Abstract: In this study, we discussed the enhancement of thermal conductivity of elasto-viscous fluid filled with nanoparticles, due to the implementation of radiation and convective boundary condition. The flow is considered impinging obliquely in the region of oblique stagnation point on the stretching surface. The obtained governing partial differential equations are transformed into a system of ordinary differential equations by employing a suitable similarity transformation. The solution of the resulting equations is computed numerically using Chebyshev Spectral Newton Iterative Scheme (CSNIS). An excellent agreement with the results available in literature is obtained and shown through tables. The effects of involving parameters on the fluid flow and heat transfer are observed and shown through graphs. It is importantly noted that the larger values of Biot number imply the enhancement in heat transfer, thermal boundary layer thickness and concentration boundary layer thickness.

Keywords: Thermal conductivity, elasto-viscous fluid; oblique stagnation point; spectral method.

1. Introduction

Oblique stagnation-point flow appears when fluid from any source impinges obliquely on a rigid wall at an arbitrary angle of incidence as shown in Fig. 1. Many researchers have studied the steady two-dimensional oblique stagnation-point flow of a Newtonian fluid. Stuart [1] did the pioneer work in this field, later studied by Tamada [2] and Dorrepaal [3]. Recently, Reza and Gupta [4] generalized the problem of Chiam [5] by introducing a stretching surface. In their paper, they ignored the displacement thickness and pressure gradient. This was partially rectified in a paper by Lok et al. [6]. Very recently, Reza and Gupta [7] gave a correct solution to the above problem by fixing the errors in [4] and [6]. Drazin and Riley [8], Tooke and Blyth [9] reviewed the problem and included a free parameter associated with the shear flow component related to the pressure gradient. Weidman and Putkaradze [10, 11] studied the steady oblique stagnation-point flow impinging on a circular cylinder. The flow is described using a coupled set of ordinary differential equations. Recently, Erfani et al. [12], Husain et al. [13], Mahapatra et al. [14], Lok et al. [15] and Yajun and Liancun [16] and Javed et al. [17] have done notable work on orthogonal and oblique stagnation point flow.

In last few decades, heat transfer in nanofluids has become a topic of major interest. Many researchers contributed in this area due its significance in pharmaceutical and food processes, hyperthermia, fuel cells, microelectronics, hybrid-powered engines, coolants for advanced nuclear Power Plants [18] and many others. The basic idea of using nano-sized particle to enhance the thermal conductivity of the fluid was given by Maxwell [19]. Choi [20] was the first who introduce the term nanofluid in 1995. He studied the characteristics of nanofluids and deduced that the thermal conductivity of the base fluid (water, oil, bio-fluids, organic liquids, ethylene glycol etc.) can be enhanced by introducing metallic particles (average size about 10 nanometers). Nanoparticles are made of different metals (Al, Cu, Ag, Au, Fe), metal carbides (SiC) non-metals (graphite carbon nanotubes), oxides (Al₂O₃, CuO, TiO₂), nitrides (AlN, SiN) etc. In 2006, Buongiorno [21] has studied the convective transport in fluid and he considered seven slip mechanisms (thermophoresis, diffusiophoresis, Brownian diffusion, inertia, Magnus effect, gravity and fluid drainage) to discuss the relative velocity of the fluid and nano-particles and he concluded that among these seven slip mechanisms only two are important. Recently
Kuznetsov and Nield [22, 23] studied the Double-diffusive and natural convective boundary-layer flow of a nanofluid past a vertical plate, they found the analytical solution of the problems. Makinde and Aziz [24] studied the convective heat transfer in nanofluid past a stretching sheet and they discussed Brownian motion and thermophoresis effects in detail. There is extensive literature available on the topic through different aspects. Few representative recent studies on the topic may be seen in the refs. [25-38].

Literature survey witnessed that much attention in the past has been accorded to the flow of viscous nanofluids. However in real situation the base fluids in the nanomaterials is not viscous. No doubt, the base fluid in reality is viscoelastic. Mention may be made to some viscoelastic nanofluids like ethylene glycol-CuO, ethylene glycol-Al₂O₃, ethylene glycol-ZnO. Keeping such preference in view the viscoelastic nanofluid is considered in this paper. Many viscoelastic fluids models have been proposed but here constitutive equations of Walter-B fluid [39-41] are employed in the mathematical formulation. Our intention here is to compute the oblique stagnation point flow of viscoelastic nanofluid. To the best of our knowledge, such problem has not been attempted before. An efficient approach namely the CSNIS is implemented for the numerical solution. The graphical results are interpreted with respect to various parameters of interest. A comparison with the previously published results in limiting sense is given. Heat transfer rate and mass diffusion flux are also analyzed.

