In this paper, variable separation method combined with the properties of Mittag-Leffler function is used to solve a variable-coefficient time fractional advection-dispersion equation with initial and boundary conditions. As a result, an explicit exact solution is obtained. It is shown that the variable separation method can provide a useful mathematical tool for solving the time fractional heat transfer equations.

Key words: advection-dispersion equation, variable separation method, Mittag-Leffler function, exact solution

Introduction

It is well known that many phenomena in engineering, physics, chemistry, economics, and other fields can be described very successfully using fractional differential equations. This make fractional calculus play a significant role in describing these phenomena. One of the most important fractional differential equation often used in engineering is the fractional advection-diffusion equation [1]. Recently, solving fractional differential equations has attached much attentions [2-5] and some effective methods have been proposed for fractional advection-diffusion equation, such as finite element method [6], Adomian decomposition method [7], homotopy perturbation method [8], and variational iteration method [9]. However, searching for exact analytical solutions of non-linear fractional differential equations is still on a preliminary stage [10]. When the inhomogeneities of media and non-uniformities of boundaries are taken into account, the variable-coefficient equations could describe more realistic physical phenomena than their constant-coefficient counterparts [11]. Therefore, how to construct exact solutions of fractional differential equations with variable coefficients is worth studying. The present paper is motivated by the desire to extend the variable separation method [12] to the following variable-coefficient time fractional advection-dispersion equation [1]:

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = -v(x,t)u_{x}(x,t) + d(x,t)u_{x}(x,t) + f(x,t), \quad 0 < t \leq T, \quad L < x < R \]  

subject to new initial and boundary conditions:

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\[ u(x,0) = 0, \quad L < x < R \] (2)
\[ u(L, t) = 0, \quad u(x,0) = 0, \quad u_t(R, t) = \varphi(t) \] (3)

where \( 0 < \alpha \leq 1 \) is a parameter describing the fractional derivative in the Caputo sense, \( \nu, d \geq 0 \), i.e., liquid is from left to right. The Caputo's fractional derivative is defined [13]:

\[ \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x, \tau)}{\partial \tau^m} \, d\tau \] (4)

for \( m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0 \). In this paper, some properties of the Caputo time-fractional derivative are used:

\[ D_t^\alpha [\lambda f(t) + \mu g(t)] = \lambda D_t^\alpha f(t) + \mu D_t^\alpha g(t) \] (5)
\[ D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \quad \gamma > 0 \] (6)
\[ D_t^\alpha E_\alpha (qt^\alpha) = qE_\alpha (t^\alpha) \] (7)

where \( \lambda, \mu, \) and \( q \) are constants or functions independent of \( t \), \( E_\alpha (\cdot) \) – the Mittag-Leffler function defined by:

\[ E_\alpha (z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)} \] (8)

**Exact solutions**

Here, we use the variable separation method [12] to solve eq. (1). To begin with, we take a transformation:

\[ u = g(t)w(x), \quad f(x, t) = f(t)w(x) \] (9)

then eq. (1) is converted into:

\[ \frac{\partial^\alpha [g(t)w(x)]}{\partial t^\alpha} = -\nu(x,t)g(t)w'(x) + d(x,t)w''(x) + f(t)w(x) \] (10)

Further supposing that:

\[ \frac{\partial^\alpha [g(t)w(x)]}{\partial t^\alpha} - f(t)w(x) = 0 \] (11)

then we get a solution of eq. (11):

\[ g(t) = \frac{A}{\Gamma(\alpha)} t^{\alpha-1} + \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(1+k+\alpha)} t^{k+\alpha}, \quad 0 < t \leq d \leq T \] (12)

Here \( f(t) \) is supposed can be expanded in Taylor series:

\[ f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k \] (13)

The \( g(t) \) has the initial condition \( D_t^{\alpha-1} g(t)|_{t=0} = A \) and converges for \( 0 < t \leq d \), \( D_t^{\alpha-1} \) – the Caputo fractional derivative operator, \( A \) – a constant, and \( d \) – the radius of convergence. Under the previous assumptions, eq. (10) is reduced to:

\[ [-\nu(x,t)w'(x) + d(x,t)w''(x)]g(t) = 0 \] (14)
Solving eq. (14), we have:

\[ w(x) = \int_{L}^{x} e^{\frac{1}{2} \frac{v(y,t)}{d(y,t)}} \, dy + c_2 \]  

(15)

and hence obtain an exact solution of eq. (1):  

\[ u = \left[ \frac{A}{\Gamma(\alpha)} t^{\alpha-1} + \sum_{k=0}^{\infty} \frac{f^{(k)}(t)}{\Gamma(1+k+\alpha)} \right] t^{\alpha} \int_{0}^{t} e^{\frac{1}{2} \frac{v(y,t)}{d(y,t)}} \, dy + c_2 \]  

(16)

where the constants \( A, c_1, c_2, \) and the function \( f(t) \) are determined by eqs. (2) and (3).

**Example**

Here, we further determine solution (16) through the example:

\[ D_t^\alpha u = -\sin h(x+t)u_x + \sin h(x+t)u_x + e^t \left[ \frac{1}{\Gamma(\alpha)} t^{\alpha-1} + \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+k+\alpha)} t^{k+\alpha} \right] \]  

(17)

subject to the initial and boundary conditions:

\[ u(x,0) = 0, \quad 0 < t < 1, \quad 0 < x < 2 \]  

(19)

\[ u(0, t) = 0, \quad u(x,0) = 0, \quad D_t^\alpha u(2, t) = (t^{-1} + e^t)(e^x - 1) \]  

(20)

In this case, it is easy to see:

\[ f(t) = e^t, \quad w(x) = e^x - 1, \quad c_1 = 0, \quad c_2 = 0 \]  

(21)

and that eq. (1) has a solution in the form:

\[ u = \left[ \frac{1}{\Gamma(\alpha)} t^{\alpha-1} + \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\alpha+1)} t^{k+\alpha} \right] (e^x - 1) \]  

(22)

When we set \( \alpha = 1 \), solution (22) becomes:

\[ u = e^t (e^x - 1) \]  

(23)

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**References**


