HEAT TRANSFER ANALYSIS OF MHD FLOW DUE TO UNSTEADY BI-DIRECTIONAL STRETCHING SHEET THROUGH POROUS SPACE

by

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In this article unsteady 3-D MHD boundary layer flow and heat transfer analysis with constant temperature and constant heat flux in a porous medium is considered. The boundary layer flow is governed by a bi-directional stretching sheet. Similarity transformations are used to transform the governing non-linear partial differential equations to ordinary differential equations. Analytical solutions are constructed using homotopy analysis method. Convergence analysis is also presented through tabular data. The quantities of interest are the velocity, temperature, skin friction coefficient, and Nusselt number. The obtained results are validated by comparisons with previously published work in special cases. The results of this parametric study are shown graphically and the physical aspects of the problem are discussed.

Key words: porous medium, MHD, stretching, unsteady, 3-D, homotropy analysis method

Introduction

Boundary layer flows over a moving surface have been extensively targeted by many researchers during past few decades due to their importance in engineering and industrial processes. These flows have abundant applications in many technological processes including paper productions, glass fiber, manufacturing plastic films, crystal growing, hot rolling, and many others. After the pioneering work done by Sakiadis [1, 2] on the boundary layer flow past a moving plate many investigators have discussed the various aspects of stretching phenomenon. Much attention in the past has been given to 2-D boundary layer flows over a steady stretching sheet [3-9], and abundant literature in this direction is available. All the studies previously mentioned were carried out when the sheet is stretched linearly in one direction. Wang [10] discussed 3-D flow of a viscous fluid due to the stretching of the elastic surface in two lateral directions. Heat transfer in fluid flows is an important phenomenon and for the details of fluid flow phenomena for in-tube/external forced convection the readers are referred to articles [11-15]. The present problem can be investigated through the recent developed techniques such as the use of inserts, non-uniform heating effect, geometrical optimization, the effect of wall axial conduction (conjugate case), generalized (power-law) fluid case, disturbed velocity, etc. [16-23].

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Heat transfer and flow through porous medium in the presence of a constant magnetic field is a phenomenon of great interest from both theoretical and practical point of view. This is due to its importance in several environmental and engineering disciplines. These applications include geothermal, petroleum resources, in situ combustion of oil shell, boiling enhancement using porous coatings, compact heat exchangers, packed bed reactors or absorbent, high performance of building insulation, etc. There is still a great deal of theoretical as well as practical interest in this area of research due to wide range of applications and importance. A glance at literature show some recent studies dealing with the MHD flow through porous medium over a stretching sheet [24-32] and references therein. Liu and Andersson [33] investigated the heat transfer characteristics over a bi-directional stretching sheet with variable thermal conditions in the presence of a temperature-dependent internal heat source (or sink). Ahmad et al. [34] analyzed heat transfer characteristics for 2-D flow of a steady viscous fluid over an exponentially stretching sheet with variable thermal conductivity through porous medium in the presence of an applied magnetic field. The problem investigated in [34] is different from the present problem in many ways firstly here we have considered a 3-D flow instead of 2-D flow, secondly the flow is steady in [34] while here we have assumed an unsteady flow, thirdly in [34] a variable thermal conductivity is taken into account as the constant thermal conductivity in the present case.

Few attempts regarding unsteady flow over a stretching sheet can be found through [35-39]. Much attention has not been given to the unsteady flow problems regarding bi-directional stretching. Mukhopadhyay [40] examined unsteady mixed convection flow and heat transfer over a porous stretching surface. The unsteady 3-D stagnation point flow of a viscoelastic fluid has been considered by Sashadri et al. [41]. Recently, Hayat et al. [42] studied the time dependent 3-D flow and mass transfer of an elastic viscous fluid over an unsteady bi-directional stretching sheet. Based on the previous review, it is noted that the heat transfer characteristic of 3-D unsteady MHD flow due to bi-directional unsteady stretching sheet in a porous medium has not been attempted or investigated yet to the best of our information. The purpose of this paper is to present the analytic solution of unsteady MHD flow and heat transfer over a bi-directional unsteady stretching sheet in a porous medium. The analytic series solution is developed using homotopy analysis method (HAM) [43, 44]. It is important to mention here that the same problem can be investigated by using other semi-analytical methods [45-47].