![Physical Model](image)

**Fig. 1: Physical Model**

### 2. Problem formulation

We consider the steady two-dimensional laminar flow of Walter-B nanofluid impinging obliquely on a stretching surface, which is placed at $\overline{y} = 0$ and the fluid occupies the upper half plane $\overline{y} > 0$ as shown in Fig. 1. The surface is heated convectively, by convective heating process, which is characterized by a temperature $T_f$ and a heat transfer coefficient $h_f$. We neglect the viscous dissipation to estimate accurately the effect of convective boundary condition because viscous dissipation would disturb the thermal boundary conditions. The velocity of the outer flow far away from the surface is $U_e(x, y) = ax + by$. The flow, energy and concentration equations are (see Beard and Walters [41])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{u}{\rho} \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{k}{\rho} \left[ \frac{\partial}{\partial x} \left( 2 \frac{\partial u}{\partial x} + 2 \frac{\partial v}{\partial y} + 4 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial v}{\partial x} \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \right) \right) + \right.$$

$$\left. \frac{\partial}{\partial y} \left[ \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right] \right] \right],$$

$$\frac{u}{\rho} \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{k}{\rho} \left[ \frac{\partial}{\partial x} \left( 2 \frac{\partial u}{\partial x} + 2 \frac{\partial v}{\partial y} + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \right.$$

$$\left. \frac{\partial}{\partial y} \left[ \frac{2 \partial u}{\partial x} \frac{\partial v}{\partial x} + 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + 4 \left( \frac{\partial v}{\partial x} \right)^2 + 2 \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y^2} \right] \right].$$
\[ \frac{\partial \bar{T}}{\partial x} + \frac{v}{\rho C_p} \left( \frac{\partial^3 \bar{T}}{\partial x^2} + \frac{\partial^3 \bar{T}}{\partial y^2} \right) - \frac{1}{\rho C_p} \frac{\partial q}{\partial y} + \tau \left[ D_b \left( \frac{\partial^2 \bar{C}}{\partial x^2} + \frac{\partial^2 \bar{C}}{\partial y^2} \right) + D_f \left( \frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} \right) \right] = \frac{\partial \bar{T}}{\partial x} + \frac{\partial \bar{T}}{\partial y} \frac{\partial \bar{T}}{\partial y}. \]  

(4)

\[ \frac{\partial \bar{C}}{\partial x} + \frac{v}{\rho C_p} \frac{\partial \bar{C}}{\partial y} = D_b \left( \frac{\partial^2 \bar{C}}{\partial x^2} + \frac{\partial^2 \bar{C}}{\partial y^2} \right) + D_f \left( \frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} \right). \]  

(5)

In the above equations, \( U(\bar{x}, \bar{y}) \) and \( V(\bar{x}, \bar{y}) \) are the velocity components in \( \bar{x} \) and \( \bar{y} \)-directions, \( \bar{C}(\bar{x}, \bar{y}) \) is the concentration, \( \bar{T}(\bar{x}, \bar{y}) \) is the temperature and \( \bar{p}(\bar{x}, \bar{y}) \) is the pressure of the fluid flow. Also \( \alpha \) is the thermal diffusivity, \( v \) is the kinematic viscosity, \( \rho \) is the density, \( k_o \) is elasticity of fluid, \( C_p \) is the specific heat, \( k \) is the thermal conductivity of the fluid and \( q \) is the radiative heat flux. \( D_f \) and \( D_b \) are the Brownian motion coefficient and thermophoretic diffusion coefficient respectively. \( \tau \) (=\((\rho C_p)/(\rho C T) \) ) is the ratio of effective heat capacity of nanoparticles materials to heat capacity of the fluid. The boundary conditions of the problem can be defined as

\[ \bar{y} = 0 : \quad \bar{u} = c \bar{x}, \quad \bar{v} = 0, -k \frac{\partial \bar{T}}{\partial \bar{y}} = h_f \left( \bar{T}_f - \bar{T} \right), \quad \bar{C} = C_w, \]  

(6)

\[ \bar{y} \rightarrow \infty: \quad \bar{u} = a \bar{x} + b \bar{y}, \quad \bar{T} = T_\infty, \quad \bar{C} = C_w, \]  

in which \( a, b \) and \( c \) are positive constants having the dimension of inverse time, \( T_\infty \) is the ambient temperature and \( h_f \) is the heat transfer coefficient. The radiative heat flux can be modeled by using Rosseland’s approximation as follows

\[ q_r = -\frac{4\sigma}{3(\alpha_r + \sigma_s)} \frac{\partial \bar{T}^4}{\partial \bar{y}} \]  

(7)

where \( \sigma \) is the Stefan-Boltzmann constant, \( \alpha_r \) is the Rosseland mean absorption coefficient and \( \sigma_s \) is the scattering coefficient. Assuming that the temperature difference within the flow is sufficient small so that \( \bar{T}^4 \) may be expressed as linear function \( \bar{T} \) such that

\[ \bar{T}^4 = 4T_\infty^3 \bar{T} - 3T_\infty^4 \]  

(8)

thus Eq. (7) takes the following form

\[ q_r = -\frac{16\sigma T_\infty^3}{3(\alpha_r + \sigma_s)} \frac{\partial \bar{T}}{\partial \bar{y}}. \]  

(9)

Upon using non-dimensional variables and stream function \( \psi \), which satisfies the continuity equation such as

\[ x = \bar{x} \sqrt{\frac{c}{v}}, \quad y = \bar{y} \sqrt{\frac{c}{v}}, \quad u = \frac{1}{\sqrt{vc}} \bar{u}, \quad v = \frac{1}{\sqrt{vc}} \bar{v}, \quad p = \frac{1}{\rho vc} \bar{p}, \quad T = \frac{\bar{T} - T_\infty}{T_f - T_\infty}, \quad C = \frac{\bar{C} - C_w}{C_w - C_\infty}, \]  

