $x(t)$, Similarly (9), we have:

\[ w'(t) \leq -\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\beta b^{\frac{\beta+1}{\alpha}}}{[\rho(t)]^{\beta+1} \sigma'(t)} \left[ w(t) \right]^{\beta+1} \]

From Lemma, we have:

\[ w'(t) \leq -\rho(t)Q(t) + \frac{\beta b^{\frac{\beta+1}{\alpha}}}{[\rho(t)]^{\beta+1} \sigma'(t)} \left[ \frac{\rho'(t)^{\beta+1} r(t)}{[\rho(t)]^{\beta+1} \sigma'(t)} \right] \]
\[ w(t) \leq w(t_1) - \int_{t_1}^{t} \rho(s)Q(s) - \left( \rho'(s) \frac{r(s)}{[\rho(s)\sigma'(s)]^{\beta + 1}} \right) ds \]

Letting \( t \to \infty \), we get \( \lim_{t \to \infty} w(t) = -\infty \), which contradicts \( w(t) > 0 \), then eq. (2) has not a positive solution. Similarly, then eq. (2) has not a negative solution. Therefore, eq. (2) is oscillatory.

**Corollary 3.** Assume that:
\[ \alpha \geq \beta > 0, \quad \int_{t_0}^{\infty} \rho(s)Q(s)ds = \infty, \quad \int_{t_0}^{\infty} \frac{[\rho'(s)]^{\beta + 1}}{\rho(s)\sigma'(s)} \frac{r(s)}{b} ds = \infty \]
then eq. (2) is oscillatory.

**Corollary 4.** Assume that \( \alpha \geq \beta > 0 \), and the following conditions are hold:

1. \( \lim_{t \to \infty} \int_{t_0}^{t} \frac{1}{\int_{s}^{\infty} Q(u)du} \frac{1}{\beta} ds = \infty \)
2. \( \int_{t_0}^{\infty} \frac{[\rho'(s)]^{\beta + 1}}{\rho(s)\sigma'(s)} \frac{r(s)}{b} \frac{1}{\beta} ds = \infty \)
3. \( \lim_{t \to \infty} \frac{[\rho(t)]^{\beta + 1}}{[\rho(t)]^{\beta + 1} r(t) b} Q(t) > \left( \frac{1}{\beta + 1} \right)^{\beta + 1} \)
then eq. (2) is oscillatory.

**Proof.** From conditions (3), we have:
\[ \frac{[\rho(t)]^{\beta + 1}[\sigma'(t)]^{\beta}}{[\rho(t)]^{\beta + 1} r(t) b} Q(t) > \left( \frac{1}{\beta + 1} \right)^{\beta + 1} + \varepsilon, \quad \forall \varepsilon > 0 \]
that is:
\[ \rho(t)Q(t) - \left( \frac{1}{\beta + 1} \right)^{\beta + 1} \frac{[\rho'(t)]^{\beta + 1} r(t) b} {[\rho(t)\sigma'(t)]^{\beta}} > \varepsilon \frac{[\rho'(t)]^{\beta + 1} r(t) b} {[\rho(t)\sigma'(t)]^{\beta}} \]
\[ \int_{t_0}^{t} \rho(s)Q(s) - \left( \frac{1}{\beta + 1} \right)^{\beta + 1} \frac{[\rho(s)]^{\beta + 1} r(s) b} {[\rho(s)\sigma'(s)]^{\beta}} ds > \varepsilon \int_{t_0}^{t} \frac{[\rho'(s)]^{\beta + 1} r(s) b} {[\rho(s)\sigma'(s)]^{\beta}} ds, \quad t > t_0 \]

Letting \( t \to \infty \), we get eq. (2) is oscillatory.

**Acknowledgments**

The work was supported by Chinese Natural Science Foundation Grant No. 11361048, 11061028, Qujing Normal University Natural Science Foundation Grant No. 2012QN016 and Yunnan Province Natural Science Foundation Grant No. 2011FB090.
References