Formulation of the problem

Consider the unsteady 3-D boundary layer flow of a viscous, incompressible fluid in a porous medium due to stretching surface in a plane at \( z = 0 \). The surface is uniformly stretched in both x- and y-directions. Flow analysis is carried out in the presence of heat generation or absorption parameter. A constant magnetic field of strength, \( B_0 \), is applied perpendicular to the flow in the z-direction. The magnetic Reynold number is assumed small so that the induced magnetic field can be neglected. The transformed form of the equations governing the unsteady MHD flow and heat transfer analysis due to bi-directional stretching sheet with constant temperature and constant heat flux are [33]:

\[
\begin{align*}
  f'''' + (f' + g) f'' - f'^2 - A \left( f' + \frac{\eta}{2} f'' \right) - (\epsilon + M^2) f' &= 0 \\
  g'''' + (f' + g) g'' - g'^2 - A \left( g' + \frac{\eta}{2} g'' \right) - (\epsilon + M^2) g' &= 0
\end{align*}
\]
\[ \frac{\partial}{\partial \eta} \left( \frac{\eta}{2} \frac{\partial \theta'}{\partial \eta} \right) \Pr(\theta - \frac{\eta}{2} \theta') + Pr(\beta - \eta' - \eta g')\theta - A \Pr = 0, \quad (CT) \] (3)

\[ \frac{\partial}{\partial \eta} \left( \frac{\eta}{2} \frac{\partial \phi'}{\partial \eta} \right) \Pr(\phi - \frac{\eta}{2} \phi') + Pr(\beta - \eta' - \eta g')\phi - A \Pr = 0, \quad (CH) \] (4)

subject to the boundary conditions:

\[ f = 0, \quad g = 0, \quad g' = \alpha, \quad \theta = 1, \quad \phi' = -1, \quad \text{at } \eta = 0, \]

\[ f' \to 0, \quad g' \to 0, \quad \theta \to 0, \quad \phi \to 0, \quad \text{as } \eta \to \infty \] (5)

where the prime denotes differentiation with respect to \( \eta \), \( \alpha = b/a \) is the stretching ratio, \( A = c/a \) – the unsteadiness parameter, \( \varepsilon = \phi_1/pa_k \) – the porosity parameter, \( M = \sigma B_0^2/\rho \) – the magnetic parameter, \( \Pr = v/k \) the Prandtl number, and \( \beta = Q/\rho C_p \alpha \) – the internal heat parameter.

The physical quantities of interest are the skin friction coefficients \( C_{fx} \) and \( C_{fy} \) along the x- and y-directions, respectively, and are given by:

\[ C_{fx} = \frac{\tau_{wx}}{\rho u_w^2}, \quad C_{fy} = \frac{\tau_{wy}}{\rho u_w^2} \] (6)

where \( \tau_{wx} \) and \( \tau_{wy} \) are the wall shear stress along x- and y-directions, respectively. In dimensionless form we get:

\[ \text{Re}_{x}^{1/2} C_{fx} = f''(0), \quad \text{Re}_{x}^{1/2} C_{fy} = \frac{v_w}{u_w} g''(0) \] (7)

where \( \text{Re}_x = u_w \nu / \nu \) is the local Reynolds number. We can obtain the flow equations for the steady case discussed in [9] when \( A = 0 \), as follows:

\[ f'' + (f + g)f' - f'' - (\varepsilon + M^2)f' = 0 \] (8)

\[ g'' + (f + g)g' - g'' - (\varepsilon + M^2)g' = 0 \] (9)

\[ \theta'' + Pr (f + g)\theta' + Pr(\beta - \eta' - \eta g')\theta = 0 \] (10)

\[ \phi'' + Pr (f + g)\phi' + Pr(\beta - \eta' - \eta g')\phi = 0 \] (11)

and the boundary conditions are given in eq. (14) of [9].