(10)

and then eliminating pressure from Eqs. (2, 3), Eqs. (2-6) take the following new form in term of \( \psi \)

\[ \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} + \text{We} \frac{\partial (\psi, \nabla^4 \psi)}{\partial (x, y)} + \nabla^4 \psi = 0, \]  

(11)

\[ \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \frac{1}{\text{Pr}} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + 16\sigma T_\infty^3 \frac{\partial^2 T}{\partial x^2} + 3k(\alpha_r + \sigma_s) \frac{\partial^2 T}{\partial y^2} \right) + N_b \left( \frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} \right) + \]  

\[ N_c \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right), \]  

(12)

\[ Sc \left( \frac{\partial \psi}{\partial x} \frac{\partial C}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial C}{\partial x} \right) = \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) + \frac{N_c}{N_b} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right), \]  

(13)

\[ y = 0 : \quad \frac{\partial \psi}{\partial y} = x, \quad \psi = 0, \frac{\partial T}{\partial y} = -Bi \left( 1 - T \right), \quad C = 1, \]  

(14)

\[ y \rightarrow \infty: \quad \psi = \frac{a}{c} xy + \frac{\tau}{c} \left( a^2 + c^2 \right), \quad T = 0, \quad C = 0, \]  

where \( \text{We}=k_o c/\rho v \) be the Weissenberg number, \( \text{Pr}=u C_r / k \) be the Prandtl number, \( Sc = v/D_b \) be the Schmidt
number, \( N_e = D \sigma / \eta \) be the thermophoresis parameter, \( N_b = D \sigma / \eta \) be the Brownian motion parameter, \( Bi = (h / k) \sqrt{\nu / \eta} \) be the Biot number and \( \gamma = b / c \) represents shear in the free stream. Suppose the solution of Eqs. (11-14) is of the form

\[
\psi = x f(y) + g(y), \quad T = \theta(y), \quad C = \phi(y),
\]

(15)

where the functions \( f(y) \) and \( g(y) \) are normal and oblique component of the flows. Using the Eq. (15) in Eqs. (11-14), and after comparing the coefficient of \( \chi^0 \) and \( \chi^1 \), we get

\[
f'' + f f' - f' + W e (f f' - f' f'') = 0, \quad (16)
\]

\[
g'' + f g' - g' f' + W e (f g' - g' f'') = 0, \quad (17)
\]

\[
(1 + 4 R d / 3) \theta + \text{Pr} \left[ f \theta' + N_b \phi' \theta' + N_1 (\theta)^2 \right] = 0,
\]

(18)

where \( A = A(a / c, \ We) \) is a constant which measures the boundary layer displacement. Constant \( A \) at free stream behave as \( (a / c \ y) \) which also corresponds to the behavior of \( f(y) \) at the free stream. For simplicity, introducing a new variable, \( g'(y) = \gamma h(y) \), then Eq. (22) with boundary conditions is written as

\[
h'' + f \ h' - f' h + W e (f h'' - f' h' + h' f'' - f'' h) = A,
\]

(23)

\[
h(0) = 0, \quad h'(\infty) = 1.
\]

(24)

Thus the system of non-linear ordinary equations becomes

\[
f'' + f f' - f' + W e (f f' - f' f'') + A^2 = 0,
\]

(21)

\[
g'' + f g' - g' f' + W e (f g' - g' f'') - A \gamma = 0,
\]

(22)

\[
(1 + 4 R d / 3) \phi'' + \text{Pr} \left[ f \phi' + N_b \phi' \phi' + N_1 (\phi)^2 \right] = 0,
\]

(27)

\[
\phi'' + S c f \phi' + (N_1 / N_b) \theta'' = 0,
\]

(28)

with boundary conditions

\[
y = 0 : \quad f(y) = 0, \quad f'(y) = 1, \quad h(y) = 0, \quad \theta'(y) = -Bi (1 - \theta(y)), \quad \phi(y) = 1,
\]

(29)

\[
y \to \infty : \quad f'(y) = a / c, \quad h'(y) = 1, \quad \theta(y) = 0, \quad \phi(y) = 0.
\]

To solve the fourth order ordinary differential equations (25, 26), we used two extra boundary conditions \( f''(y) = 0 \) and \( h''(y) = 0 \) as \( y \to \infty \). These conditions are called augmented boundary conditions [45, 46]. The dimensionless components of velocity are

\[
u = - \frac{\partial \psi}{\partial y} = x f'(y) + g'(y), \quad v = - \frac{\partial \psi}{\partial x} = -f(y).
\]

(30)

The quantities of physical interest are the skin friction coefficients \( C_f \), the local Nusselt number \( N u_s \) and the local Sherwood number \( S h_s \), can be expressed as

\[
C_f = \frac{\tau_w}{\rho u_s^2}, \quad N u_s = \frac{\tau_w (q_v + q_a)}{k(T_s - T_v)}, \quad S h_s = \frac{\tau_m q_m}{D_b (C_v - C_a)},
\]

