ABOUT LOCAL FRACTIONAL THREE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS IN CANTOR-TYPE CYLINDRICAL CO-ORDINATE SYSTEM

by

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In this article, we investigate the local fractional 3-D compressible Navier-Stokes equation via local fractional derivative. We use the Cantor-type cylindrical co-ordinate method to transfer 3-D compressible Navier-Stokes equation from the Cantorian co-ordinate system to the Cantor-type cylindrical co-ordinate system.

Key words: compressible Navier-Stokes equation, local fractional derivative, Cantor-type cylindrical co-ordinate method

Introduction

Local fractional partial differential equations were observed in different co-ordinate systems, such as heat-conduction equation [1], Helmholtz equation [2], Maxwell’s equation [3], wave equation [4], and diffusion equation [5]. The Cantor-type spherical co-ordinate [1], Cantor-type circle co-ordinate [2], and Cantor-type cylindrical co-ordinate [6] methods were proposed and developed to describe the heat transfer problems. The 3-D compressible Navier-Stokes equation in the Cantorian co-ordinate system via local fractional derivative was reported in [2, 7, 8]. In this manuscript, we use the Cantor-type cylindrical co-ordinate method to transfer 3-D compressible Navier-Stokes equation from the Cantorian co-ordinate system to the Cantor-type cylindrical co-ordinate system.

The Cantor-type cylindrical co-ordinate method

In this section, we introduce the concept of the local fractional derivative and the Cantor-type cylindrical co-ordinate method.

The local fractional partial derivative of the function \( \Phi_\kappa (\theta, \vartheta) \) of order \( \kappa \) \((0 < \kappa < 1)\) at \( \theta = \theta_0 \) is defined by [1, 2]:

\[
D^{(\kappa)}_\theta \Phi_\kappa (\theta_0, \vartheta) = \frac{d^\kappa \Phi_\kappa (\theta, \vartheta)}{d\theta^\kappa} \bigg|_{\theta=\theta_0} = \lim_{\theta \to \theta_0} \frac{\Delta^\kappa [\Phi_\kappa (\theta, \vartheta) - \Phi_\kappa (\theta_0, \vartheta)]}{(\theta - \theta_0)^\kappa} \tag{1a}
\]

where \( \Delta^\kappa [\Phi_\kappa (\theta, \vartheta) - \Phi_\kappa (\theta_0, \vartheta)] \equiv \Gamma(1 + \kappa) \Delta[\Phi_\kappa (\theta, \vartheta) - \Phi_\kappa (\theta_0, \vartheta)] \).

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The Cantor-type cylindrical co-ordinates can be written \([2, 4]\):

\[
\begin{align*}
\mu^\kappa &= R^\kappa \cos \kappa (\theta^\kappa), \\
\eta^\kappa &= R^\kappa \sin \kappa (\theta^\kappa), \\
\sigma^\kappa &= \sigma^\kappa
\end{align*}
\]

where \( R \in (0, +\infty), \sigma \in (-\infty, +\infty), \theta \in (0, 2\pi), \mu^2 + \eta^2 = R^2 \).

By using eq. (1), we have \([2, 4]\):

\[
\nabla^\kappa \vec{t} = \frac{\partial^\kappa r_R}{\partial R^\kappa} + \frac{1}{R^\kappa} \frac{\partial^\kappa r_\theta}{\partial \theta^\kappa} + \frac{r_R}{R^\kappa} + \frac{\partial^\kappa r_\sigma}{\partial \sigma^\kappa}
\]

(2a)

and

\[
\nabla^\kappa \times \vec{t} = \left( \frac{1}{R^\kappa} \frac{\partial^\kappa r_\theta}{\partial \sigma^\kappa} - \frac{\partial^\kappa r_\sigma}{\partial \theta^\kappa} \right) \vec{e}_R + \left( \frac{\partial^\kappa r_\sigma}{\partial R^\kappa} - \frac{\partial^\kappa r_\theta}{\partial \sigma^\kappa} \right) \vec{e}_\theta + \left( \frac{\partial^\kappa r_\theta}{\partial R^\kappa} + \frac{r_R}{R^\kappa} - \frac{1}{R^\kappa} \frac{\partial^\kappa r_\sigma}{\partial \theta^\kappa} \right) \vec{e}_\sigma
\]

(2b)

where

\[
r = R^\kappa \cos \kappa (\theta^\kappa) \vec{e}_R^\kappa + R^\kappa \sin \kappa (\theta^\kappa) \vec{e}_\theta^\kappa + \sigma^\kappa \vec{e}_\sigma^\kappa = r_R \vec{e}_R + r_{\theta \theta} \vec{e}_\theta + r_{\theta \sigma} \vec{e}_\sigma
\]

(2c)