**The HAM solutions**

Based on the rules of solution expressions and the boundary conditions, eq. (5), the initial approximations \( f_0(\eta), g_0(\eta), \theta_0(\eta), \) and \( \phi_0(\eta) \) for the functions \( f(\eta), g(\eta), \theta(\eta), \) and \( \phi(\eta) \) are:

\[ f_0(\eta) = 1 - \exp(-\eta) \] (12)

\[ g_0(\eta) = \alpha[1 + \exp(-\eta)] \] (13)

\[ \theta_0(\eta) = \exp(-\eta) \] (14)

\[ \phi_0(\eta) = \exp(-\eta) \] (15)
and the auxiliary linear operators are:

\begin{align}
\mathcal{L}_1(f) &= \frac{d^3 f}{d\eta^3} - \frac{df}{d\eta} \\
\mathcal{L}_2(f) &= \frac{d^2 f}{d\eta^2} - f
\end{align}

(16)

(17)

satisfying

\begin{align}
\mathcal{L}_1 \left[ C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta) \right] &= 0 \\
\mathcal{L}_2 \left[ C_4 \exp(\eta) + C_5 \exp(-\eta) \right] &= 0
\end{align}

(18)

(19)

in which \(C_i (i = 1, 2, \ldots, 5)\) are the arbitrary constants. From eqs. (1)-(4), the non-linear operators \(\mathcal{N}_f, \mathcal{N}_g, \mathcal{N}_\theta, \) and \(\mathcal{N}_\phi\) are defined:

\begin{align}
\mathcal{N}_f(\hat{f}(\eta, p), \hat{g}(\eta, p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p), p, h_1, h_2, h_3, h_4, h_5, s) &= \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + (\varepsilon + M^2) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left[ \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right]^2 + \\
&+ \left[ \hat{f}(\eta, p) + \hat{g}(\eta, p) \right] \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - A \left[ \frac{\eta}{2} \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} + \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right]
\end{align}

(20)

\begin{align}
\mathcal{N}_g(\hat{f}(\eta, p), \hat{g}(\eta, p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p), p, h_1, h_2, h_3, h_4, h_5, s) &= \frac{\partial^3 \hat{g}(\eta, p)}{\partial \eta^3} + (\varepsilon + M^2) \frac{\partial^2 \hat{g}(\eta, p)}{\partial \eta^2} - \left[ \frac{\partial^2 \hat{g}(\eta, p)}{\partial \eta^2} \right]^2 + \\
&+ \left[ \hat{f}(\eta, p) + \hat{g}(\eta, p) \right] \frac{\partial^2 \hat{g}(\eta, p)}{\partial \eta^2} - A \left[ \frac{\eta}{2} \frac{\partial^2 \hat{g}(\eta, p)}{\partial \eta^2} + \frac{\partial \hat{g}(\eta, p)}{\partial \eta} \right]
\end{align}

(21)

\begin{align}
\mathcal{N}_\theta(\hat{f}(\eta, p), \hat{g}(\eta, p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p), p, h_1, h_2, h_3, h_4, h_5, s) &= \frac{\partial^3 \hat{\theta}(\eta, p)}{\partial \eta^3} + \text{Pr} \left[ \hat{f}(\eta, p) + \hat{g}(\eta, p) \right] \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} + \\
&+ \text{Pr} \left[ \beta \frac{\partial \hat{f}(\eta, p)}{\partial \eta} - s \frac{\partial \hat{g}(\eta, p)}{\partial \eta} \right] \hat{\theta}(\eta, p) - A \left[ \frac{\eta}{2} \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} + \hat{\theta}(\eta, p) \right]
\end{align}

(22)

\begin{align}
\mathcal{N}_\phi(\hat{f}(\eta, p), \hat{g}(\eta, p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p), p, h_1, h_2, h_3, h_4, h_5, s) &= \frac{\partial^3 \hat{\phi}(\eta, p)}{\partial \eta^3} + \text{Pr} \left[ \hat{f}(\eta, p) + \hat{g}(\eta, p) \right] \frac{\partial \hat{\phi}(\eta, p)}{\partial \eta} + \\
&+ \text{Pr} \left[ \beta \frac{\partial \hat{f}(\eta, p)}{\partial \eta} - s \frac{\partial \hat{g}(\eta, p)}{\partial \eta} \right] \hat{\phi}(\eta, p) - A \left[ \frac{\eta}{2} \frac{\partial \hat{\phi}(\eta, p)}{\partial \eta} + \hat{\phi}(\eta, p) \right]
\end{align}

(23)