(31)

where \( \tau_w \) is shear stress at the wall, \( q_v \) is the radiative heat flux, \( q_a \) and \( q_m \) represents local heat flux, and local mass diffusion flux at the wall are
\[ z = \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - 2k_0 \left( -\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{1}{2} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right) \plus \frac{1}{2} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right) \right] \phi_{,,0} \),

\[ q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}, \quad q_w = -D_b \left( \frac{\partial \phi}{\partial x} \right)_{y=0}, \quad q_r = -\frac{4\sigma}{3(\alpha_1 + \sigma_1)} \frac{\partial T^4}{\partial y^r}. \] (32)

After using Eqs. (10) and (15), the skin friction coefficients \( C_f \), the local Nusselt number \( Nu \), and the local Sherwood number \( Sh \), takes the following form

\[ \text{Re}_x C_f = x(1 - 3We) f''(0) + (1 - 2We\gamma h'(0)), \]

\[ \text{Re}_x^{1/2} Nu = - \left( 1 + \frac{4}{3} Rd \right) \theta'(0), \quad \text{Re}_x^{1/2} Sh = -\phi'(0). \] (33)

where \( \text{Re}_x = u_x / \nu \).

### 3. Numerical method

Exact solutions of the nonlinear differential equations (25-28) subject to the boundary conditions (29) are very rare due to the nonlinearity. Some authors have used analytical semi-analytical techniques to solve these equations [33, 39]. In the present study, we used a numerical technique called Chebyshev Spectral Newton Iterative Scheme (CSNIS). In this scheme, we first convert the system of nonlinear differential equation into a linear form by using Newton iterative scheme. For \((i+1)th\) iterates, we write

\[ f_{i+1} = f_i + \delta f_i, \quad \theta_{i+1} = \theta_i + \delta \theta_i, \quad \phi_{i+1} = \phi_i + \delta \phi_i, \] (34)

for all dependent variables, where \( \delta f_i, \delta \theta_i \) and \( \delta \phi_i \) represents a very small change in \( f_i, \theta_i \) and \( \phi_i \) respectively.

The equations (25-28) in linearized form are

\[ a_{i,j} \delta f_j + a_{i,1} \delta f_j'' + a_{i,2} \delta f_j'' + a_{i,3} \delta f_j' + a_{i,4} \delta f_j = R_{i,j}, \]

\[ b_{i,j} \delta f_j'' + b_{i,1} \delta f_j'' + b_{i,2} \delta f_j' + b_{i,3} \delta h_j'' + b_{i,4} \delta h_j' + b_{i,5} \delta h_j = R_{i,j}, \]

\[ c_{i,1} \delta f + c_{i,1} \delta \theta_j' + c_{i,2} \delta \theta_j' + c_{i,3} \delta \phi_j' = R_{i,j}, \]

\[ d_{i,1} \delta f + d_{i,1} \delta \theta_j' + d_{i,2} \delta \phi_j' + d_{i,3} \delta \phi_j' = R_{i,j}, \]

subject to boundary conditions

\[ \delta f_j(0) = -f_j(0), \quad \delta f_j''(0) = a_{i,1}^2 - f_j'(0), \quad \delta f_j''(\infty) = 1 - f_j'(\infty), \quad \delta f_j''(\infty) = -f_j''(\infty), \]

\[ \delta h_j(0) = -h_j(0), \quad \delta h_j''(\infty) = 0 - h_j'(\infty), \quad \delta h_j''(\infty) = -h_j''(\infty), \]

\[ \delta \theta_j'(0) = B_i \delta \theta_j'(0) = -\theta_j'(0) - Bi(1 - \theta_j'(0)), \quad \delta \theta_j'(\infty) = -\theta_j'(\infty), \]

The coefficients \( a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j} \) and \( R_{i,j} (j = 0, 1, 2, 3, \ldots) \) are

\[ a_{0,j} = We \frac{f_j}{c - f_j}, \quad a_{1,j} = -2We f_j', \quad a_{2,j} = -f_j', \quad a_{3,j} = -2f_j', \quad a_{4,j} = f_j'' + We f_j', \]

\[ b_{0,j} = -We h_j, \quad b_{1,j} = We h_j', \quad b_{2,j} = -h_j - We h_j', \quad b_{3,j} = h_j' + We h_j', \quad b_{4,j} = We f_j, \quad b_{5,j} = 1 - We f_j, \]

\[ b_{6,j} = f_j + We f_j', \quad b_{7,j} = -f_j - We f_j', \quad c_{0,j} = Pr \theta_j', \quad c_{1,j} = (1 + 4Rd / 3), \quad c_{2,j} = Pr \left( N_b \theta_j' \right), \]

\[ c_{2,j} = Pr \left( f_j + N_b \theta_j' + 2N_b \theta_j' \right), \quad d_{0,j} = Sc \phi_j', \quad d_{1,j} = \left( N_i / N_b \right), \quad d_{2,j} = 1, \quad d_{3,j} = Sc f_j, \]

\[ R_{i,j} = -We \left( f_j f_j'' - 2f_j' f_j' + (f_j')^2 \right) - f_j'' - f_j f_j'' - (f_j')^2 - a^2 / c^2, \]

\[ R_{2,j} = -We \left( f_j h_j'' - f_j' h_j' + f_j' h_j' - h_j'' f_j' h_j + A, \right), \]

\[ R_{3,j} = -\left( 1 + 4Rd / 3 \right) \theta_j' - Pr \left( f_j \theta_j' + N_b \theta_j' \theta_j' + N_i (\theta_j')^2 \right), \]

\[ R_{4,j} = -\phi_j' - Sc f_j \phi_j' - \left( N_i / N_b \right) \theta_j'. \]