The local fractional gradient and Laplace operators in the Cantor-type cylindrical co-ordinate system are written as \([2, 4]\):

\[
\nabla^\kappa \phi (R, \theta, \sigma) = \vec{e}_R^\kappa \frac{\partial^\kappa \phi}{\partial R^\kappa} + \vec{e}_\theta^\kappa \frac{1}{R^\kappa} \frac{\partial^\kappa \phi}{\partial \theta^\kappa} + \vec{e}_\sigma^\kappa \frac{\partial^\kappa \phi}{\partial \sigma^\kappa}
\]

(3a)

\[
\nabla^{2\kappa} \phi (R, \theta, \sigma) = \vec{e}_R^{2\kappa} \frac{\partial^{2\kappa} \phi}{\partial R^{2\kappa}} + \frac{1}{R^{2\kappa}} \vec{e}_\theta^{2\kappa} \frac{\partial^{2\kappa} \phi}{\partial \theta^{2\kappa}} + \frac{1}{R^{2\kappa}} \vec{e}_\sigma^{2\kappa} \frac{\partial^{2\kappa} \phi}{\partial \sigma^{2\kappa}}
\]

(3b)

where a local fractional vector is \([2, 4]\):

\[
\vec{e}_R^\kappa = \cos \kappa (\theta^\kappa) \vec{e}_R + \sin \kappa (\theta^\kappa) \vec{e}_\theta + \vec{e}_\sigma^\kappa
\]

\[
\vec{e}_\theta^\kappa = -\sin \kappa (\theta^\kappa) \vec{e}_R + \cos \kappa (\theta^\kappa) \vec{e}_\theta + \vec{e}_\sigma^\kappa
\]

\[
\vec{e}_\sigma^\kappa = \vec{e}_\sigma^\kappa
\]

(3c)

The local fractional operator is written:

\[
\nabla^\kappa (V^\kappa \vec{A}) = \frac{\partial^{2\kappa} A_R}{\partial R^{2\kappa}} + \frac{1}{R^{2\kappa}} \frac{\partial^{2\kappa} A_\theta}{\partial \theta^{2\kappa}} + \frac{A_R}{R^{2\kappa}} + \frac{\partial^{2\kappa} A_\sigma}{\partial \sigma^{2\kappa}} + \frac{1}{\Gamma(1-2\kappa)R^{2\kappa}} \frac{\partial^{2\kappa} \vec{A}_{\theta}}{\partial \theta^{2\kappa}} + \frac{\vec{A}_{\sigma}}{\Gamma(1-2\kappa)R^{2\kappa}}
\]

(3d)

**Transferring the 3-D compressible Navier-Stokes equation**

For compressible fluid, the 3-D Navier-Stokes equation on Cantor sets without the specific fractal body force is written in the form:

\[
\frac{\rho \partial^\kappa \vec{u}}{\partial t^\kappa} = -\nabla^\kappa p + \mu^\kappa \nabla^\kappa \left[ (\nabla^\kappa \vec{u})^\kappa \right] + \nu^\kappa \nabla^\kappa \vec{u} - \rho \vec{g}(\nabla^\kappa \vec{u})
\]

(4a)
which reduces to:

$$\frac{\partial\tilde{\sigma}}{\partial\xi} = -\nabla^\xi p + \frac{\sigma}{3} \nabla^\xi [\nabla^\xi \tilde{\sigma}] + \sigma \nabla^2\tilde{v} - \tilde{u}(\nabla^\xi \tilde{v})$$  \hspace{1cm} (4b)

where $p$ is the fractal pressure field, $\rho$ – the fractal mass density, $\tilde{u}$ – the fractal flow velocity, and $\mu$ – the dynamic viscosity.

The 3-D compressible Navier-Stokes equations on Cantor sets without the specific fractal body force in the Cantorian co-ordinate system using eq. (4b) are written:

$$\frac{\partial\tilde{v}_x}{\partial\xi} = -\frac{\partial}{\partial\xi} P + \underline{\underline{v}} \left( \frac{\partial^2 \tilde{v}_x}{\partial\xi^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_x}{\partial\theta^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_x}{\partial\phi^2} \right) + \underline{\underline{v}} \left( \frac{\partial^2 \tilde{v}_x}{\partial\xi^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_x}{\partial\theta^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_x}{\partial\phi^2} \right) + \nabla^2 \tilde{v}_x - \tilde{u}(\nabla^\xi \tilde{v}) \hspace{1cm} (4c)

$$

$$\frac{\partial\tilde{v}_y}{\partial\xi} = -\frac{\partial}{\partial\xi} P + \underline{\underline{v}} \left( \frac{\partial^2 \tilde{v}_y}{\partial\xi^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_y}{\partial\theta^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_y}{\partial\phi^2} \right) + \underline{\underline{v}} \left( \frac{\partial^2 \tilde{v}_y}{\partial\xi^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_y}{\partial\theta^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_y}{\partial\phi^2} \right) + \nabla^2 \tilde{v}_y - \tilde{u}(\nabla^\xi \tilde{v}) \hspace{1cm} (4d)

$$

$$\frac{\partial\tilde{v}_z}{\partial\xi} = -\frac{\partial}{\partial\xi} P + \underline{\underline{v}} \left( \frac{\partial^2 \tilde{v}_z}{\partial\xi^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_z}{\partial\theta^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_z}{\partial\phi^2} \right) + \underline{\underline{v}} \left( \frac{\partial^2 \tilde{v}_z}{\partial\xi^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_z}{\partial\theta^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}_z}{\partial\phi^2} \right) + \nabla^2 \tilde{v}_z - \tilde{u}(\nabla^\xi \tilde{v}) \hspace{1cm} (4e)

$$

where $v_x$, $v_y$, and $v_z$, are the components of the fractal flow velocity $\tilde{v}$ in the directions x, y, and z, respectively.