If \(p \in [0, 1]\) is the embedding parameter, and \(h_1, h_2, h_3, h_4, h_5, s\), are the non-zero auxiliary parameters, the \(0^{th}\)-order deformation problems are of the following form:

\begin{align}
(1-p)\mathcal{L}_1(\hat{f}(\eta, p) - f_0(\eta)) &= p h_1 \mathcal{N}_f(\hat{f}(\eta, p), \hat{g}(\eta, p)) \\
(1-p)\mathcal{L}_1(\hat{g}(\eta, p) - g_0(\eta)) &= p h_2 \mathcal{N}_g(\hat{f}(\eta, p), \hat{g}(\eta, p))
\end{align}

(24)

(25)
\[ (1 - p)L_2[\eta_0(\eta, p) - \eta_0(\eta)] = ph_\eta N_\eta[\hat{f}(\eta, p) + \hat{g}(\eta, p)] \]  
\[ (1 - p)L_2[\eta_0(\eta, p) - \eta_0(\eta)] = ph_\eta N_\eta[\hat{f}(\eta, p) + \hat{g}(\eta, p)] \]  
\[ \hat{f}(0, p) = \hat{g}(0, p) = 0, \quad \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \bigg|_{\eta=0} = 0, \quad \frac{\partial \hat{g}(\eta, p)}{\partial \eta} \bigg|_{\eta=0} = -1 \]  
\[ \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \bigg|_{\eta=+\infty} = \frac{\partial \hat{g}(\eta, p)}{\partial \eta} \bigg|_{\eta=+\infty} = 0 \]

Previous equations clearly imply that for \( p = 0 \) and \( p = 1 \) these have the following solutions:

\[ \hat{f}(\eta, 0) = f_0(\eta), \quad \hat{f}(\eta, 1) = f(\eta) \]  
\[ \hat{g}(\eta, 0) = g_0(\eta), \quad \hat{g}(\eta, 1) = g(\eta) \]  
\[ \hat{\theta}(\eta, 0) = \theta_0(\eta), \quad \hat{\theta}(\eta, 1) = \theta(\eta) \]  
\[ \hat{\phi}(\eta, 0) = \phi_0(\eta), \quad \hat{\phi}(\eta, 1) = \phi(\eta) \]

As \( p \) increase from 0 to 1 continuously, \( \hat{f}(\eta, p), \hat{g}(\eta, p), \hat{\theta}(\eta, p), \) and \( \hat{\phi}(\eta, p) \) vary from \( f_0(\eta), g_0(\eta), \theta_0(\eta), \) and \( \phi_0(\eta) \) to the solutions \( f(\eta), g(\eta), \theta(\eta), \) and \( \phi(\eta) \). The Taylor’s series thus suggest that:

\[ \hat{f}(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \quad f_m(\eta) = \frac{1}{m!} \frac{\partial^m \hat{f}(\eta, p)}{\partial p^m} \bigg|_{p=0} \]  
\[ \hat{g}(\eta, p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta)p^m, \quad g_m(\eta) = \frac{1}{m!} \frac{\partial^m \hat{g}(\eta, p)}{\partial p^m} \bigg|_{p=0} \]  
\[ \hat{\theta}(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \quad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \hat{\theta}(\eta, p)}{\partial p^m} \bigg|_{p=0} \]  
\[ \hat{\phi}(\eta, p) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta)p^m, \quad \phi_m(\eta) = \frac{1}{m!} \frac{\partial^m \hat{\phi}(\eta, p)}{\partial p^m} \bigg|_{p=0} \]

The convergence of previous series depend upon the auxiliary parameters \( h_\eta, h_n, h_\theta, \) and \( h_\phi \). Assuming that \( h_\eta, h_n, h_\theta, \) and \( h_\phi \) are chosen such that the series in eqs. (34)-(37) are convergent at \( p = 1 \). Therefore:

\[ f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) \]
Equations (38)-(41) have the general solutions in the form:

\[ g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta) \]  

\[ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) \]  

\[ \phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta) \] 

where \( f_0^*(\eta), g_0^*(\eta), \theta_0^*(\eta), \text{and} \phi_0^*(\eta) \) denote the special solutions.