The system of linear Eqs. (35) subject to boundary conditions (36) is solved using the Chebyshev Spectral Collocation method [43, 44]. For this purpose, the physical domain \([0, \infty]\) is truncated to finite domain \([0, L]\), where \(L\) is chosen sufficiently large. The reduced domain is transformed to \([-1, 1]\) by using transformation \( \xi = 2\eta L \). Nodes from -1 to 1 are defined as \( \xi_j = \cos(\pi j/N), j = 0, 1, 2, \ldots, N \), which are known as Gauss-Lobatto collocation points. The Chebyshev Spectral Collocation Method is based on differentiation matrix D, which can
be computed in different ways. Here we used D as suggested by Trefethen [42]. Applying Collocation method to Eqs. (35, 36), the following matrix is obtained

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}
\begin{bmatrix}
\delta f_i \\
\delta \theta_i \\
\delta \phi_i
\end{bmatrix} =
\begin{bmatrix}
R_{i,j} \\
R_{i,j} \\
R_{i,j} \\
R_{i,j}
\end{bmatrix}
\tag{38}
\]

Where

\[
\begin{align*}
A_{11} = & a_{0,i}D^4 + a_{1,i}D^3 + a_{2,i}D^2 + a_{3,i}D + a_{4,i}, A_{12} = 0, A_{13} = 0, A_{14} = 0, \\
A_{21} = & b_{0,i}D^3 + b_{1,i}D^2 + b_{2,i}D + b_{3,i}, A_{22} = b_{4,i}D^4 + b_{5,i}D^3 + b_{6,i}D + b_{7,i}, A_{23} = 0, A_{24} = 0, \\
A_{31} = & c_{0,i}, A_{32} = 0, A_{33} = c_{1,i}D^2 + c_{2,i}D, A_{34} = c_{3,i}D, \\
A_{41} = & d_{0,i}, A_{42} = 0, A_{43} = d_{1,i}D^2, A_{44} = d_{2,i}D^2 + d_{3,i}D,
\end{align*}
\tag{39}
\]

\(I\) is identity matrix, \(a_{j,i}, b_{j,i}, c_{j,i}, d_{j,i}\) and \(R_{j,i}(j = 0, 1, 2, 3...)\) are given in (37).

4. Results and discussion

The non-linear differential equations (25-28) subject to the boundary conditions (29) are solved numerically for the different values of dimensionless parameters namely Weissenberg number (We), velocities ratio parameter \(a/c\), radiation (\(Rd\)), thermophoresis (\(N_t\)), Brownian motion (\(N_b\)), Prandtl number (\(Pr\)), Schmidt number (\(Sc\)) and Biot number (\(Bi\)). The values of \(f'(0), -\theta'(0)\) and \(-\phi'(0)\) shown in limiting case through Tables 1 and 2. It is found that the results are in excellent agreement with previous investigations published in the literature. The results in term of velocity profile, temperature profile, concentration profile, \(f'(0), -\theta'(0)\) and \(-\phi'(0)\) for sundry parameters are shown through graphs. In most cases, the values of the parameters are taken as \(Pr=6.8, Sc=1.5, Bi=0.5\ We = 0.1, N_t=N_b=0.3\ a/c=0.3, 1.2\ and \(Rd=1\) or otherwise mentioned.

### Table 1: Comparison of \(-\theta'(0)\) for the various values of \(a/c\) and \(Pr\) in the absence Thermophoresis effects and Brownian motion of nanoparticles when \(We=0, Rd=0\) and \(Bi\rightarrow\infty\).

<table>
<thead>
<tr>
<th>(a/c)</th>
<th>Present work</th>
<th>Labropulu et al. [47]</th>
<th>Present work</th>
<th>Labropulu et al. [47]</th>
<th>Present work</th>
<th>Labropulu et al. [47]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Pr=1)</td>
<td>(Pr=10)</td>
<td>(Pr=20)</td>
<td>(Pr=10)</td>
<td>(Pr=20)</td>
<td>(Pr=20)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.60215</td>
<td>0.60281</td>
<td>2.31693</td>
<td>2.31684</td>
<td>3.36196</td>
<td>3.36172</td>
</tr>
<tr>
<td>0.3</td>
<td>0.64728</td>
<td>0.64732</td>
<td>2.34841</td>
<td>2.34841</td>
<td>3.39149</td>
<td>3.39148</td>
</tr>
<tr>
<td>0.8</td>
<td>0.75710</td>
<td>0.75709</td>
<td>2.46778</td>
<td>2.46778</td>
<td>3.51054</td>
<td>3.51054</td>
</tr>
<tr>
<td>1.0</td>
<td>0.79788</td>
<td>0.79788</td>
<td>2.52313</td>
<td>2.52313</td>
<td>3.56825</td>
<td>3.56825</td>
</tr>
<tr>
<td>2.0</td>
<td>0.97873</td>
<td>0.97872</td>
<td>2.81389</td>
<td>2.81389</td>
<td>3.88689</td>
<td>3.88689</td>
</tr>
<tr>
<td>3.0</td>
<td>1.13209</td>
<td>1.13209</td>
<td>3.09751</td>
<td>3.09751</td>
<td>4.21307</td>
<td>4.21307</td>
</tr>
<tr>
<td>4.0</td>
<td>1.26733</td>
<td>1.26733</td>
<td>3.36440</td>
<td>3.36441</td>
<td>4.52808</td>
<td>4.52810</td>
</tr>
</tbody>
</table>