The 3-D compressible Navier-Stokes equations on Cantor sets without the specific fractal body force in the Cantor-type cylindrical co-ordinate system are written:

$$\frac{\partial \tilde{v}}{\partial\xi} = -\left( \frac{\partial^2 \tilde{v}}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}}{\partial\theta^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}}{\partial\phi^2} \right) + \underline{\underline{v}} \left( \frac{\partial^2 \tilde{v}}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}}{\partial\theta^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{v}}{\partial\phi^2} \right) + \nabla^2 \tilde{v} - \tilde{u}(\nabla^\xi \tilde{v}) \hspace{1cm} (4f)

$$

where $\tilde{u}_R$, $\tilde{u}_\theta$, and $\tilde{u}_\phi$ are the components of the fractal flow velocity $\tilde{u}$ in the Cantor-type cylindrical co-ordinate system, respectively.

For compressible fluid, the Navier-Stokes equation on Cantor sets with the specific fractal body force, $b$, is written in the form:
\[
\rho \frac{\partial^x \tilde{u}}{\partial t^x} = -\nabla^x p + \frac{\mu}{3} \nabla^x [(\nabla^x \tilde{u})] + \mu \nabla^2 \tilde{u} + \rho \tilde{b} - \rho \tilde{u} (\nabla^x \tilde{u}), \tag{5a}
\]

which leads to:

\[
\frac{\partial^x \tilde{v}}{\partial t^x} = -\nabla^x P + \frac{\sigma}{3} \nabla^x [(\nabla^x \tilde{u})] + \sigma \nabla^2 \tilde{v} - \tilde{u} (\nabla^x \tilde{u}) + \tilde{b} \tag{5b}
\]

where \( \tilde{b} \) is the body acceleration, and \( P = p/\rho \) and \( \sigma = \mu/\rho \).

The 3-D compressible Navier-Stokes equations on Cantor sets without the specific fractal body force in the Cantor-type cylindrical co-ordinate system are written as:

\[
\frac{\partial^x v_x}{\partial t^x} = -\frac{\partial^x P}{\partial x^x} + \sigma \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \frac{\sigma}{3} \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + b_x \tag{5c}
\]

\[
\frac{\partial^x v_y}{\partial t^x} = -\frac{\partial^x P}{\partial y^x} + \sigma \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \frac{\sigma}{3} \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + b_y \tag{5d}
\]

\[
\frac{\partial^x v_z}{\partial t^x} = -\frac{\partial^x P}{\partial z^x} + \sigma \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \frac{\sigma}{3} \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + b_z \tag{5e}
\]

The 3-D compressible Navier-Stokes equations on Cantor sets without the specific fractal body force in the Cantor-type cylindrical co-ordinate system are written as:

\[
\frac{\partial^x \tilde{v}}{\partial t^x} = -\left( \frac{\partial^2 \tilde{v}}{\partial R^2} + \frac{\partial}{\partial \theta} \frac{1}{R^2} \frac{\partial^2 \tilde{v}}{\partial \theta^2} + \frac{\partial^2}{\partial \sigma^2} \frac{\partial^x P}{\partial \sigma^x} \right) + \frac{\sigma}{3} \left( \frac{\partial^2 \tilde{u}_R}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 \tilde{u}_R}{\partial \theta^2} + \frac{\partial^2 \tilde{u}_\theta}{\partial \sigma^2} \right) + \frac{\tilde{u}_R}{\Gamma (1 - 2\kappa) R^{2k}} + \frac{\tilde{u}_\theta}{\Gamma (1 - 2\kappa) R^{2k}} + \frac{\tilde{u}_\sigma}{\Gamma (1 - 2\kappa) R^{2k}} + \tilde{b} \tag{5f}
\]
Conclusion

In this work, we investigated the 3-D compressible Navier-Stokes equations on Cantor sets with and without the specific fractal body forces in the Cantorian co-ordinate systems. We applied the Cantor-type cylindrical co-ordinate method to transfer the 3-D compressible Navier-Stokes equations on Cantor sets in the Cantorian co-ordinate system into the Cantor-type cylindrical co-ordinate system.

Nomenclature

\[ x, y, z \] – space co-ordinate, [m]
\[ \rho \] – fractal pressure field, [Pa·m\(^{-3}\)]

Greek symbols

\[ \kappa \] – fractal dimension, [-]
\[ \rho \] – fractal mass density, [kg·m\(^{-3}\)]

References


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