**Convergence of the HAM solutions**

The series solutions given in eqs. (38)-(41) contain the auxiliary parameters \( h_f, h_g, h_\theta, \text{and} h_\phi \). As pointed out by Liao [43, 44], the rate of approximation and the convergence of the solutions strongly depends upon the values of \( h_f, h_g, h_\theta, \text{and} h_\phi \). In the present case the proper values of \( h_f, h_g, h_\theta, \text{and} h_\phi \) are chosen through \( h \)-curves of \( f''(0), g''(0), \theta'(0), \text{and} \phi'(0) \) for 15th-order of approximation in fig. 1. The \( h \)-curves suggest that we can take \( h_f = h_g = h_\theta = h_\phi = h \), and thus the range of the admissible values of \( h \) is \(-1.15 \leq h \leq 0.25\).

Table 1 has been prepared to see how many orders of approximations are required for obtaining a convergent solution up to six decimal places. The results show that for velocities \( f \) and \( g \) a 12th order solution is sufficient however for \( \theta \) and \( \phi \) the required convergence will be achieved at 35th order solution.

**Results and discussion**

The HAM solutions in the form of an infinite series are obtained using symbolic software MATHEMATICA. The values of \( h \) are chosen in such a way that the obtained series is convergent for the chosen set of fluid parameters appearing in the problem. In the present discussion we are only focusing on the discussion of temperature profile and heat transfer across the surface for both constant surface temperature and constant heat flux cases. To depict the

**Figure 1.** The \( h \)-curves for \( f''(0), g''(0), \theta'(0), \text{and} \phi'(0) \) at 15th-order of approximation
influence of different parameters on the temperature profiles figs. 2-9 have been plotted. In all figs. 2-9, on 2(a), is referred to the constant surface temperature (CT) case and 9(b), is referred to constant heat (CH) flux case. The influence of parameter $A$ on temperature profiles $\theta$ and $\phi$ for 3-D flow situation is portrayed in fig. 2.

This figure shows that temperature is a decreasing function of $A$ for both CT and CH cases. It is also noted that the thermal boundary layer decreases with an increase in $A$. Figure 3 describe the effect of porosity parameter on the temperature profiles. It is evident from these figures that the temperature is high when fluid is passing through a porous medium. Figure 4 shows the effects of the magnetic parameter, $M$, on the dimensionless temperature profiles. The temperature increases by increasing the values of the magnetic parameter in both the cases of CT and flux CH, respectively. Figure 5 elucidates the influences of the stretching ratio, $\alpha$, on the temperature profiles. It is observed that the temperature profile decreases with increasing values of the stretching ratio in both CT and CH cases. It is also observed that the thermal boundary layer is decreased for large values of the stretching ratio. It is further noted that these results are in qualitatively similar with the temperature profiles shown by Liu and Andersson [33] in the presence of the magnetic field and porous medium. The impact of pow-

<table>
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<th>Order of approximation</th>
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<th>$-g'(0)$</th>
<th>$-\theta(0)$</th>
<th>$-\phi(0)$</th>
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</table>

![Figure 2](image-url)
er index \( r \) and \( s \) on the temperature profiles is seen through figs. 6 and 7. It is observed that both the indices have similar effect on the temperature profiles. The \( r \) and \( s \) decreases the temperature and thermal boundary layer thickness. Figure 8 shows the effects of the heat source/sink parameter, \( \beta \), on the temperatures \( \theta \) and \( \phi \). As expected, the temperature increases with increasing heat source \( \beta > 0 \), and decreases in the case of heat sink \( \beta < 0 \). The behavior of Prandtl number, is same is as that of unsteadiness parameter \( A \) as shown in fig. 9.

Table 2 shows the values of the heat transfer rate at the surface \( \theta'(0) \) for different values of \( r \) and \( s \) with \( \beta = 0 \), and \( \text{Pr} = 1 \) in the case of \( M = \epsilon = A = 0 \). It is found that the temperature gradient at the surface \( \theta'(0) \) becomes positive and decreases for \( r = -2 \) and \( s = 0 \), and
Figure 6. Influence of $r$ on temperature; (a) CT case, (b) CH case