### Table 2: Comparison of \(-\theta'(0)\) and \(-\phi'(0)\) for the various values of \(N_t\) and \(N_b\) when \(We=0, a/c=0, Rd=0, Pr=10, Sc=10\), and \(Bi=0.1\). The results in small brackets are reported by Makinde and Aziz [24].

<table>
<thead>
<tr>
<th>(N_t)</th>
<th>(N_b=0.1)</th>
<th>(N_b=0.3)</th>
<th>(N_b=0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(-\theta'(0))</td>
<td>(-\phi'(0))</td>
<td>(-\theta'(0))</td>
</tr>
<tr>
<td>0.1</td>
<td>(0.0929)</td>
<td>(2.2774)</td>
<td>(0.0769)</td>
</tr>
<tr>
<td>0.3</td>
<td>(0.0925)</td>
<td>(2.2228)</td>
<td>(0.0729)</td>
</tr>
<tr>
<td>0.5</td>
<td>(0.0921)</td>
<td>(2.1783)</td>
<td>(0.0700)</td>
</tr>
</tbody>
</table>
Table 3: Numerical values of $A$ for various values of $We$ and $a/c$.

<table>
<thead>
<tr>
<th>$a/c$</th>
<th>0</th>
<th>0.05</th>
<th>0.1</th>
<th>0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000</td>
<td>0.9747</td>
<td>0.9487</td>
<td>0.8944</td>
</tr>
<tr>
<td>0.1</td>
<td>0.7917</td>
<td>0.7663</td>
<td>0.7402</td>
<td>0.6854</td>
</tr>
<tr>
<td>0.2</td>
<td>0.6407</td>
<td>0.6161</td>
<td>0.5906</td>
<td>0.5369</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5195</td>
<td>0.4962</td>
<td>0.4720</td>
<td>0.4205</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4173</td>
<td>0.3959</td>
<td>0.3735</td>
<td>0.3254</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3286</td>
<td>0.3096</td>
<td>0.2896</td>
<td>0.2459</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2499</td>
<td>0.2338</td>
<td>0.2167</td>
<td>0.1789</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1791</td>
<td>0.1664</td>
<td>0.1527</td>
<td>0.1220</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1145</td>
<td>0.1056</td>
<td>0.0960</td>
<td>0.0738</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0551</td>
<td>0.0505</td>
<td>0.0454</td>
<td>0.0334</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Table 4: Numerical values of $Re_x^{1/2} Nu_x$ and $Re_x^{1/2} Sh_x$ for wider range of $Pr$.

<table>
<thead>
<tr>
<th>$We$</th>
<th>$a/c$</th>
<th>$Rd$</th>
<th>$N_t$</th>
<th>$N_b$</th>
<th>$Bi$</th>
<th>$Sc$</th>
<th>$Pr$</th>
<th>$Re_x^{1/2} Nu_x$</th>
<th>$Re_x^{1/2} Sh_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.10</td>
<td>1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>1</td>
<td>0.7</td>
<td>0.1681</td>
<td>0.5539</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td>0.8182</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10.00</td>
<td>0.5489</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>50.00</td>
<td>0.5225</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100.00</td>
<td>0.5107</td>
</tr>
<tr>
<td>0.20</td>
<td>0.50</td>
<td>2</td>
<td>0.3</td>
<td>0.3</td>
<td>1</td>
<td>5</td>
<td>0.7</td>
<td>0.7404</td>
<td>1.5984</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td>1.5899</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10.00</td>
<td>1.5803</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>50.00</td>
<td>1.9263</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100.00</td>
<td>2.0450</td>
</tr>
<tr>
<td>0.3</td>
<td>1.0</td>
<td>5</td>
<td>0.5</td>
<td>0.5</td>
<td>$\infty$</td>
<td>10</td>
<td>0.7</td>
<td>1.7315</td>
<td>2.5104</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td>2.5078</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10.00</td>
<td>2.6083</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>50.00</td>
<td>2.9136</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100.00</td>
<td>2.9281</td>
</tr>
</tbody>
</table>

Fig. 2: Variation of $f''(0)$ against $We$ for the different values of $a/c$.

Fig. 3: Variation of $-\theta'(0)$ against $We$ for the different values of $a/c$. 
Fig. 4: Variation of $-\phi'(0)$ against $We$ for the different values of $a/c$.