Figure 7. Influence of $s$ on temperature; (a) CT case, (b) CH case

Figure 8. Influence of $\beta$ on temperature; (a) CT case, (b) CH case

Figure 9. Influence of Prandtl number on temperature; (a) CT case, (b) CH case
Table 2. Numerical values of $\theta(0)$, when $Pr = 1$, $\alpha = 0$, and $\varepsilon = M = 0 = A$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$r = 0$, $s = 0$</th>
<th>$r = -2$, $s = 0$</th>
<th>$r = 2$, $s = 0$</th>
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<th>$r = 0$, $s = 2$</th>
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<td>-1.292010</td>
</tr>
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Table 3. Numerical values of $\theta(0)$ with $\beta = 0$, $\varepsilon = 0.2$, $M = 0.5 = A = 0.5$, and $Pr = 1$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$r = 0$, $s = 0$</th>
<th>$r = -2$, $s = 0$</th>
<th>$r = 2$, $s = 0$</th>
<th>$r = 0$, $s = -2$</th>
<th>$r = 0$, $s = 2$</th>
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<td>-1.55989</td>
</tr>
</tbody>
</table>

Table 4. Numerical values of $\theta(0)$ and $\phi(0)$, when $\varepsilon = 0$, $M = 0$, $A = 0$, $r = 1.0$, $s = 1$, and $\alpha = 0.5$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\theta(0)$ for CT</th>
<th>$\phi(0)$ for CH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.741805</td>
<td>0.796317</td>
<td>0.870355</td>
</tr>
<tr>
<td>$-0.2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.741807</td>
<td>0.796319</td>
<td>0.870373</td>
</tr>
<tr>
<td>0.330257</td>
<td>0.315360</td>
<td>0.333069</td>
</tr>
<tr>
<td>0.330257</td>
<td>0.315365</td>
<td>0.333073</td>
</tr>
<tr>
<td>0.207807</td>
<td>0.217527</td>
<td>0.228754</td>
</tr>
<tr>
<td>0.207808</td>
<td>0.217530</td>
<td>0.228757</td>
</tr>
<tr>
<td>0.207808</td>
<td>0.217530</td>
<td>0.228757</td>
</tr>
<tr>
<td>0.207808</td>
<td>0.217530</td>
<td>0.228757</td>
</tr>
</tbody>
</table>
Table 5. Numerical values of $\theta'(0)$ and $\Phi(0)$, when $\varepsilon = 0.2$, $M = 0.5$, $r = 1.0$, $s = 1$, and $\alpha = 0.5$

<table>
<thead>
<tr>
<th>$\theta'(0)$ for CT</th>
<th>$\Phi(0)$ for CH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>$\beta = -0.2$</td>
<td>$\beta = 0$</td>
</tr>
<tr>
<td>$\Pr = 1$</td>
<td>$-1.59799$</td>
</tr>
<tr>
<td>$\Pr = 5$</td>
<td>$-4.09759$</td>
</tr>
<tr>
<td>$\Pr = 10$</td>
<td>$-5.80320$</td>
</tr>
</tbody>
</table>

Concluding remarks

The MHD 3-D flow and heat transfer characteristics of a viscous fluid due to an unsteady bi-directional stretching sheet through a porous medium is investigated in this paper. For the heat transfer analysis the heating process of CT, and CH are taken into account. The influence of the various parameters of interest is analyzed through similarity solution of the governing equations. The convergence of the developed series solution is explicitly discussed. A comparison with the existing results in the literature is also made and found in excellent agreement.

Nomenclature

$A$ – unsteadiness parameter

$a, b, c$ – stretching constant [T$^{-1}$]

$B_0$ – magnetic field strength

$C_i - C_{10}$ – integration constants

$C_{f_x}$ – skin friction coefficient in x-direction

$C_{f_y}$ – skin friction coefficient in y-direction

$C_p$ – specific heat

$f, g$ – dimensionless velocities

$k$ – thermal conductivity

$k_1$ – permeability of the medium

$M$ – magnetic parameter

$Pr$ – Prandtl number

$p$ – embedding parameter

$Q$ – heat source or sink

$r, s$ – power indices

$x, y, z$ – Cartesian co-ordinates

$\alpha$ – stretching ratio

$\beta$ – internal heat parameter

$\varepsilon$ – porosity parameter

$\eta$ – dimensionless independent co-ordinate

$\theta, \phi$ – dimensionless temperatures

$\nu$ – kinematic viscosity

$\rho$ – fluid density

$\sigma$ – electrical conductivity

$\phi_1$ – porosity of the medium

$h_x, h_y, h_\theta, h_4$ – auxiliary parameters

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