Fig. 5: Velocity profile against $y$ for the different values of $We$, $a/c$ and $\gamma$.

Fig. 6: Variation of Nusselt number against $N_i$ for the different values of $Rd$ and $a/c$.

Fig. 7: Variation of Sherwood number against $N_i$ for the different values of $Rd$ and $a/c$.

Fig. 8: Variation of Nusselt number against $N_b$ for the different values of $Rd$ and $a/c$.

Fig. 9: Variation of Sherwood number against $N_b$ for the different values of $Rd$ and $a/c$.

Fig. 10: Variation of Nusselt number against $Bi$ for the different values of $Rd$ and $a/c$.

Fig. 11: Variation of Sherwood number against $Bi$ for the different values of $Rd$ and $a/c$. 
The variation of $f''(0), -\theta'(0)$ and $-\phi'(0)$ against $We$ for $a/c = 0.8, 1.0, 1.1,$ and $1.2$ are shown in Figures 2 to 4 respectively. From these Figures, it is observed that the similarity equations (25-28) subject to the boundary conditions (29) have dual solutions in some range of the parameter $We$. There exist unique solution in particular range of the parameter $We$ and there exist a region where the solution of the equation does not exist. The solid lines show the stable solution and dashed lines show the unstable solution. For $a/c>1$, the range of solution enhances due to increase in $a/c$ and for $a/c<1$ the range of unstable solution become larger than the stable solution. There exist a unique solution at critical value $We=We_c$, dual solution exist between the range $0\leq We<We_c$ and no solution exists for $We<0$ and $We>We_c$. The critical values are $We_{c1}=0.3149$, $We_{c2}=0.3642$, $We_{c3}=0.528$ and $We_{c4}=0.33$ for different values of $a/c$ as shown in figures. It is observed that unstable solution has higher values of $f''(0), -\theta'(0)$ and $-\phi'(0)$ than that of the stable solution for given values of $We$. It is further noted that in stable solution (first solution) heat and mass transfer rate increase with increase in the values of $a/c$, where a reverse behavior has been observed for unstable solution (second solution). The stability analysis of multiple solutions has been discussed by many researcher see [48-50]. They found that first solution is applicable physically while the second solution is not. In Figure 5 the velocity profile is plotted against $y$ for the different

---

**Fig. 12:** Temperature profile against $y$ for the different values of $Bi$.

**Fig. 13:** Concentration profile against $y$ for the different values of $Bi$.

**Fig. 14:** Temperature profile against $y$ for the different values of $We$ and $a/c$.

**Fig. 15:** Concentration profile against $y$ for the different values of $We$ and $a/c$.

**Fig. 16:** Temperature profile against $y$ for the different values of $Rd$.

**Fig. 17:** Concentration profile against $y$ for the different values of $Rd$.
values of $We$, $a/c$ and $\gamma$. Here $\gamma = 0$ and $\gamma = 0.2$ correspond to the case for orthogonal stagnation point flow and non-orthogonal stagnation point flow respectively. It is noted that the velocity of the fluid is increasing with increase in the values of $We$ when $a/c>1$. An opposite behavior is observed for the case when $a/c<1$. It is also seen that with increase in the values of $\gamma$ the velocity of the fluid increases. In Figures 6 and 7 the variation of local Nusselt ($Re_x^{-1/2}Nu_x$) and local Sherwood ($Re_x^{-1/2}Sh_x$) numbers are plotted against thermophoresis parameter ($N_t$) for the different values of $R_d$ and $a/c$. It is clear from Figure 6 that with increase in the values of $N_t$, a very slight decrease in local Nusselt number is observed for both cases of $a/c$ ($a/c>1$, $a/c<1$). Consequently, temperature profile and thermal boundary layer thickness increase with increase of thermophoresis parameter ($N_t$) near the wall. Figure 7 elucidates that the local Sherwood number decreases with increase of $N_t$, as a consequence the concentration profile and concentration boundary layer increase with increase of $N_t$. From Figures 6 and 7, an increase in local Nusselt and local Sherwood numbers is observed due to enhancement of radiation. In Figures 8 and 9, the values of local Nusselt ($Re_x^{-1/2}Nu_x$) and local Sherwood ($Re_x^{-1/2}Sh_x$) numbers are plotted against Brownian motion parameter ($N_b$) for the different values of $R_d$ and $a/c$. It is seen that with increase in Brownian motion the local Nusselt number decreases but the local Sherwood number increases. This increase in local Sherwood number is very rapid in the range $0<N_t<0.2$. This phenomenon leads to increase the temperature and thermal boundary layer thickness but decrease in concentration profile. In Figures 10 and 11, the variation of local Nusselt ($Re_x^{-1/2}Nu_x$) and local Sherwood ($Re_x^{-1/2}Sh_x$) numbers are plotted against Biot number $Bi$ (depending on the heat transfer coefficient) for the different values of $R_d$ and $a/c$. It is seen that the local Nusselt number increases and local Sherwood number decreases for initial values of $Bi$ and for the larger values of $Bi$ both quantities become constant. Due to the larger values of Biot number $Bi$ ($Bi \to \infty$) the surface become heated and heat transfer rate increases. In fact, the larger values of Biot number imply the strong surface convection result in high surface temperature; therefore, increase in Biot number enhanced the temperature and thermal boundary layer thickness. This behavior can be predicted from Figure 12. In Figure 13, concentration profile is plotted against $y$ for different values of $Bi$ when $R_d=2$ and $a/c = 0.3$. Concentration profile increases with increase in the values of $Bi$ because concentration distribution depends upon the temperature field hence the lager Biot number helps to increase the concentration of nanoparticles in fluid. In Figures 14 and 15 temperature and concentration profiles are plotted against $y$ for the different values of $We$ and $a/c$ when $R_d=1$, $Bi=0.1$. For $a/c<1$, it is observed that the temperature and concentration profiles are increasing functions of $We$ but for $a/c>1$ an opposite behavior is noted. In Figures 16 and 17, temperature and concentration profiles are plotted against $y$ for the different values of $R_d$ when $Bi=0.1$ and $a/c = 0.3$. With increase in the values of radiation parameter, temperature of the fluid increases where the concentration profile decreases near the wall but for the larger value of $y$, it increases with increase in the values of radiation parameter.

5. Conclusion

The combined effect of radiation and convective boundary condition in the region of oblique stagnation point flow of elasto-viscous fluid saturated with nanoparticles is considered. The governing partial differential equations are transformed into system of ordinary differential equations by using the similarity transformation. The obtained system of equations is solved numerically by using the Chebyshev Spectral Newton Iterative Scheme (CSNIS). The present numerical results are in excellent agreement with the previously obtained results. It is observed that the similarity equations (25-28) subject to the boundary conditions (29) have unique solution, dual solution and no solution in different region of the parameter $We$. For $a/c>1$, the range of existence of solution increases due to increase in $a/c$ and for $a/c<1$, the range of unstable solution become larger than that of the stable solution. It is also concluded that

- The velocity of the fluid intensifies due to increase in $We$ when $a/c>1$ but an opposite behavior is observed for $a/c<1$.
- The velocity of the fluid is found an increasing function of $\gamma$.
- Temperature profile and thermal boundary layer thickness enhance due to increase in the values thermophoresis parameter ($N_t$).
- Concentration profile and concentration boundary layer thickness increase with increase of $N_t$. 

10
- Brownian motion enhanced the thermal boundary layer thickness.
- Brownian motion decreases the concentration boundary layer thickness.
- The larger values of Biot number imply the enhancement in heat transfer and thermal boundary layer thickness.
- Concentration profile increases with increase in the values of Biot number.
- Temperature is noted an increasing function of radiation.

### Nomenclature

- $a, b, c$: positive constants
- $Bi$: Biot number
- $C$: Solutal concentration
- $C_w$: Solutal concentration at the wall
- $C_\infty$: Ambient solutal concentration
- $C_f$: Skin friction coefficient
- $C_p$: Specific heat constant
- $D_B$: Brownian diffusion coefficient
- $Dr$: Thermophoretic diffusion coefficient
- $f$: Dimensionless stream function
- $h_f$: convective heat transfer coefficient
- $k$: thermal conductivity of the nanofluid
- $k_o$: elasticity of fluid
- $Sc$: Schmidt number
- $N_b$: Brownian motion parameter
- $N_t$: thermophoresis parameter
- $Nu$: Nusselt number
- $Pr$: Prandtl number
- $q_m$: mass flux at the wall
- $q_w$: heat flux at the wall
- $R_e$: local Reynolds number
- $Rd$: Radiation conduction parameter or Planck number
- $Sh_x$: local nanoparticle Sherwood number
- $T$: Temperature of the fluid in the boundary layer
- $T_f$: temperature of the hot fluid
- $T_\infty$: ambient fluid temperature
- $T_w$: surface temperature
- $U_e$: Free stream velocity
- $u, v$: dimensionless velocity components in x and y directions
- $We$: Weissenberg number
- $x, y$: coordinates along and normal to the plate
- $\alpha_r$: Rosseland mean absorption
- $\beta$: Thermal expansion coefficient
- $\rho$: Fluid density
- $(\rho C)_f$: heat capacity of the fluid
- $(\rho C)_p$: effective heat capacity of the nanoparticle material
- $\tau$: ratio between the effective heat capacity of the nanoparticle material and heat capacity of the fluid
- $\nu$: kinematic viscosity
- $\sigma$: Stephen-Boltzmann constant
- $\sigma_s$: scattering coefficient
- $\theta$: dimensionless temperature
- $\Psi$: Stream function

### Greek symbols

- $k$: thermal conductivity of the nanofluid
- $\rho$: Fluid density
- $C_p$: Specific heat constant
- $\alpha_r$: Rosseland mean absorption
- $\beta$: Thermal expansion coefficient
- $\rho$: Fluid density
- $\sigma$: Stephen-Boltzmann constant
- $\sigma_s$: scattering coefficient
- $\theta$: dimensionless temperature
- $\Psi$: Stream function

### Acknowledgement

The authors are grateful to the editor and anonymous reviewers for their valuable suggestions that helped to improve the manuscript. The first author is thankful to HEC for financial assistance.

### References